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Electron Radial Wave Functions and Nuclear Beta-decay

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PREFACE

AS THERE ARE already a number of excellent books on the subject of beta decay, all published in the late sixties, or early seventies, one could ask the question: why another book of this kind?

Well, these last years have, after all, brought some basic changes in the way of looking at the whole complex of beta decay, and, at the same time, weak interactions as a whole. As far as the beta decay treatment is concerned, problems are tackled nowadays with the so-called elementary particle treatment, i.e. one now uses methods which first emerged in elementary particle physics for low energy beta decay as well. The background for this change in philosophy, which is quite contrary to former practice (of considering the decaying nucleons inside the nucleus as independent), is to be seen in the fact that today there is a great tendency to achieve model-independent statements from experiments; 'model-independent' in this case means independent of details of the underlying nuclear structure. In conflict with this model independency, where observables are expressed in terms of form factors, is the Coulomb final state interaction between the charged electron and the charged nucleus. Because of this, a lot of model dependencies crop up which could otherwise be ignored. These dependencies arise because the number of form factors becomes too great. For this reason it is particularly important to pay attention to the Coulomb interaction.

Of course, there is another reason. The precision of experiments in the last decade has increased so much for various observables that even small effects have to be considered nowadays, whereas in the past these could be omitted. Sometimes it is particularly the Coulomb interaction which is either directly responsible for such small effects or which makes the interpretation of observables, which are difficult to measure, more complicated. Thus, the interdependence of Coulomb interaction and beta decay is one of the main aspects of the book.

The present book is certainly not an elementary introduction to the subject of nuclear beta decay, and the prospective reader is expected to have already some knowledge of the basic physical phenomena and their elementary theoretical description. Further prerequisites are some knowledge of the techniques of the quantum-mechanical theory of angular momenta, of nuclear physics and of the physical aspects of relativistic quantum mechanics. From the topic of special functions, the gamma function and spherical Bessel functions are assumed to be well-known subjects, but lack of familiarity with them is not expected to be too prohibitive since all properties and equations for these quantities are

explicitly stated where they are needed. Similarly, familiarity with linear ordinary differential equations and complex integration would be helpful but is not necessary.

The composite title *Electron radial wave functions and nuclear beta decay* has been chosen so as to indicate that the electron radial wave functions, in fact, play a major role for our treatise on nuclear beta decay, and that in this respect the present book is significantly different from the well-known earlier books on beta decay. Also, the composite title is to indicate that the book contains a more detailed and further-reaching account of electron radial wave functions than would necessarily be needed in the context of nuclear beta decay. In particular, Chapter 3 is hoped also to be of interest to readers who do not work on beta decay but who deal with electron radial wave functions in another context.

In fact, Chapters 3 and 4 can be read independently of the other parts of the book. Chapter 4, of course, uses many results from Chapter 3, but we expect that the user who is mainly interested in Chapter 4 can read it independently of Chapter 3 if he is willing to accept the statements based on and the results quoted from Chapter 3, and in any case for a more detailed understanding he would not need the whole of Chapter 3. Similarly, the chapters on beta decay can be read independently of the chapters on electron wave functions if the reader is willing to accept the results quoted from Chapter 4. In order to support such independent reading we have sometimes repeated important formulae which are needed from other chapters (or even other sections, if far away, of the same chapter).

What the book contains may best be seen from the detailed table of contents, so only a few additional remarks are needed. Chapters 12 and 13 merely report, without any kind of derivation, some results on gauge theories and radiative corrections, topics which to some extent have been included for the convenience of the reader although they are essentially beyond the scope of our treatise. For in order to treat these phenomena in detail, we would need quite different techniques, such as those based on Feynman graphs, which otherwise have not been applied at all in this book. Other topics which are not dealt with are double beta decay, inner bremsstrahlung, atomic effects accompanying beta decay, and gross theories.

Although this is a theoretical book, occasionally the formulae are also compared with corresponding experiments. It should be pointed out, however, that the book does not contain a compilation of all existing experiments. The authors have just selected suitable ones when opportune.

To choose an appropriate notation is a particularly cumbersome task for a book of this scope. We would have liked to have used many more

different symbols than are available. Many of our symbols therefore have a local meaning for one or a few sections only, and are used in other sections for different quantities. The coefficients of various different power series, for instance, are all denoted by a_n. For the global variables, on the other hand, which have the same meaning throughout the whole book, we sometimes use a simplified notation, omitting subscripts or indexes which are kept fixed. The momentum of the electron p_e, for instance, is simply denoted by p in Chapter 4 where almost exclusively electrons are considered, or the scattering phase shift Δ_κ is simply denoted by Δ as long as the angular momentum quantum number κ need not be specified. As for the more mathematical parts of the book we have decided not to use the modern notation, like $x \in]0, 1[$, for instance, in place of $0 < x < 1$, in view of the fact that many of the workers on beta decay belong, like ourselves, to an older generation.

Although our list of references is not short, we want to point out that we did not intend to give a complete bibliography, but have included when appropriate some of those references which are of immediate relevance to our text in giving support to our statements or in providing additional information beyond our treatment. In addition some of the references have been included which treat a subject for the first time or otherwise seem to be interesting for the historical development.

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March, 1981

H. B.
W. B.

CONTENTS

1
INTRODUCTION

FOR A LONG time the strong, electromagnetic, weak, and gravitational interactions were believed to be the four basic interactions. Generally, their processes were treated separately. A unified theory which was able to describe them all did not exist. As far as weak interaction theory is concerned, to which our interest is directed in the following, its development is strongly connected with the understanding of beta-decay. Many of the basic ideas in weak interactions were originally discovered from theoretical and experimental investigations of nuclear beta-decay. The theoretical game started when Fermi (1934) first suggested his famous theory of beta-decay. In his attempt only the vector interaction was considered. At that early stage it was also Fermi who recognized that beta-decay is strongly influenced by the Coulomb interaction between the beta-particle and the charged nucleus. The function which takes care of the essential effects in that context, and later on was usually called the 'Fermi function', was already explicit in his article. It should also be mentioned that even the existence of forbidden beta-transitions was globally discussed in that paper. Shortly after Fermi's first formulation of beta-decay theory the formalism was completed by showing that the Hamiltonian must be a linear combination of the five possible relativistic invariants, i.e. scalar, vector, tensor, axial vector, and pseudoscalar interaction (Bethe and Bacher 1936, Fierz 1937). Corresponding selection rules for beta-disintegrations were additionally discussed (Gamow and Teller 1936). Electron capture was also soon included (Yukawa and Sakata 1935, 1936, 1937; Bethe and Bacher 1936; Möller 1937). Thus, the form of the beta-decay or weak interaction theory, which is today usually called the phenomenological one, had finally been worked out by about 1937.

This form has been remarkably stable over the last forty years. With some minor changes it remains up to now a fundamental corner-stone for a theoretical description of nuclear beta-decay. It has also been applied with success to many other weak processes. The question, however, which, or which combination, of the five possible interactions (scalar, vector, tensor, axial vector, pseudoscalar) really contributes could only be decided experimentally. This point was not definitively cleared up until twenty years later. In the meantime, the theory of forbidden transitions was elaborated in detail for beta-decay (Konopinski and Uhlenbeck 1941) and for electron capture (Marshak 1942). In that context for the

first time the necessary beta-decay Coulomb functions were additionally considered in more detail because of the greater sensitivity of the relevant observables to such details. A summary of the status of knowledge at that time was given by Konopinski (1943).

With the end of the forties, beta-radioactive sources of high specific activities became available. A very active era of experimental beta-spectroscopy began. One after another, more and more puzzles in the theory were experimentally solved (Konopinski and Langer 1953). A further modification of the theory became necessary, however, when parity violation in beta-decay was discovered by Wu and co-workers in 1956 (Wu *et al.* 1957). Their experiment was initiated by Lee and Yang who claimed that no experimental proof of the hypothesis of parity conservation for beta-decay existed at that time (Lee and Yang 1956). Two years later, by the end of 1958, the final form of the V–A interaction was then conclusively established, mainly by electron neutrino angular correlation experiments and by the determination of the electron asymmetry of polarized neutrons. After the concept of the universal Fermi interaction had been proposed in 1948 to 1949 the near equality of the coupling constants G_β in beta-decay and of the Fermi constant g_μ in μ-decay led to the introduction of the conserved vector current (CVC) hypothesis (Gershtein and Zeldovich 1956; Feynmann and Gell-Mann 1958). Experimental evidence for the CVC theory was then obtained up to 1964 (Wu 1964). A second symmetry concept which makes use of the behaviour of the weak interacting currents under a G-parity transformation was suggested by Weinberg (1958). In this framework one distinguishes between first and second class currents, both types being distinguished by their G-parity transformation properties. The fifties and the sixties can be considered as the most fruitful epoch of beta-spectroscopy. Thus a number of excellent reviews (Konopinski 1959; Weidenmüller 1961; Wu *et al.* 1965; Blin-Stoyle 1966) and textbooks (Wu and Moszkowski 1966; Konopinski 1966; Schopper 1966; Morita 1973; Blin-Stoyle 1973) about beta-decay appeared in or shortly after that period, and all aspects were fully elucidated. Electron capture especially had been considered in detail (Brysk and Rose 1958; Bouchez and Depommier 1960; Berenyi 1968; Bambynek *et al.* 1977). The progress of experimental techniques in beta-decay increased the accuracy of the experimental data which, on the other hand, required more accurate theoretical calculations for their interpretation. Thus, greater attention was also given to a more exact treatment of the necessary Coulomb functions (Bühring 1963, 1965; Behrens and Bühring 1971). Also, a number of comprehensive tables for these beta-decay Coulomb functions (Fermi function, etc.) were published (Dzhelepov and Zyrianova 1956; Bhalla and Rose 1960, 1962; Behrens and Jänecke 1969; Dzhelepov and Zyrianova 1972; Bambynek *et al.* 1977).

In the period before 1964 the approximation of considering the nucleons inside the nucleus to be independent (impulse approximation) was introduced at the beginning of every theoretical treatment of beta-decay. All formulae obtained up till then were therefore expressed in terms of nuclear matrix elements. But the results derived in this way are, of course, burdened by approximations and not of general validity. In view of this situation the theory was reformulated by expressing all observables model-independently in terms of form factors, a method which was well-known before from elementary particle physics (Stech and Schülke 1964; Kim and Primakoff 1965). This elementary particle theory (EPT) formalism, which, so to speak, forgets that the nucleus consists of nucleons, became more and more used (see Schopper 1966). The reason was that this approach offered excellent possibilities for establishing model-independent constraints. In that context a particular version of this formalism introduced by Holstein (Holstein *et al.* 1972; Holstein 1974*a*) should be mentioned especially since it was this special formulation which was applied to various analyses of allowed transitions.

Now we come to the last ten years. As far as beta-decay and the weak interaction are concerned the activities within the seventies can, *cum grano salis*, be summarized under the heading 'symmetries'. In the field of beta-decay where many original ideas were intimately connected with the name of Wilkinson, one was interested in a possible failure of the CVC hypothesis, in the renormalization of the axial vector coupling constant in complex nuclei, in the validity of time reversal invariance, and in the possible existence of second class currents. For that purpose, experiments of the highest precision have been carried out in order to determine ft-values of the 0^+–0^+ superallowed transitions (for a review see Wilkinson 1978; Wilkinson *et al.* 1978) and in order to search for small deviations from the usual behaviour of allowed transitions (see, for example, Wu 1977; Wilkinson 1978; Calaprice 1978; Behrens *et al.* 1978; Telegdi 1979). Naturally, the highest precision was now also required for the theoretical interpretation of these results. Coulomb effects, of which the main part can be taken care of via the electron radial wave functions, had to be treated more and more accurately. Even very small terms which could be omitted in earlier times without any problems are of importance today. No dramatic changes, however, emerged in the end. Up to now, CVC and time reversal invariance are believed to be valid and second class currents to be absent, as was assumed before. But it should be noted that a certain increase in precision with respect to every aspect should be achieved in order to come to definite conclusions.

From a universal point of view, on the other hand, dramatic changes have occurred in the last decade. The so-called phenomenological theory of weak interactions which was originally developed by Fermi, as mentioned before, is in fact not renormalizable. That means, in a usual

perturbation expansion, every order beyond the first diverges but these infinities can not be made to disappear as in quantum electromagnetics. In fact, no basic theory is normally expressed by the name phenomenological. That problem was not solved until the so-called gauge theories, first proposed by Weinberg (1967) and Salam (1968). These gauge theories predicted that the weak interaction density contains neutral currents in addition to charged ones. Indeed, these neutral currents have been found (Hasert *et al.* 1973). This discovery can be considered as the breakthrough for gauge theories, but this type of theory is able to bring about even more. Actually, they overcame the other old problem of how to treat the four fundamental interactions separately and to consider them to be basically different ones. As a first step, both weak and electromagnetic interactions could be described in an unified way on the basis of a spontaneously broken gauge symmetry (Weinberg 1980). Next, a synthesis of weak, electromagnetic, and strong interactions, which is usually called the grand unified theory, was established by assuming all these interactions to be gauge ones (Glashow 1980). But beyond that, the great vision of an ultimate unification of all forces could be seen on the horizon for the first time (Salam 1980). Although the topic of gauge theories is beyond the scope of this book, we would not omit at least a mention of these fundamentally important developments.

2

DIRAC EQUATION

2.1. Dirac equation, units, and conventions

SEVERAL different conventions are in use for writing the Dirac equation and related quantities. It is therefore the main purpose of this chapter to establish the conventions and definitions to be used in this book, rather than to given an introduction to Dirac theory.

The Dirac equation for a free particle of mass m may be written in Hamiltonian form as

$$\{-(\boldsymbol{\alpha}\not{p})-\beta m\}\psi(\mathbf{r}, t)=\mathcal{W}\psi(\mathbf{r}, t) \tag{2.1}$$

where $\not{p}$ is the vector differential operator

$$\not{p}=-\mathrm{i}\,\mathrm{grad} \tag{2.2}$$

with respect to the space co-ordinates $\mathbf{r}=(x, y, z)$ and $\mathcal{W}$ the differential operator

$$\mathcal{W}=\mathrm{i}\,\partial/\partial t \tag{2.3}$$

with respect to time. Natural units are being used such that

$$\hbar=1,$$

$$c=1,$$

$$m_e=1,$$

where $2\pi\hbar=h$ is Planck's constant, c the velocity of light, and m_e the rest mass of the electron. The term $(\boldsymbol{\alpha}\not{p})$ denotes a scalar product with

$$\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \alpha_3)$$

a vector the components of which are four by four hermitian matrices. Similarly, β is a hermitian four by four matrix. Equation (2.1) therefore represents a system of four coupled linear partial differential equations. The Dirac matrices $\alpha_1, \alpha_2, \alpha_3, \beta$ satisfy

$$\alpha_1^2=\alpha_2^2=\alpha_3^2=\beta^2=1, \tag{2.4}$$

$$\alpha_k\beta+\beta\alpha_k=0, \tag{2.5}$$

$$\alpha_k\alpha_l+\alpha_l\alpha_k=2\delta_{kl}1 \tag{2.6}$$

with

$$\delta_{kl}=\begin{cases}1 & \text{if } k=l\\ 0 & \text{if } k\neq l\end{cases} \tag{2.7}$$

$$k, l=1, 2, 3.$$

Here 0 and 1 are the four by four zero and unit matrix, respectively. While eqns (2.4)–(2.6) describe the essential properties of the Dirac matrices, they do not fix them uniquely. By convention we may choose the following explicit representation:

$$\beta = \begin{Bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{Bmatrix}, \tag{2.8}$$

$$\alpha_1 = \begin{Bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{Bmatrix}, \tag{2.9a}$$

$$\alpha_2 = \begin{Bmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{Bmatrix}, \tag{2.9b}$$

$$\alpha_3 = \begin{Bmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{Bmatrix}, \tag{2.9c}$$

where the elements not filled in are zero. These matrices may be represented conveniently in partitioned form

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.10}$$

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1^{\mathrm{P}} \\ \sigma_1^{\mathrm{P}} & 0 \end{pmatrix}, \tag{2.11a}$$

$$\alpha_2 = \begin{pmatrix} 0 & \sigma_2^{\mathrm{P}} \\ \sigma_2^{\mathrm{P}} & 0 \end{pmatrix}, \tag{2.11b}$$

$$\alpha_3 = \begin{pmatrix} 0 & \sigma_3^{\mathrm{P}} \\ \sigma_3^{\mathrm{P}} & 0 \end{pmatrix}, \tag{2.11c}$$

in terms of the two by two submatrices

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.12a}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.12b}$$

$$\sigma_1^{\mathrm{P}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.13a}$$

$$\sigma_2^{\mathrm{P}} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{2.13b}$$

$$\sigma_3^{\mathrm{P}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.13c}$$

where the σ_k^{P} with $k = 1, 2, 3$ are the well-known Pauli spin matrices. It is convenient to introduce the four by four matrices

$$\sigma_k = \begin{pmatrix} \sigma_k^{\mathrm{P}} & 0 \\ 0 & \sigma_k^{\mathrm{P}} \end{pmatrix}, \tag{2.14}$$

and the matrix

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.15}$$

so that we have

$$\alpha_k = \gamma_5 \sigma_k = \sigma_k \gamma_5 \tag{2.16a}$$

or

$$\sigma_k = \gamma_5 \alpha_k = \alpha_k \gamma_5 \tag{2.16b}$$

with

$$(\gamma_5)^2 = 1. \tag{2.17}$$

Equations (2.16), which apply for $k = 1, 2, 3$, are often written as 'vector' equations

$$\boldsymbol{\alpha} = \gamma_5 \boldsymbol{\sigma} = \boldsymbol{\sigma} \gamma_5, \tag{2.18a}$$

$$\boldsymbol{\sigma} = \gamma_5 \boldsymbol{\alpha} = \boldsymbol{\alpha} \gamma_5. \tag{2.18b}$$

Furthermore, we can introduce the matrix

$$\gamma_4 = -\beta \tag{2.19}$$

and for $k = 1, 2, 3$ the matrices

$$\gamma_k = -\mathrm{i}\beta\alpha_k = \mathrm{i}\gamma_4 \alpha_k \tag{2.20}$$

such that for $\mu, \nu = 1, 2, 3, 4$ we have

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} 1. \tag{2.21}$$

It then follows that

$$\gamma_1 \gamma_2 \gamma_3 \gamma_4 = \gamma_5. \tag{2.22}$$

Introducing the four-vector

$$x = (x_1, x_2, x_3, x_4) = (\mathbf{r}, \mathrm{i}t) \tag{2.23}$$

and the four-vector operator

$$P=(P_1, P_2, P_3, P_4)=(\not{p}, \mathrm{i}\mathcal{W}) \tag{2.24}$$

we may write the Dirac equation in the relativistically covariant form

$$\left(\sum_{\mu=1}^{4} \gamma_\mu P_\mu - 1\mathrm{i}m\right)\psi(x)=0. \tag{2.25}$$

This equation is obtained if eqn (2.1) is multiplied from the left by $\mathrm{i}\beta$.

From the Dirac equation for free particles in the form of eqn (2.25), the corresponding Dirac equation for the case when the particle is charged and an external electromagnetic field is present may be obtained by the substitution

$$P_\mu \rightarrow P_\mu - (e/c)A_\mu. \tag{2.26}$$

Here e is the charge of the particle with $e<0$ in the case of an electron or $e>0$ in the case of a positron. The external field is represented by the four-vector potential

$$A=(A_1, A_2, A_3, A_4)=(\mathbf{A}, \mathrm{i}\Phi) \tag{2.27}$$

with the vector potential $\mathbf{A}$ and the scalar potential Φ. Particularly important in our context is the case when $\mathbf{A}\equiv 0$ and Φ does not depend on time, for which the Dirac equation becomes†

$$\left(\sum_{\mu=1}^{4} \gamma_\mu P_\mu - 1\mathrm{i}m\right)\psi(x)=\mathrm{i}e\Phi\gamma_4\psi(x). \tag{2.28}$$

Introducing the Dirac potential

$$V(\mathbf{r})=e\Phi \tag{2.29}$$

and returning to the Hamiltonian form we obtain the Dirac equation for a particle in an external potential

$$[-(\boldsymbol{\alpha}\not{p})-\beta m-1\{\mathcal{W}-V(\mathbf{r})\}]\psi(\mathbf{r}, t)=0. \tag{2.30}$$

By a product ansatz for the solution the time and space coordinates may be separated easily. It follows that, with

$$\psi(\mathbf{r}, t)=\psi(\mathbf{r})\exp(-\mathrm{i}Wt), \tag{2.31}$$

$\psi(\mathbf{r})$ is a solution of the time-independent Dirac equation

$$[-(\boldsymbol{\alpha}\not{p})-\beta m-1\{W-V(\mathbf{r})\}]\psi(\mathbf{r})=0, \tag{2.32}$$

where, because of Planck's relation between frequency and energy, W is the total energy of the particle (including its rest mass).

† We may recall that $c=1$ in our units.

Let us consider solutions of eqn (2.32) for the simple case when the potential vanishes identically. By means of the ansatz

$$\psi(\mathbf{r}) = u(\mathbf{p})\exp\{i(\mathbf{pr})\} \tag{2.33}$$

the differential equation immediately reduces to an algebraic equation

$$\{-(\boldsymbol{\alpha}\not{p}) - \beta m - 1W\}u(\mathbf{p}) = 0 \tag{2.34}$$

for the four-spinor $u(\mathbf{p})$ which is a column matrix. Its dependence on $\mathbf{p}$ is indicated explicitly for later convenience, although $\mathbf{p}$ here is a constant vector rather than a variable of the problem. Writing the four-spinor in terms of two-spinors

$$u(\mathbf{p}) = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{2.35}$$

and the eqn (2.34) in partitioned form in terms of two by two matrices, we have

$$\begin{pmatrix} -1(W+m) & -(\boldsymbol{\sigma}^{\mathrm{P}}\mathbf{p}) \\ -(\boldsymbol{\sigma}^{\mathrm{P}}\mathbf{p}) & -1(W-m) \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.36}$$

While the first line of this matrix equation yields†

$$\chi_1 = -(W+m)^{-1}(\boldsymbol{\sigma}\mathbf{p})\chi_2, \tag{2.37a}$$

the second one yields

$$\chi_2 = -(W-m)^{-1}(\boldsymbol{\sigma}\mathbf{p})\chi_1. \tag{2.37b}$$

These two equations are compatible only if

$$\{(W-m)(W+m)\}^{-1}(\boldsymbol{\sigma}\mathbf{p})(\boldsymbol{\sigma}\mathbf{p}) = 1 \tag{2.38}$$

or, because‡

$$(\boldsymbol{\sigma}\mathbf{p})(\boldsymbol{\sigma}\mathbf{p}) = 1p^2, \tag{2.39}$$

if

$$W^2 - m^2 = p^2. \tag{2.40}$$

The vector $\mathbf{p}$ therefore is seen to be the momentum of the particle. If p and W are related according to eqn (2.40), eqns (2.37) are linearly dependent so that either χ_1 or χ_2 may be chosen arbitrarily. Since both are two-component quantities, we need two basis spinors

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.41a}$$

$$\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.41b}$$

† We shall no longer carry the superscript P which indicates that the two by two Pauli spin matrices are needed rather than the four by four ones.

‡ In this chapter we use $p = |\mathbf{p}|$ and a corresponding notation for other vectors (three-vectors).

which span the corresponding space. Since these spinors are eigen functions of the Pauli spin matrix σ_3^P, they describe spin states with a definite spin component of $\frac{1}{2}$ or $-\frac{1}{2}$, respectively, in the space direction x_3.

For a given momentum vector $\mathbf{p}$, two corresponding values of the total energy W, differing in sign, have to be considered according to eqn (2.40). The four linearly independent solutions, properly normalized, of eqn (2.36) may then be written

$$u_{(+)}^{(s)}(\mathbf{p}) = \{(|W|+m)/(2|W|)\}^{1/2} \begin{pmatrix} -(|W|+m)^{-1}(\boldsymbol{\sigma}\mathbf{p})\xi^{(s)} \\ \xi^{(s)} \end{pmatrix}, \tag{2.42a}$$

$$u_{(-)}^{(s)}(\mathbf{p}) = \{(|(W)|+m)/(2|W|)\}^{1/2}(-1)^r \begin{pmatrix} \xi^{(r)} \\ (|W|+m)^{-1}(\boldsymbol{\sigma}\mathbf{p})\xi^{(r)} \end{pmatrix}, \tag{2.42b}$$

with $s = 1, 2$, $r = 1, 2$, but $r \neq s$. The first two of these solutions corresponding to eqn (2.42a) are for $W > 0$, while the other two corresponding to eqn (2.42b) are for $W < 0$. It is convenient to have a unified notation showing no explicit difference between the case when $W > 0$ or $W < 0$, respectively. We therefore introduce the four four-component spinors $w^{(t)}$ with $t = 1, 2, 3, 4$ by

$$w^{(s)} = u_{(+)}^{(s)}(\mathbf{p}), \tag{2.43a}$$

$$w^{(s+2)} = u_{(-)}^{(s)}(\mathbf{p}). \tag{2.43b}$$

Denoting the components of the spinors by $w_k^{(t)}$ where $k = 1, 2, 3, 4$, the closure property of the basis spinors may be written

$$\sum_{t=1}^{4} w_k^{(t)*} w_l^{(t)} = \delta_{kl} \tag{2.44a}$$

or in matrix notation

$$\sum_{t=1}^{4} w^{(t)} w^{(t)+} = 1, \tag{2.44b}$$

where * means complex conjugate and $^+$ means adjoint, that is transposed and complex conjugate. Furthermore, the basis spinors are orthogonal and normalized according to

$$\sum_{k=1}^{4} w_k^{(s)*} w_k^{(t)} = \delta_{st} \tag{2.45a}$$

or in matrix notation

$$w^{(s)+} w^{(t)} = \delta_{st}. \tag{2.45b}$$

According to Dirac's theory the states of negative energy are normally all filled with particles. If, however, such a state is empty, it is to be interpreted as an antiparticle of opposite momentum and spin. In order to avoid trouble with the signs depending on whether the hole or the corresponding antiparticle is referred to, we may introduce, besides the particle spinors

$$u^{(s)}(\mathbf{p}) = u^{(s)}_{(+)}(\mathbf{p}) \quad \text{for} \quad s = 1, 2, \tag{2.46}$$

the antiparticle spinors

$$v^{(s)}(\mathbf{p}) = u^{(s)}_{(-)}(-\mathbf{p}) \quad \text{for} \quad s = 1, 2, \tag{2.47}$$

so that we are no longer concerned with spin† and momentum of the holes.

It may be verified from the various definitions that particle and antiparticle spinors are related by

$$v^{(s)}(\mathbf{p}) = -\gamma_2 u^{(s)*}(\mathbf{p}) = \mathrm{i}\beta\gamma_5\sigma_2 u^{(s)*}(\mathbf{p}). \tag{2.48}$$

The operation transforming a particle into an antiparticle is called charge conjugation, but more generally we have to consider the wave function including the plane wave factor rather than the spinors alone. We may then define a charge conjugation operator C, which transforms a particle wave function into an antiparticle wave function, by

$$C\psi^{(s)}(\mathbf{p}, \mathbf{r}) = \mathrm{i}\beta\gamma_5\sigma_2\psi^{(s)*}(\mathbf{p}, \mathbf{r}), \tag{2.49}$$

such that, if

$$\psi^{(s)}(\mathbf{p}, \mathbf{r}) = u^{(s)}(\mathbf{p})\exp\{\mathrm{i}(\mathbf{pr})\} \tag{2.50}$$

is a particle wave function, $C\psi^{(s)}$ is the corresponding antiparticle wave function, and vice versa.

Finally, we note that, when matrix elements are formed, in place of the adjoint wave function ψ^+ or spinor u^+ in the final state position, often the quantities $\bar{\psi}$ or $\bar{u}$ defined by

$$\bar{\psi} = \psi^+\gamma_4, \tag{2.51a}$$

$$\bar{u} = u^+\gamma_4 \tag{2.51b}$$

are used.

† The spin of $u^{(s)}_{(-)}$ is already opposite to the spin of $u^{(s)}_{(+)}$ by definition, eqn (2.42).

2.2. Separation of the Dirac equation in spherical co-ordinates

Let us consider the Dirac equation with an external electromagnetic field present, but immediately restrict our attention to the important special case when the vector potential vanishes identically and the scalar potential is static and spherically symmetric.† The time independent Dirac equation then is

$$[-(\boldsymbol{\alpha}\not{p})-\beta m-1\{W-V(r)\}]\psi(\mathbf{r})=0. \tag{2.52}$$

Because of the symmetry of the potential it is possible to separate this differential equation in spherical co-ordinates, but to do so is more complicated than in the case of the Schrödinger equation. For if

$$\mathbf{L}=[\mathbf{r}\times\not{p}] \tag{2.53}$$

denotes the orbital angular momentum operator, the operator L^2 no longer commutes with the Hamiltonian. Instead, the appropriate operator to be considered here is

$$K=\beta\{(\boldsymbol{\sigma}\mathbf{L})+1\} \tag{2.54}$$

besides J_z which is the z-component of the total angular momentum operator

$$\mathbf{J}=1\mathbf{L}+\tfrac{1}{2}\boldsymbol{\sigma} \tag{2.55}$$

as usual (but with four by four rather than two by two matrices).

We are going to construct special solutions ψ of the Dirac equation (2.52) which are eigen functions of K with eigen values κ and of J_z with eigen values μ such that

$$K\psi_\kappa^\mu=\kappa\psi_\kappa^\mu, \tag{2.56}$$

$$J_z\psi_\kappa^\mu=\mu\psi_\kappa^\mu, \tag{2.57}$$

where κ is an integer zero excepted, as will be seen below, and μ half an odd integer of a finite set.

It is convenient to decompose ψ_κ^μ as

$$\psi_\kappa^\mu=\underbrace{\tfrac{1}{2}(1+\beta)\psi_\kappa^\mu}_{\Phi_\kappa^{\mathrm{I}\mu}}+\underbrace{\tfrac{1}{2}(1-\beta)\psi_\kappa^\mu}_{\Phi_\kappa^{\mathrm{II}\mu}}, \tag{2.58}$$

This means that we have projected out the upper two or the lower two components, respectively, so that in $\Phi_\kappa^{\mathrm{I}\mu}$ the lower two, and in $\Phi_\kappa^{\mathrm{II}\mu}$ the

† The scalar potential is no longer spherically symmetric if it is generated by a nucleus possessing an electric quadrupole moment. A treatment of this case is beyond the scope of the present book, although recently the influence of a nuclear quadrupole moment on nuclear beta-decay has, in fact, been investigated in a paper by Bergkvist (1975). References to the earlier attempts may be found in that paper. Furthermore, the scalar potential is no longer spherically symmetric if the influence of the nuclear magnetic moment is taken into account. This effect has been discussed in the context of internal conversion (Church and Weneser 1960) and is expected to be very small.

upper two, components are zero. For in partitioned form the matrices are given explicitly by

$$\tfrac{1}{2}(1+\beta)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \tag{2.59a}$$

$$\tfrac{1}{2}(1-\beta)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{2.59b}$$

and therefore are seen to have the desired projecting property. Because

$$\beta\tfrac{1}{2}(1+\beta)=\tfrac{1}{2}(1+\beta), \tag{2.60a}$$

$$\beta\tfrac{1}{2}(1-\beta)=-\tfrac{1}{2}(1-\beta), \tag{2.60b}$$

it follows from eqn (2.56) and eqn (2.58) that

$$\{(\boldsymbol{\sigma}\mathbf{L})+1\}\Phi_\kappa^{\mathrm{I}\mu}-\{(\boldsymbol{\sigma}\mathbf{L})+1\}\Phi_\kappa^{\mathrm{II}\mu}=\kappa\Phi_\kappa^{\mathrm{I}\mu}+\kappa\Phi_\kappa^{\mathrm{II}\mu}$$

or, after multiplication with $\frac{1}{2}(1+\beta)$ or $\frac{1}{2}(1-\beta)$,

$$\{(\boldsymbol{\sigma}\mathbf{L})+1\}\Phi_\kappa^{\mathrm{I}\mu}=\kappa\Phi_\kappa^{\mathrm{I}\mu}, \tag{2.61a}$$

$$\{(\boldsymbol{\sigma}\mathbf{L})+1\}\Phi_\kappa^{\mathrm{II}\mu}=-\kappa\Phi_\kappa^{\mathrm{II}\mu}. \tag{2.61b}$$

Introducing two-component spin angular wave functions χ_κ^μ which satisfy†

$$\{(\boldsymbol{\sigma}\mathbf{L})+1\}\chi_\kappa^\mu=-\kappa\chi_\kappa^\mu, \tag{2.62}$$

$$J_z\chi_\kappa^\mu=\mu\chi_\kappa^\mu, \tag{2.63}$$

we may identify the non-vanishing components of $\Phi_\kappa^{\mathrm{I}\mu}$ with $\chi_{-\kappa}^\mu$ and those of $\Phi_\kappa^{\mathrm{II}\mu}$ with χ_κ^μ. In partitioned form we may then write‡

$$\Phi_\kappa^{\mathrm{I}\mu}=\mathrm{sign}(\kappa)f_\kappa(r)\begin{pmatrix}\chi_{-\kappa}^\mu\\ 0\end{pmatrix}, \tag{2.64a}$$

$$\Phi_\kappa^{\mathrm{II}\mu}=g_\kappa(r)\begin{pmatrix}0\\ \chi_\kappa^\mu\end{pmatrix}, \tag{2.64b}$$

where the factor $\mathrm{sign}(\kappa)$ has been inserted for later convenience and $f_k(r)$, $g_\kappa(r)$ are functions of the radial co-ordinate, still arbitrary at this stage, but to be determined later by the requirement that we want to construct a solution of the differential eqn (2.52).

†Again we here need the two by two spin matrices, so that now

$$\mathbf{J}=1\mathbf{L}+\tfrac{1}{2}\boldsymbol{\sigma}$$

is a two by two matrix.

‡Here 0 means the two-component spinor $\begin{pmatrix}0\\0\end{pmatrix}$.

From

$$J^2=(\mathbf{1L}+\tfrac{1}{2}\boldsymbol{\sigma})^2=\mathbf{1}L^2+(\boldsymbol{\sigma}\mathbf{L})+\tfrac{3}{4}\mathbf{1}$$

it follows that

$$(\boldsymbol{\sigma}\mathbf{L})+\mathbf{1}=\{J^2-L^2+\tfrac{1}{4}\}\mathbf{1},$$

so that the χ_κ^μ satisfy

$$\{J^2-L^2+\tfrac{1}{4}\}\chi_\kappa^\mu=-\kappa\chi_\kappa^\mu. \tag{2.65}$$

We therefore may identify the χ_κ^μ with the usual spin-angular momentum wave functions describing the coupling of an orbital angular momentum l and a spin $\frac{1}{2}$ to give a total angular momentum j with z-component μ. Besides eqn (2.63) they also satisfy

$$J^2\chi_\kappa^\mu=j(j+1)\chi_\kappa^\mu, \tag{2.66}$$

$$L^2\chi_\kappa^\mu=l(l+1)\chi_\kappa^\mu. \tag{2.67}$$

From eqn (2.65) our eigenvalue κ may now be seen to be

$$\kappa=-j(j+1)+l(l+1)-\tfrac{1}{4}$$

so that

$$\kappa=-(l+1)=-(j+\tfrac{1}{2}) \quad \text{if} \quad j=l+\tfrac{1}{2} \tag{2.68a}$$

or

$$\kappa=l=j+\tfrac{1}{2} \quad \text{if} \quad j=l-\tfrac{1}{2}. \tag{2.68b}$$

Explicitly the spin-angular wave functions are given by†

$$\chi_\kappa^\mu=i^l\sum_m C(l\tfrac{1}{2}j;\ \mu-m\ m)\,Y_l^{\mu-m}(\hat{r})\chi^m \tag{2.69}$$

with

$$l=l(\kappa)=\begin{cases}\kappa & \text{if} \quad \kappa>0\\ |\kappa|-1 & \text{if} \quad \kappa<0\end{cases}. \tag{2.70}$$

Combining the results we have

$$\psi_\kappa^\mu=\begin{pmatrix}\operatorname{sign}(\kappa)f_\kappa(r)\chi_{-\kappa}^\mu\\ g_\kappa(r)\chi_\kappa^\mu\end{pmatrix}, \tag{2.71}$$

and this is the appropriate ansatz for a special solution of the Dirac equation with a spherically symmetric potential. We now need the iden-

† Here $C(\ldots;..)$ is a Clebsch–Gordan coefficient, $Y_l^{\mu-m}(\hat{r})$ a spherical harmonic depending on the radial unit vector $\hat{r}=\mathbf{r}/r$, and χ^m a two-component spinor. But while the spinors, eqn (2.41), above have been numbered consecutively by the indices 1 or 2, the index m here denotes the appropriate magnetic quantum number, so that we need $\chi^{1/2}=\begin{pmatrix}1\\0\end{pmatrix}$ or $\chi^{-1/2}=\begin{pmatrix}0\\1\end{pmatrix}$.

tity (Rose 1961)

$$(\boldsymbol{\alpha}\not{\mathbf{r}}) = -\mathrm{i}r^{-1}(\boldsymbol{\alpha}\mathbf{r})\{1(\partial/\partial r) - r^{-1}(\boldsymbol{\sigma}\mathbf{L})\} \tag{2.72}$$

in order to rewrite the Dirac eqn (2.52) in the form

$$[\mathrm{i}r^{-1}(\boldsymbol{\alpha}\mathbf{r})\{1(\partial/\partial r) - r^{-1}(\boldsymbol{\sigma}\mathbf{L})\} - \beta m - 1\{W - V(r)\}]\psi_\kappa^\mu = 0. \tag{2.73}$$

Because $\boldsymbol{\alpha} = \boldsymbol{\sigma}\gamma_5$ and

$$\gamma_5\psi_\kappa^\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{sign}(\kappa) f_\kappa \chi_{-\kappa}^\mu \\ g_\kappa \chi_\kappa^\mu \end{pmatrix} = \begin{pmatrix} g_\kappa \chi_\kappa^\mu \\ \mathrm{sign}(\kappa) f_\kappa \chi_{-\kappa}^\mu \end{pmatrix}$$

eqn (2.73) in partitioned form reads

$$\{\mathrm{i}r^{-1}(\boldsymbol{\sigma}\mathbf{r})(\partial/\partial r) - \mathrm{i}r^{-1}(\boldsymbol{\sigma}\mathbf{r})r^{-1}(\boldsymbol{\sigma}\mathbf{L})\} g_\kappa \chi_\kappa^\mu - 1\{W + m - V(r)\}\mathrm{sign}(\kappa) f_\kappa \chi_{-\kappa}^\mu = 0, \tag{2.74a}$$

$$\{\mathrm{i}r^{-1}(\boldsymbol{\sigma}\mathbf{r})(\partial/\partial r) - \mathrm{i}r^{-1}(\boldsymbol{\sigma}\mathbf{r})r^{-1}(\boldsymbol{\sigma}\mathbf{L})\}\mathrm{sign}(\kappa) f_\kappa \chi_{-\kappa}^\mu - 1\{W - m - V(r)\} g_\kappa \chi_\kappa^\mu = 0. \tag{2.74b}$$

Using

$$(\boldsymbol{\sigma}\mathbf{L})\chi_\kappa^\mu = -(\kappa + 1)\chi_\kappa^\mu \tag{2.75}$$

according to eqn (2.62) and the identity† (Rose 1961)

$$r^{-1}(\boldsymbol{\sigma}\mathbf{r})\chi_\kappa^\mu = -\mathrm{i}\,\mathrm{sign}(\kappa)\chi_{-\kappa}^\mu, \tag{2.76}$$

we obtain from eqn (2.74)

$$[(\mathrm{d}g_\kappa/\mathrm{d}r) + r^{-1}(\kappa + 1)g_\kappa(r) - \{W + m - V(r)\}f_\kappa(r)]\,\mathrm{sign}(\kappa)\chi_{-\kappa}^\mu = 0,$$
$$[(\mathrm{d}f_\kappa/\mathrm{d}r) - r^{-1}(\kappa - 1)f_k(r) + \{W - m - V(r)\}g_\kappa(r)]\,(-1)\chi_\kappa^\mu = 0.$$

Since the spin angular functions $\chi_{-\kappa}^\mu$, χ_κ^μ do not vanish identically, the radial-dependent factors have to be zero. This requirement leads to the Dirac radial equations

$$\mathrm{d}g_\kappa/\mathrm{d}r + r^{-1}(\kappa + 1)g_\kappa(r) - \{W + m - V(r)\}f_\kappa(r) = 0, \tag{2.77a}$$

$$\mathrm{d}f_\kappa/\mathrm{d}r - r^{-1}(\kappa - 1)f_\kappa(r) + \{W - m - V(r)\}g_\kappa(r) = 0 \tag{2.77b}$$

for the radial wave functions $g_\kappa(r)$, $f_\kappa(r)$, which will be investigated in detail in Chapters 3 and 4.

2.3. Orders of magnitude

In several places in this book we will need a qualitative discussion based on order of magnitude estimates for various quantities. Therefore we are going to collect the information needed for this purpose.

† The factor $\mathrm{i}\,\mathrm{sign}(\kappa)$ is a consequence of the fact that our definition of the spin angular functions χ_κ^μ includes the phase factor i^l. In this point our conventions are different from those of Rose (1961).

TABLE 2.1 *Relation between energy and momentum for a particle of unit mass*

p	W	p/W	*Kin* [keV]
0·1	1·00499	0·0995	2·55
0·2	1·01980	0·1961	10·1
0.3	1·04403	0·2873	22·5
0·4	1·07703	0·3714	39·4
0·5	1·11803	0·4472	60·3
1·0	1·41421	0·7071	211·7
2·0	2·23607	0·8944	631·6
5·0	5·09902	0·9806	2095
50·0	50·01000	0·9998	25044

For electrons or positrons the mass m_e is equal to unity, and therefore the energy momentum relation, eqn (2.40), becomes

$$W^2 = p^2 + 1. \tag{2.78}$$

This means that the energy W is given in units of $m_e c^2$ and the momentum $\mathbf{p}$ in units of $m_e c$. Also of interest is the quantity p/W, which is equal to the velocity of the electron or positron in units of the velocity of light. With

$$m_e c^2 = 511{\cdot}006 \text{ keV}, \tag{2.79}$$

the kinetic energy, Kin, in keV is

$$\text{Kin} = (W-1) * 511{\cdot}006. \tag{2.80}$$

In order to give a rough idea of values, we have included Table 2.1, which contains some values for these quantities.

The Coulomb potential of a nuclear point charge $Z|e|$ for an electron is

$$V(r) = -\alpha Z/r, \tag{2.81}$$

where

$$\alpha = 1/137{\cdot}036 = e^2/(\hbar c) \tag{2.82}$$

denotes the fine structure constant.†

Another most important quantity, depending on the nuclear mass number A, is the nuclear radius R, for which we approximately have, in units of $\hbar/(m_e c)$,

$$R = r_0 A^{1/3} \tag{2.83}$$

with a value of

$$r_0 = 0{\cdot}42587 * \alpha = 0{\cdot}0031077 \tag{2.84}$$

† It should be noted that in the tables by Behrens and Jänecke (1969) the slightly different value of $\alpha = 1/137{\cdot}0388$ has been used.

corresponding to 1·2 fm. For a heavy nucleus eqn (2.83) yields a value of $R \simeq 0{\cdot}02$.

As to the extension of the atoms, we note that the Bohr radius is equal to $1/\alpha \simeq 137$ in our units. If the atomic potential is approximated in the simplest way by a Coulomb potential with a one-term exponential screening factor according to

$$V(r) = -(\alpha Z/r)\exp(-\beta r), \tag{2.85}$$

we have for the screening constant

$$\beta = N(Z)\alpha Z^{1/3}, \tag{2.86}$$

where $N(Z)$ is a slowly increasing function with values between 1 and 2.

3

METHODS OF SOLUTION FOR THE DIRAC RADIAL EQUATIONS

3.1. The differential equations

RADIAL wave functions $f(r)$, $g(r)$ appear as solutions of the Dirac radial equations

$$rf'(r)-(\kappa-1)f(r)+r\{T-V(r)\}g(r)=0, \tag{3.1a}$$

$$rg'(r)+(\kappa+1)g(r)-r\{T+2m-V(r)\}f(r)=0, \tag{3.1b}$$

a two-dimensional linear system of ordinary differential equations depending on the real parameters κ, T, m and a function $V(r)$ which may contain one or several additional parameters. Concurrently with the kinetic energy T of the particle, its total energy

$$W=T+m \tag{3.2}$$

may and will be used. The symbol m for the particle mass has been retained for later convenience, although it may be set equal to 1 without loss of generality and, in fact, is equal to 1 in our units for electrons. For the angular momentum quantum number κ, only integer values, zero excepted, are of interest. Defining

$$k=|\kappa| \tag{3.3}$$

we therefore have to consider $\kappa=k$ or $\kappa=-k$ where $k=1, 2, 3, \ldots$ is a natural number. The potential function $V(r)$ is assumed to be real-valued on the real r axis. The solution $\{f(r), g(r)\}$, which is needed on the interval $0\leqslant r\leqslant\infty$, may then also be assumed to be real-valued on this interval. Since the method of solution depends to a large extent on the type of the potential function $V(r)$ under consideration, we will treat the differential equations for several important classes of potentials in different subsections, proceeding from simple models of $V(r)$ to sufficiently general ones. Also there will be some differences according to whether we are concerned with the continuum state solutions ($T>0$) or with the bound state solutions ($-2m<T<0$). Independent of such details, however, are some general properties of the differential equations and their solutions, which will be considered first.

3.1.1. Wronskian relations

Let us consider two different solutions $\{f_1(r), g_1(r)\}$ and $\{f_2(r), g_2(r)\}$. Writing the differential equations, eqns (3.1), with $f_1(r)$, $g_1(r)$ or

$f_2(r)$, $g_2(r)$, respectively, we have four equations from which we may eliminate the terms with $T-V$ and $T+2m-V$ by forming suitable linear combinations. The resulting two equations

$$r\{g_2 f_1' - g_1 f_2'\} - (\kappa - 1)\{f_1 g_2 - g_1 f_2\} = 0$$
$$r\{f_2 g_1' - f_1 g_2'\} + (\kappa + 1)\{g_1 f_2 - f_1 g_2\} = 0,$$

when subtracted from each other, yield the differential equation

$$r\{f_1 g_2 - g_1 f_2\}' + 2\{f_1 g_2 - g_1 f_2\} = 0,$$

which is easily integrated to give

$$f_1 g_2 - g_1 f_2 = C/r^2$$

where C is the constant of integration. Therefore we have shown that for any two solutions, labelled by the indexes 1 or 2, of the differential equations (3.1), the bilinear combination

$$r^2\{f_1(r) g_2(r) - g_1(r) f_2(r)\} \equiv C \tag{3.4}$$

is identically constant, irrespective of the parameters κ, T, m and the potential function $V(r)$. It is the constant C only which *does* depend on the choice of the solutions 1, 2, on the parameter κ, and on some of the potential parameters. Since $f_1 g_2 - g_1 f_2$ plays the role of the Wronskian of the two solutions, we refer to equations of the type of eqn (3.4) as Wronskian relations.

3.2. Continuum state solutions

3.2.1. Identically vanishing potential

The simplest case is $V(r) \equiv 0$, which is of interest for $T > 0$. It is then useful to introduce the momentum p of the particle by

$$p = (W^2 - m^2)^{1/2} \tag{3.5a}$$

or

$$p = \{(2m + T)T\}^{1/2} \tag{3.5b}$$

and the integer $l(n)$, which corresponds to the orbital angular momentum of the particle,† by

$$l(n) = \begin{cases} n & \text{if } n > 0 \\ |n| - 1 & \text{if } n < 0 \end{cases} \tag{3.6}$$

where n stands for $+\kappa$ or $-\kappa$. Conveniently normalized solutions of the

† The integer $l = l(n)$ is here defined mathematically by eqn (3.6), but it should be remembered that physically the orbital angular momentum of the relativistic particle is not well-defined since $f(r)$ and $g(r)$ have different values of l.

differential equations are

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \begin{Bmatrix} \text{sign}(\kappa)(T/W)^{1/2} p z_{l(-\kappa)}(pr) \\ \{(T+2m)/W\}^{1/2} p z_{l(+\kappa)}(pr) \end{Bmatrix} \tag{3.7}$$

where $z_l(x)$ is a solution of the differential equation

$$x^2 z_l'' + 2x z_l' + \{x^2 - l(l+1)\} z_l(x) = 0 \tag{3.8}$$

such that it obeys, with respect to the index l, the recurrence relations

$$z_{l-1}(x) + z_{l+1}(x) = (2l+1)x^{-1} z_l(x), \tag{3.9}$$

$$l z_{l-1}(x) - (l+1) z_{l+1}(x) = (2l+1) z_l'(x). \tag{3.10}$$

The functions $z_l(x)$ therefore are seen to be spherical Bessel functions in the broad sense. In particular, $z_l(x)$ stands for either the (spherical) Bessel function in the narrow sense or Bessel function of the first kind $j_l(x)$, or the Neumann function or function of the second kind $n_l(x)$, or any of the two Hankel functions or functions of the third kind $h_l^{(s)}(x)$ with $s = 1, 2$. Each choice of $z_l(x)$ in eqn (3.7) gives a special solution of the differential equations. In order to prove that eqn (3.7) represents the solution we may substitute eqn (3.7) into the differential eqn (3.1) with $V(r) \equiv 0$. Using again $x = pr$, writing derivatives with respect to x, and dropping common factors we obtain

$$\text{sign}(\kappa)\{x z_{l(-\kappa)}' - (\kappa - 1) z_{l(-\kappa)}\} + x z_{l(\kappa)}(x) = 0,$$

$$x z_{l(\kappa)}' + (\kappa + 1) z_{l(\kappa)} - \text{sign}(\kappa) x z_{l(-\kappa)}(x) = 0.$$

Since

$$\left.\begin{aligned} l(\kappa) &= \kappa = k \\ l(-\kappa) &= \kappa - 1 = k - 1 \end{aligned}\right\} \quad \text{if} \quad \kappa > 0$$

according to eqn (3.6), these differential equations may be written as

$$x z_{k-1}' - (k-1) z_{k-1} + x z_k(x) = 0$$

$$x z_k' + (k+1) z_k - x z_{k-1}(x) = 0.$$

The same differential equations, although interchanged, follow if $\kappa < 0$. Replacing k by $l+1$ in the first and by l in the second of these differential equations we see that each of them is true since they are just linear combinations of eqns (3.9) and (3.10).

3.2.2. Coulomb potential

We may write the Coulomb potential

$$V(r) = -\alpha Z / r \tag{3.11}$$

where $-\alpha Z$, with $\alpha = 1/137{\cdot}036$ and Z the nuclear charge number, is the

coupling constant which enters as another parameter of the differential equations. The sign is appropriate for electrons and so are the solutions of the differential equations presented below. The corresponding results for positrons can be obtained by using a formally negative nuclear charge number. The differential equations, eqns (3.1), now read

$$rf' - (\kappa - 1)f + \{Tr + \alpha Z\}g(r) = 0, \tag{3.12a}$$

$$rg' + (\kappa + 1)g - \{(T + 2m)r + \alpha Z\}f(r) = 0. \tag{3.12b}$$

They can be solved analytically in terms of confluent hypergeometric functions. These closed-form solutions are easily accessible (Rose 1961), but they are complicated, and detailed knowledge of the properties of the confluent hypergeometric function is necessary for their proper use and interpretation. For these reasons we prefer to treat the differential equations, eqns (3.12), directly without recourse to confluent hypergeometric functions, but we should emphasize that, although the present treatment is selfcontained, it is to complement rather than to replace the work of Rose.

As in the case with no potential, the differential equations (3.12) still have one regular singular point at $r = 0$ and one irregular singular point of rank 1 at $r = \infty$, all the other points of the complex r-plane being ordinary ones.† However, although the type of the singular points did not change, the behaviour of the solutions at the singular points is significantly modified by the Coulomb potential in a way which, from a physical point of view, can even be considered as pathological.

The first step in constructing solutions is to obtain local solutions, i.e. power series solutions defined by their behaviour in a neighbourhood of any given point, in particular a singular point. The second, more difficult step is to find the connection between the solutions defined relative to different points.

3.2.2.1. Power series solutions relative to $r = 0$

Since $r = 0$ is a regular singular point of the differential equations, we consider the power series solutions

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = N_0 \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^{-1+s+n} \tag{3.13}$$

where N_0 is an arbitrary normalization factor and the coefficients a_n, b_n and the exponent s have to be determined from the differential equations. The exponent -1 has been added for later convenience when s is defined. Substituting the power series, eqn (3.13), into the differential equations

† Since we are going to investigate the analytic properties of the solutions, it is appropriate to consider the differential equations in the complex domain, although we finally are interested only in the real interval $[0, \infty]$.

(3.12) and collecting the terms with equal powers of r, we have, with $a_{-1} = b_{-1} = 0$,

$$\sum_{n=0}^{\infty} r^{-1+s+n} \begin{Bmatrix} (s+n-\kappa)a_n + \alpha Z b_n + T b_{n-1} \\ -\alpha Z a_n + (s+n+\kappa)b_n - (T+2m)a_{n-1} \end{Bmatrix} = 0,$$

and since this has to be valid identically in r we obtain, with $n = 0, 1, 2, \ldots$, the recursive system of algebraic equations

$$(s+n-\kappa)a_n + \alpha Z b_n = -T b_{n-1}, \tag{3.14a}$$

$$-\alpha Z a_n + (s+n+\kappa)b_n = (T+2m)a_{n-1}. \tag{3.14b}$$

For every n the determinant of this system of linear equations for the unknown coefficients a_n, b_n is

$$D(n) = (s+n)^2 - \kappa^2 + (\alpha Z)^2. \tag{3.15}$$

For $n = 0$ the equations reduce to

$$(s-\kappa)a_0 + \alpha Z b_0 = 0, \tag{3.16a}$$

$$-\alpha Z a_0 + (s+\kappa)b_0 = 0. \tag{3.16b}$$

It is reasonable to assume that of the initial coefficients a_0, b_0 at least one is different from zero, since otherwise the still undetermined exponent s could be shifted by an integer until this condition is met. Compatibility of the eqns (3.16) then requires that

$$D(0) = (s-\kappa)(s+\kappa) + (\alpha Z)^2 = 0. \tag{3.17}$$

This is the indicial equation of the differential equations with respect to the point $r = 0$, the roots of which determine the two possible values of s, the characteristic exponents

$$s = -\gamma, +\gamma \tag{3.18}$$

where

$$\gamma = \{\kappa^2 - (\alpha Z)^2\}^{1/2} = \{k^2 - (\alpha Z)^2\}^{1/2}. \tag{3.19}$$

Because of eqn (3.17) the eqns (3.16) are now linearly dependent and therefore determine the ratio of a_0 and b_0 only, so that one of them plays the role of a constant of integration and may be assigned arbitrarily, especially since we have explicitly made allowance for a normalization factor N_0 in eqn (3.13). It is then reasonable to choose one of the initial coefficients equal to 1, but in such a way that they remain well-defined even when $\alpha Z \to 0$. For this reason we make a different choice according to whether $\kappa s > 0$ or $\kappa s < 0$, namely

$$a_0 = 1, b_0 = \alpha Z/(\kappa + s) \quad \text{if} \quad \kappa s > 0, \tag{3.20a}$$

$$a_0 = \alpha Z/(\kappa - s), b_0 = 1 \quad \text{if} \quad \kappa s < 0. \tag{3.20b}$$

After the possible values of s have been determined by eqns (3.18) and

(3.19) so as to satisfy eqn (3.17), the determinant becomes

$$D(n) = n(2s+n)$$

which is different from zero† for all the values of $n \geq 1$. Therefore the eqns (3.14) may be solved uniquely for a_n, b_n, the result being

$$a_n = \{n(n+2s)\}^{-1}\{-\alpha Z(T+2m)a_{n-1} - (s+n+\kappa)Tb_{n-1}\}, \tag{3.21a}$$

$$b_n = \{n(n+2s)\}^{-1}\{(s+n-\kappa)(T+2m)a_{n-1} - \alpha ZTb_{n-1}\}. \tag{3.21b}$$

All the coefficients may now be evaluated recursively. For later convenience we dispose of the normalization factor N_0 according to‡

$$N_0 = \begin{cases} \text{Sign}(\alpha Z)\text{sign}(\kappa)\{(\kappa+s)(\kappa W - sm)\}^{1/2} & \text{if} \quad \kappa s > 0 \\ \{(\kappa - s)(\kappa W - sm)\}^{1/2} & \text{if} \quad \kappa s < 0. \end{cases} \tag{3.22}$$

Here the factor $(\kappa W - sm)$, which has been included for later convenience, helps to keep the radicand of the square root positive. Moreover, this choice of N_0 guarantees that our solutions have been defined in the same way irrespective of the sign of κs and αZ.

We therefore have two standard solutions

$$\begin{Bmatrix} f_{0j}(r) \\ g_{0j}(r) \end{Bmatrix} = N_0 r^{-1+s} \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^n \tag{3.23}$$

with the coefficients a_n, b_n, which depend on s, from eqns (3.20) and (3.21) and a label $j = 1$ if $s = \gamma$ or $j = 2$ if $s = -\gamma$. The Wronskian relation for these solutions is

$$r^2\{f_{01}(r)g_{02}(r) - g_{01}(r)f_{02}(r)\} \equiv \text{Sign}(\alpha Z)2\gamma\{(kp)^2 + (\alpha Z)^2 m^2\}^{1/2}. \tag{3.24}$$

We may verify this result by evaluating the Wronskian in the limit $r \to 0$

† Except when $\alpha Z = 0$ and $s < 0$ so that s becomes a negative integer. But this case need not be considered here since it has been treated separately in Section 3.2.1. The other exception, occurring for $s = -\gamma$ whenever γ becomes equal to half an odd integer >0, need not concern us as long as we are interested in integer values of Z only.

‡ The function Sign(x) needed here is

$$\text{Sign}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ \text{undefined} & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

rather than the more familiar function

$$\text{sign}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

which is not useful in the present context.

so that the terms with a_0, b_0 only need to be considered. Their contribution is $\text{sign}(\kappa)2\gamma/(k+\gamma)$. This still has to be multiplied by the product of the normalization factors, which is

$$\text{Sign}(\kappa\alpha Z)(k+\gamma)\{(kW)^2-(\gamma m)^2\}^{1/2}.$$

Finally, the useful identity

$$(kW)^2-(\gamma m)^2=(kp)^2+(\alpha Zm)^2 \tag{3.25}$$

has been applied in order to obtain eqn (3.24).

3.2.2.2. Power series solutions relative to $r=\infty$

Since $r=\infty$ is an irregular singular point, of rank 1, of the differential equations, we know (Ince 1956; Jurkat 1978) that there are formal solutions of the type

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = N_\infty \exp(t_0 r) \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^{-\nu-n}, \tag{3.26}$$

which, although the series diverge, represent the solutions asymptotically as $r\to\infty$ (in suitable sectors of the complex plane) and are very useful for analytic as well as numerical work. Again we allow for a still arbitrary normalization factor N_∞. In order to determine the coefficients† a_n, b_n and the exponents t_0, ν we substitute the ansatz, eqn (3.26), into the differential equations (3.12) and collect the terms with equal powers of r. With $a_{-1}=b_{-1}=0$ we then have‡

$$\sum_{n=0}^{\infty} \begin{Bmatrix} -(\nu+n-2+\kappa)a_{n-1}+\alpha Z b_{n-1}+t_0 a_n+T b_n \\ -\alpha Z a_{n-1}-(\nu+n-2-\kappa)b_{n-1}-(T+2m)a_n+t_0 b_n \end{Bmatrix} r^{-\nu-n+1}=0,$$

and since this has to hold identically in r, we obtain for the coefficients a_n, b_n, the recursive system of linear equations

$$t_0 a_n + T b_n = (\nu+n-2+\kappa)a_{n-1}-\alpha Z b_{n-1}, \tag{3.27a}$$

$$-(T+2m)a_n+t_0 b_n=\alpha Z a_{n-1}+(\nu+n-2-\kappa)b_{n-1}, \tag{3.27b}$$

where $n=0,1,2,\ldots$.

For $n=0$ the equations reduce to

$$t_0 a_0 + T b_0 = 0, \tag{3.28a}$$

$$-(T+2m)a_0+t_0 b_0=0. \tag{3.28b}$$

Assuming, without loss of generality, that at least one of the initial coefficients a_0, b_0 is different from zero, compatibility of the equations

† The coefficients a_n, b_n are defined locally for this section only and are different from those of other sections.

‡ Omitting the common factor $N_\infty \exp(t_0 r)$.

requires that their determinant be zero,

$$t_0^2 + T(T+2m) = 0,$$

or

$$t_0^2 + p^2 = 0. \tag{3.29}$$

There are, therefore, just two possible values of t_0, namely $t_0 = +ip$ or $t_0 = -ip$, thus generating two linearly independent solutions, as expected. The ratio of the initial coefficients may now be determined from any one of the two linearly dependent equations (3.28), while the normalization is arbitrary. We may fix the normalization so that

$$a_0 = t_0, \tag{3.30a}$$

$$b_0 = T + 2m. \tag{3.30b}$$

A convenient choice of the common normalization factor N_∞ is then

$$N_\infty = \{(T+m)(T+2m)\}^{-1/2} = \{W(W+m)\}^{-1/2}. \tag{3.31}$$

For every $n > 0$ the determinant of the system (3.27) remains zero. Compatibility then requires that two other determinants† be zero too, namely

$$\begin{vmatrix} (\nu+n-2+\kappa)a_{n-1} - \alpha Z b_{n-1} & T \\ \alpha Z a_{n-1} + (\nu+n-2-\kappa)b_{n-1} & t_0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} t_0 & (\nu+n-2+\kappa)a_{n-1} - \alpha Z b_{n-1} \\ -(T+2m) & \alpha Z a_{n-1} + (\nu+n-2-\kappa)b_{n-1} \end{vmatrix} = 0.$$

Writing these equations explicitly we have

$$\{(\nu+n-2+\kappa)t_0 - \alpha Z T\}a_{n-1} + \{-\alpha Z t_0 - (\nu+n-2-\kappa)T\}b_{n-1} = 0, \tag{3.32a}$$

$$\{\alpha Z t_0 + (\nu+n-2+\kappa)(T+2m)\}a_{n-1} + \{(\nu+n-2-\kappa)t_0 - \alpha Z(T+2m)\}\, b_{n-1} = 0 \tag{3.32b}$$

where $n = 1, 2, 3, \ldots$. The determinant of this system of homogeneous linear equations for a_{n-1}, b_{n-1}, which turns out to be

$$(t_0^2 + p^2)\{(\nu+n-2)^2 - \kappa^2 + (\alpha Z)^2\},$$

vanishes for all the n because of the first factor. The two equations (3.32), are therefore linearly dependent, so that we have to keep only one of them, which may be rewritten with $n-1$ replaced by n. Similarly, we have to keep one of the eqns (3.27). The system to be further considered, for

† Each of which is the original determinant, but with one of its columns replaced by the right-hand side of the equations.

$n = 0, 1, 2, \ldots$, then reads

$$t_0 a_n + T b_n = (\nu + n - 2 + \kappa) a_{n-1} - \alpha Z b_{n-1}, \tag{3.33a}$$

$$\{\alpha Z t_0 + (\nu + n - 1 + \kappa)(T + 2m)\} a_n + \{(\nu + n - 1 - \kappa) t_0 - \alpha Z (T + 2m)\} b_n = 0. \tag{3.33b}$$

Alternatively we might further consider the system consisting of the two other equations

$$-(T + 2m) a_n + t_0 b_n = \alpha Z a_{n-1} + (\nu - n - 2 - \kappa) b_{n-1}, \tag{3.34a}$$

$$\{-(\nu + n - 1 + \kappa) t_0 + \alpha Z T\} a_n + \{\alpha Z t_0 + (\nu + n - 1 - \kappa) T\} b_n = 0. \tag{3.34b}$$

In each case the determinant of the system is

$$D(n) = 2 t_0 \{(\nu + n - 1) t_0 - \alpha Z (T + m)\}. \tag{3.35}$$

For $n = 0$ the system (3.34) becomes homogeneous so that $D(0) = 0$ or

$$(\nu - 1) t_0 - \alpha Z (T + m) = 0 \tag{3.36}$$

is required for compatibility. This is the equation which determines the exponent

$$\nu = 1 + \alpha Z (T + m)/t_0 = 1 - (\alpha Z W / p^2) t_0$$

or

$$\nu = 1 - y t_0 / p \tag{3.37}$$

if we introduce the so-called Coulomb parameter

$$y = \alpha Z W / p. \tag{3.38}$$

Because of eqn (3.36) the determinant, eqn (3.35), then becomes

$$D(n) = 2 n t_0^2 = -2 n p^2, \tag{3.39}$$

which is different from zero for all the values of $n > 0$, so that eqn (3.33) or (3.34) can be solved for a_n, b_n uniquely. In order to obtain the recurrence relations in the most convenient† form, we use eqn (3.33) to

† These recurrence relations, eqn (3.40), for the coefficients a_n, b_n have the property to decouple in the case of $\alpha Z = 0$. Other recurrence relations could be obtained by solving eqn (3.34) for a_n and eqn (3.33) for b_n, which, although looking different, generate the same coefficients, of course. Furthermore, it would be possible to derive decoupled recurrence relations even for $\alpha Z \neq 0$, for it follows from eqn (3.32b), by means of eqn (3.36), that

$$b_{n-1} = -\frac{T + 2m}{t_0} \frac{(n - 1 + \kappa) t_0 + \alpha Z m}{(n - 1 - \kappa) t_0 - \alpha Z m} a_{n-1},$$

a relation which enables us to eliminate b_{n-1} from the first or a_{n-1} from the second of eqn (3.40).

get a_n and eqn (3.34) to get b_n. The result then is

$$a_n = \{(n-1+\kappa-\mathrm{y}t_0/p)a_{n-1} - \alpha Z b_{n-1}\}\{(n-\kappa)-\alpha Z m/t_0\}/(2nt_0), \tag{3.40a}$$

$$b_n = \{\alpha Z a_{n-1} + (n-1-\kappa-\mathrm{y}t_0/p)b_{n-1}\}\{(n+\kappa)+\alpha Z m/t_0\}/(2nt_0). \tag{3.40b}$$

This completes our construction of a set of two formal solutions relative to $r=\infty$,

$$\begin{Bmatrix} f_{\infty j}(r) \\ g_{\infty j}(r) \end{Bmatrix} \sim N_\infty r^{-1+\mathrm{y}t_0/p} \exp(t_0 r) \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^{-n}. \tag{3.41}$$

The solutions which they represent asymptotically as $r \to \infty$ are labelled by the index j according to $j=1$ for $t_0=+\mathrm{i}p$ or $j=2$ for $t_0=-\mathrm{i}p$, respectively.

In addition it is useful to have another standard set of solutions, relative to $r=\infty$, which are real (when r is real and positive). Introducing for brevity

$$x = pr + \alpha Z(W/p)\ln(2pr) - (\pi/2)\{l(\kappa)+1\} \tag{3.42}$$

and forming suitable linear combinations of the solutions (3.41) like

$$(1/2)\left[(2p)^{\mathrm{i}\mathrm{y}} \exp[-\mathrm{i}(\pi/2)\{l(\kappa)+1\}]\begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + (2p)^{-\mathrm{i}\mathrm{y}} \exp[+\mathrm{i}(\pi/2)\{l(\kappa)+1\}]\begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}\right] = \begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix}, \tag{3.43a}$$

$$(i/2)\left[(2p)^{\mathrm{i}\mathrm{y}} \exp[-\mathrm{i}(\pi/2)\{l(\kappa)+1\}]\begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} - (2p)^{-\mathrm{i}\mathrm{y}} \exp[+\mathrm{i}(\pi/2)\{l(\kappa)+1\}]\begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}\right] = \begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}, \tag{3.43b}$$

we obtain

$$\begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} \sim \{W(T+2m)\}^{-1/2} r^{-1}\left[\cos(x) \sum_{n=0}^{\infty} \begin{Bmatrix} A_n^{\mathrm{I}} \\ B_n^{\mathrm{I}} \end{Bmatrix} r^{-n} + \sin(x) \sum_{n=0}^{\infty} \begin{Bmatrix} A_n^{\mathrm{II}} \\ B_n^{\mathrm{II}} \end{Bmatrix} r^{-n}\right], \tag{3.44a}$$

$$\begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix} \sim \{W(T+2m)\}^{-1/2} r^{-1}\left[\cos(x) \sum_{n=0}^{\infty} \begin{Bmatrix} A_n^{\mathrm{II}} \\ B_n^{\mathrm{II}} \end{Bmatrix} r^{-n} - \sin(x) \sum_{n=0}^{\infty} \begin{Bmatrix} A_n^{\mathrm{I}} \\ B_n^{\mathrm{I}} \end{Bmatrix} r^{-n}\right], \tag{3.44b}$$

with

$$A_n^{\mathrm{I}} = (a_n + a_n^*)/2, \tag{3.45a}$$

$$A_n^{\mathrm{II}} = -\mathrm{i}(a_n - a_n^*)/2, \tag{3.45b}$$

$$B_n^{\mathrm{I}} = (b_n + b_n^*)/2, \tag{3.45c}$$

$$B_n^{\mathrm{II}} = -\mathrm{i}(b_n - b_n^*)/2, \tag{3.45d}$$

where a_n means $a_n(t_0 = -\mathrm{i}p)$, a_n^* means $a_n(t_0 = +\mathrm{i}p)$ and similarly for b_n. The initial coefficients are, according to eqn (3.30),

$$A_0^{\mathrm{I}} = 0, \tag{3.46a}$$

$$A_0^{\mathrm{II}} = -p, \tag{3.46b}$$

$$B_0^{\mathrm{I}} = T + 2m, \tag{3.46c}$$

$$B_0^{\mathrm{II}} = 0, \tag{3.46d}$$

and the following real recurrence relations for the coefficients with $n > 0$ can be derived from eqn (3.40) by a straightforward but space consuming calculation:

$$A_n^{I} = [\{-(n-1+\kappa)(n-\kappa) - (\alpha Z)^2(mW/p^2)\}A_{n-1}^{\mathrm{II}} + \alpha Z(n-\kappa)B_{n-1}^{\mathrm{II}} - (\alpha Z/p)\{W(n-\kappa) - m(n-1+\kappa)\}A_{n-1}^{\mathrm{I}} - \{(\alpha Z)^2 m/p\}B_{n-1}^{\mathrm{I}}]/(2np), \tag{3.46e}$$

$$A_n^{\mathrm{II}} = [\{n-1+\kappa)(n-\kappa) + (\alpha Z)^2(mW/p^2)\}A_{n-1}^{\mathrm{I}} - \alpha Z(n-\kappa)B_{n-1}^{\mathrm{I}} - (\alpha Z/p)\{W(n-\kappa) - m(n-1+\kappa)\}A_{n-1}^{\mathrm{II}} - \{(\alpha Z)^2 m/p\}B_{n-1}^{\mathrm{II}}]/(2np), \tag{3.46f}$$

$$B_n^{\mathrm{I}} = [\{-(n-1-\kappa)(n+\kappa) + (\alpha Z)^2(mW/p^2)\}B_{n-1}^{\mathrm{II}} - \alpha Z(n+\kappa)A_{n-1}^{\mathrm{II}} - (\alpha Z/p)\{W(n+\kappa) + m(n-1-\kappa)\}B_{n-1}^{\mathrm{I}} - \{(\alpha Z)^2 m/p\}A_{n-1}^{\mathrm{I}}]/(2np), \tag{3.46g}$$

$$B_n^{\mathrm{II}} = [\{(n-1-\kappa)(n+\kappa) - (\alpha Z)^2(mW/p^2)\}B_{n-1}^{\mathrm{I}} + \alpha Z(n+\kappa)A_{n-1}^{\mathrm{I}} - (\alpha Z/p)\{W(n+\kappa) - m(n-1-\kappa)\}B_{n-1}^{\mathrm{II}} - \{(\alpha Z)^2 m/p\}A_{n-1}^{\mathrm{II}}]/(2np). \tag{3.46h}$$

The Wronskian relations for the two sets of solutions relative to $r = \infty$ are

$$r^2\{f_{\infty 1}(r)g_{\infty 2}(r) - g_{\infty 1}(r)f_{\infty 2}(r)\} \equiv 2\mathrm{i}p/W \tag{3.47a}$$

or

$$r^2\{f_{\infty 3}(r)g_{\infty 4}(r) - g_{\infty 3}(r)f_{\infty 4}(r)\} \equiv p/W, \tag{3.47b}$$

respectively. Again we may verify these results by evaluating the Wronskian in the limit $r \to \infty$ so that the leading terms only need to be considered..

If these solutions have to be evaluated numerically for some fixed value of r, the Wronskian relation may be used as an inexpensive check and monitor against possible loss of accuracy which might have occurred for some unforeseen reason.

3.2.2.3. Connection between the solutions of different type

3.2.2.3.1. Contour integral solutions. In order to obtain the connection between the different local solutions introduced so far, we need an integral representation for the solution, which, while the integrand remains the same, reproduces the different local solutions if the limits of the integral are chosen differently. We start again from the differential equations (3.12), but in order to gain some flexibility we first extract out of the solution a power of r, with an exponent L which is still arbitrary and will be fixed later according to what is most convenient. With

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = r^L \begin{Bmatrix} F(r) \\ G(r) \end{Bmatrix} \tag{3.48}$$

the differential equations, because

$$\begin{Bmatrix} f' \\ g' \end{Bmatrix} = r^L \left[\begin{Bmatrix} F' \\ G' \end{Bmatrix} + (L/r) \begin{Bmatrix} F \\ G \end{Bmatrix} \right],$$

then read

$$rF' - (\kappa - 1 - L)F + \{Tr + \alpha Z\}G(r) = 0, \tag{3.49a}$$

$$rG' + (\kappa + 1 + L)G - \{(T + 2m)r + \alpha Z\}F(r) = 0. \tag{3.49b}$$

The type of the singular point $r = \infty$ of the differential equations suggests that a suitable kernel for the integral representation is the Laplace kernel. We therefore try the contour integral

$$\begin{Bmatrix} F(r) \\ G(r) \end{Bmatrix} = (2\pi \mathrm{i})^{-1} \int_C \exp(rt) \begin{Bmatrix} v(t) \\ w(t) \end{Bmatrix} \mathrm{d}t \tag{3.50}$$

with the path of integration C, which is to be independent of r, not yet fixed at this stage. It is our hope that the differential equations we will obtain for $v(t)$ and $w(t)$ are tractable. However, as is known from the theory of the Laplace transform, and as will also be seen below, powers of r give rise to derivatives with respect to t, and since both of the eqns (3.49) contains terms with the factor r as well as terms without the factor r, the resulting differential equations for $v(t)$ and $w(t)$ would be rather difficult to solve. It would be helpful if we could replace one of the differential eqns (3.49) by another one which contains terms of one type only. We therefore try to eliminate the terms without r by a suitable choice of L. Multiplying the first of eqns (3.49) by $(-\alpha Z)$, the second by $(\kappa - 1 - L)$, and adding, we obtain

$$r\{(\kappa - 1 - L)G' - \alpha Z F' - (\kappa - 1 - L)(T + 2m)F - \alpha Z T G\} + \{\kappa^2 - (1 + L)^2 - (\alpha Z)^2\}G = 0. \tag{3.51}$$

We now dispose of L so that the last term vanishes. We therefore define

$$L = -1 - s \tag{3.52}$$

where $s=-\gamma$, $+\gamma$ and $\gamma=\{\kappa^2-(\alpha Z)^2\}^{1/2}$ as above according to eqns (3.18) and (3.19). The two possible values of L needed here are not unexpected since they are just the characteristic exponents relative to the point $r=0$. Equation (3.51) then reduces to an equation with all its terms multiplied by r. Omitting this common factor we have an equation of the desired type. Using this one together with one of the eqns (3.49), the second one for instance, we have to consider the system

$$(\kappa+s)G'-\alpha ZF'-(\kappa+s)(T+2m)F-\alpha ZTG(r)=0, \tag{3.53a}$$

$$rG'+(\kappa-s)G-\{(T+2m)r+\alpha Z\}F(r)=0. \tag{3.53b}$$

Inserting the integrals, eqn (3.50), and assuming that the integration over t and the differentiation with respect to r may be interchanged, we obtain

$$\int_C \exp(rt)[\{(\kappa+s)t-\alpha ZT\}w(t)-\{\alpha Zt+(\kappa+s)(T+2m)\}v(t)]\,dt=0$$

and

$$\int_C \exp(rt)[r\{tw(t)-(T+2m)v(t)\}+\{(\kappa-s)w(t)-\alpha Zv(t)\}]\,dt=0.$$

The integrand of the first integral is the product of two factors, one depending on r and t but neither on $v(t)$ nor $w(t)$, and the other depending on $v(t)$ and $w(t)$ but not on r. Since the integrals are to vanish irrespective of r, we can immediately conclude from the first one that

$$\{(\kappa+s)t-\alpha ZT\}w(t)-\{\alpha Zt+(\kappa+s)(T+2m)\}v(t)\equiv 0.$$

This relationship between $v(t)$ and $w(t)$ is satisfied if we put

$$v(t)=\{(\kappa+s)t-\alpha ZT\}u(t), \tag{3.54a}$$

$$w(t)=\{\alpha Zt+(\kappa+s)(T+2m)\}u(t), \tag{3.54b}$$

so that it remains to determine one function $u(t)$ rather than $v(t)$ and $w(t)$. The second integral then yields

$$\int_C \exp(rt)\{r(t^2+p^2)-2st+2\alpha Z(T+m)\}u(t)\,dt=0,$$

where we have simplified the terms using eqns (3.5) and (3.17) and have omitted a common factor αZ. Since the term containing the factor r is prohibitive for obtaining information on $u(t)$, we remove it by partial integration and obtain

$$\{\exp(rt)(t^2+p^2)u(t)\}\,|_C+\int_C \exp(rt) * [-(t^2+p^2)u'+\{-2(s+1)t+2\alpha Z(T+m)\}u]\,dt=0.$$

The integral vanishes identically in r if $u(t)$ satisfies the differential

equation

$$(t^2+p^2)u'+\{2(s+1)t-2\alpha Z(T+m)\}u(t)=0, \tag{3.55}$$

while the integrated term can be made to vanish by a suitable choice of the contour C such that $\exp(rt)(t^2+p^2)u(t)$ have the same value at both the termini of the contour. Integration of the differential eqn (3.55) then yields

$$u(t)=K(t^2+p^2)^{-1-s}(t+\mathrm{i}p)^{+\mathrm{i}y}(t-\mathrm{i}p)^{-\mathrm{i}y}$$

where K is an arbitrary constant of integration and $y=\alpha Z(T+m)/p$, the Coulomb parameter as above according to eqns (3.38) and (3.2). With K suitably chosen for later convenience we have

$$u(t)=(2p)^{1+s}(t-\mathrm{i}p)^{-1-s-\mathrm{i}y}(t+\mathrm{i}p)^{-1-s+\mathrm{i}y}. \tag{3.56}$$

The solutions of the original differential eqn (3.12) then appear in the form

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = (2p)^{1+s}r^{-1-s}(2\pi\mathrm{i})^{-1}\int_C \exp(rt)(t+\mathrm{i}p)^{-1-s+\mathrm{i}y} * (t-\mathrm{i}p)^{-1-s-\mathrm{i}y}\begin{Bmatrix} (\kappa+s)t-\alpha ZT \\ \alpha Zt+(\kappa+s)(T+2m) \end{Bmatrix}\mathrm{d}t \tag{3.57}$$

with any contour C such that, identically in r, the value which the expression

$$\exp(rt)(t+\mathrm{i}p)^{-s+\mathrm{i}y}(t-\mathrm{i}p)^{-s-\mathrm{i}y} \tag{3.58}$$

has at the starting point of the contour is resumed at its end after the contour has been described. It is by choosing, in accordance with this requirement, different contours, that we may reproduce the solutions of different type considered above and find the connection between them.

By inspection of the expresion (3.58) we may find the possible contours, and we are going to list them (for $r>0$) as far as they are essentially different:

(1) The contour C_1 which starts at or near $t=-\infty$, encircles both the singular points of the integrand, $t=-\mathrm{i}p$ and $t=+\mathrm{i}p$, once in the positive sense and returns to $t=-\infty$ as shown in Fig. 3.1(a). The integral with this contour will be denoted by

$$\int_{C_1}\ldots\mathrm{d}t \quad \text{or by} \quad \int_{-\infty}^{(-\mathrm{i}p+,\mathrm{i}p+)}\ldots\mathrm{d}t.$$

(2) The closed double loop contour C_2 which starts at $t=0$, encircles the singular point $t=\mathrm{i}p$ once in the positive sense, encircles the point $t=-\mathrm{i}p$ once in the positive sense, encircles the point $t=\mathrm{i}p$ once in the negative sense, encircles the point $t=-\mathrm{i}p$ once in the

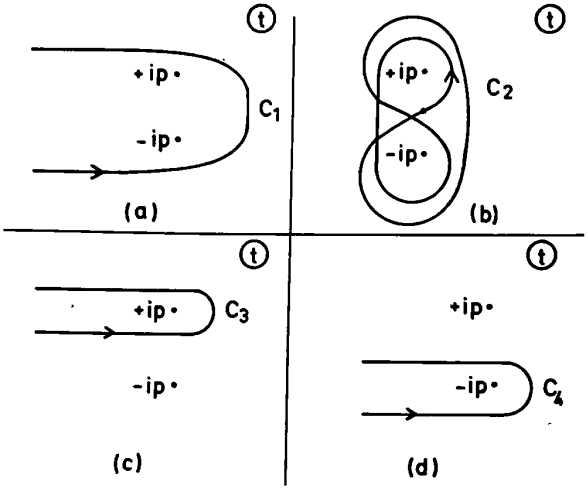


FIG. 3.1. t-plane contours suitable for the integral representation.

negative sense, and returns to the starting point as shown in Fig. 3.1(b). The integral with this contour will be denoted by

$$\int_{C_2} \ldots dt \quad \text{or} \quad \int_0^{(ip+,-ip+,ip-,-ip-)} \ldots dt.$$

(3) The contour C_3 which starts at or near $t=-\infty+ip$, encircles the singular point $t=ip$ once in the positive sense, and returns to $t=-\infty+ip$ as shown in Fig. 3.1(c). The integral with this contour will be denoted by

$$\int_{C_3} \ldots dt \quad \text{or} \quad \int_{-\infty+ip}^{(ip+)} \ldots dt.$$

(4) The contour C_4 which starts at or near $t=-\infty-ip$, encircles the singular point $t=-ip$ once in the positive sense, and returns to $t=-\infty-ip$ as shown in Fig. 3.1(d). The integral with this contour will be denoted by

$$\int_{C_4} \ldots dt \quad \text{or} \quad \int_{-\infty-ip}^{(-ip+)} \ldots dt.$$

If s is restricted to its negative value $s=-\gamma$, some contours become possible which are simply straight lines starting from one singular point and ending in another singular point, rather than encircling singular

points. Such contours are the straight line from $-ip$ to $+ip$ which generates, apart from a multiplicative factor, the same solution as the contour C_2, the straight line from $-\infty+ip$ to ip which corresponds to the contour C_3, and the straight line from $-\infty-ip$ to $-ip$ which corresponds to C_4.

In order to have the integrals well-defined it is, because of the non-integer powers in the integrand, necessary to define by convention the phases of the powers at some point of each contour. Our convention, corresponding to principal values, is: $\arg(t)=0$ when t is real and positive, in particular for that point where C_1 traverses the positive real axis; $\arg(t+ip)=\pi/2$ and $\arg(t-ip)=-\pi/2$ when t is purely imaginary and $|t|<p$, in particular for the starting point of C_2 and for the points where, while $|t|<p$, C_3 or C_4 traverses the imaginary axis.

This convention also defines the integrand on the straight line contour from $-ip$ to ip. In the case of the other straight line contours it is more reasonable to change the integrand by an appropriate phase factor. For the straight line contour from $-\infty+ip$ to $+ip$ we then adopt $\arg\{-(t-ip)\}=0$, $\pi>\arg(t+ip)\geq\pi/2$, and for the straight line contour from $-\infty-ip$ to $-ip$ we adopt $\arg\{-(t+ip)\}=0$, $-\pi<\arg(t-ip)\leq-\pi/2$.

We are now prepared to investigate the contour integral solutions with the different contours C_j, which will be denoted by $\{f_j(r), g_j(r)\}$ where $j=1,2,3,4$. In order to find the connection between the different power series solutions, it is sufficient to consider three different contours rather than four; C_1 or C_2 together with C_3 and C_4. This is because we have two possible values for the exponent L in eqn (3.48) at our disposal according to eqn (3.52). We decided to omit the contour C_2, which technically is slightly more complicated than C_1. Since the contour C_1 is equivalent to the sum of the contours C_3 and C_4 according to Cauchy's integral theorem, and since the phases of the integrand have been agreed upon in the same way in all three cases, it follows immediately that

$$\begin{Bmatrix} f_1(r) \\ g_1(r) \end{Bmatrix} = \begin{Bmatrix} f_3(r) \\ g_3(r) \end{Bmatrix} + \begin{Bmatrix} f_4(r) \\ g_4(r) \end{Bmatrix}. \tag{3.59}$$

We now have to identify these solutions as power series solutions by performing suitable series expansions of the integrands and integrating the series term by term. Since we already know the power series solutions and are only looking for the normalization factors, it suffices to obtain the leading term of each integral. In order to economize this task we introduce the function

$$\psi(r)=(2p)^{1+s}r^{-1-s}(2\pi i)^{-1}\int_C \exp(rt) \ast(t+ip)^{-1-s+iy}(t-ip)^{-1-s-iy}(At+B)\,dt, \tag{3.60}$$

which is equal to $f(r)$ or $g(r)$ depending on the choice of A and B according to

$$A=\kappa+s, \quad B=-\alpha ZT \quad \text{in the case of } f(r), \tag{3.61a}$$

$$A=\alpha Z, \quad B=(\kappa+s)(T+2m) \quad \text{in the case of } g(r). \tag{3.61b}$$

Equation (3.59) then simply reads

$$\psi_1(r)=\psi_3(r)+\psi_4(r). \tag{3.62}$$

With the contour C_1 we have to consider the integrand for large values of t, which correspond to small values of r. The contour C_1 may be continuously deformed so that it remains outside the circle $|t|=p$. The expansion of the integrand in powers of $1/t$ then converges uniformly everywhere on the contour, so that term by term integration of the series is justified and the resulting series converges too. Introducing a new variable of integration, $z=rt$, we see that the result is a series in powers of r. The contour C_1 therefore essentially yields one of the solutions defined relative to the point $r=0$. Restricting our attention to the leading term we have

$$\psi_1(r)\sim(2p)^{1+s}r^{-1-s}A(2\pi i)^{-1}\int_{-\infty}^{(0+)}e^{rt}t^{-1-2s}\,dt$$

$$=(2p)^{1+s}r^{-1+s}A(2\pi i)^{-1}\int_{-\infty}^{(0+)}e^{z}z^{-1-2s}\,dz.$$

Since the integral is related to the gamma function $\Gamma(x)$ by the formula

$$\{\Gamma(x)\}^{-1}=(2\pi i)^{-1}\int_{-\infty}^{(0+)}e^{z}z^{-x}\,dz, \tag{3.63}$$

we finally have, as $r\to 0$,

$$\psi_1(r)\sim(2p)^{1+s}\{\Gamma(1+2s)\}^{-1}Ar^{-1+s}. \tag{3.64}$$

With the contour C_3 the behaviour of the integrand in a neighbourhood of $t=ip$ is relevant. We therefore have to consider the expansion of the integrand in powers of $t-ip$. This expansion, convergent for $|t-ip|<2p$, does not converge everywhere on the contour, however. Consequently, term by term integration of the expansion yields a divergent series in powers of $1/r$, which, by Watson's lemma (Olver 1974), represents the integral asymptotically as $r\to\infty$. The contour C_3 therefore essentially generates one of the formal solutions defined relative to the point $r=\infty$. Restricting our attention to the leading term we have

$$\psi_3(r)\sim(2p)^{1+s}r^{-1-s}(2pi)^{-1-s+iy}(ipA+B)$$

$$*\,(2\pi i)^{-1}\int_{-\infty+ip}^{(+ip+)}e^{rt}(t-ip)^{-1-s-iy}\,dt. \tag{3.65}$$

With a new variable of integration $x = r(t - ip)$ and subsequent use of eqn (3.63) the integral becomes

$$\exp(ipr)r^{s+iy}\int_{-\infty}^{(0+)} e^x x^{-1-s-iy}\,dx = \exp(ipr)r^{s+iy}(2\pi i)\{\Gamma(1+s+iy)\}^{-1}.$$

Using

$$\Gamma(1+x) = x\Gamma(x)$$

and collecting the factors we finally have, as $r \to \infty$,

$$\psi_3(r) \sim i^{-1-s+iy}\{(ipA+B)/(s+iy)\}\{\Gamma(s+iy)\}^{-1}(2pr)^{iy}r^{-1}\exp(ipr). \tag{3.66}$$

We want to rewrite the complex factor $(ipA+B)/(s+iy)$ so that it will later be easy to express by its modulus and phase. For this purpose some manipulations with frequent use of the identities

$$W = T + m,$$

$$yp = \alpha ZW, \tag{3.67}$$

$$p^2 = T(T+2m) = (W-m)(W+m), \tag{3.68}$$

$$(\alpha Z)^2 = \kappa^2 - s^2 = (\kappa - s)(\kappa + s) \tag{3.69}$$

are necessary, the most promising strategy for obtaining simplification being to replace $(\alpha Z)^2$ and p^2. Let us first look at the numerator, which depends on whether we consider $f(r)$ or $g(r)$. We have by definition, eqn (3.61),

$$ipA + B = \begin{Bmatrix} ip(\kappa+s) - \alpha ZT \\ ip\alpha Z + (\kappa+s)(T+2m) \end{Bmatrix}.$$

$$= K\begin{cases} -T^{1/2} & \text{in the case of } f(r) \\ i(T+2m)^{1/2} & \text{in the case of } g(r) \end{cases} \tag{3.70}$$

where

$$K = \alpha ZT^{1/2} - i(\kappa+s)(T+2m)^{1/2}. \tag{3.71}$$

We therefore have to consider

$$K/(s+iy) = (s^2+y^2)^{-1}(s-iy)K.$$

We find

$$s^2 + y^2 = p^{-2}(\kappa W - sm)(\kappa W + sm),$$

$$(s-iy)K = -p^{-1}T^{-1/2}(\kappa W + sm)\{\alpha Zp + i(\kappa+s)T\},$$

and consequently

$$K/(s+iy) = -T^{-1/2}p(\kappa W - sm)^{-1}\{\alpha Zp + i(\kappa+s)T\}.$$

Extracting a factor $-\alpha Z/(\kappa - s)$ out of the complex factor, for later

convenience, we may also write

$$K/(s+\mathrm{i}y) = T^{-1/2}p\alpha Z\{(\kappa-s)(\kappa W-sm)\}^{-1}\{-(\kappa-s)p-\mathrm{i}\alpha ZT\}.$$

For the remaining complex factor we find

$$|-(\kappa-s)p-\mathrm{i}\alpha ZT|^2 = 2T(\kappa-s)(\kappa W-sm).$$

It is now reasonable to introduce a real phase η by†

$$\exp(\mathrm{i}\eta) = \{-(\kappa-s)p-\mathrm{i}\alpha ZT\}/\{2T(\kappa-s)(\kappa W-sm)\}^{1/2}. \tag{3.72}$$

We then have

$$K/(s+\mathrm{i}y) = 2^{1/2}\alpha Zp\{(\kappa-s)(\kappa W-sm)\}^{-1/2}\exp(\mathrm{i}\eta). \tag{3.73}$$

By means of eqns (3.70), (3.71) and (3.73), eqn (3.66) may now be rewritten as

$$\psi_3(r) = \begin{Bmatrix} f_3(r) \\ g_3(r) \end{Bmatrix} \sim \begin{Bmatrix} \mathrm{i}T^{1/2} \\ (T+2m)^{1/2} \end{Bmatrix} \mathrm{i}^{-s}\exp\{-(\pi/2)y\}2^{1/2}\alpha Zp$$
$$*\{(\kappa-s)(\kappa W-sm)\}^{-1/2}\exp(\mathrm{i}\eta)\{\Gamma(s+\mathrm{i}y)\}^{-1}(2pr)^{\mathrm{i}y}r^{-1}\exp(\mathrm{i}pr). \tag{3.74}$$

With the contour C_4 the calculation proceeds in a similar way as with C_3, i being replaced by $-\mathrm{i}$ everywhere except in the factor $(2\pi\mathrm{i})^{-1}$ in front of the integral since this factor corresponds to the positive orientation, which is not changed, of the contour with regard to the singular point. The result for $\psi_4(r)$ therefore is simply the complex conjugate of $\psi_3(r)$ and need not be written down.

Comparing the leading terms obtained here for $\psi_1(r)$, $\psi_3(r)$, $\psi_4(r)$ with the corresponding leading terms of our standard solutions, we may conclude that

$$\psi_1(r) = (2p)^{1+s}\{\Gamma(1+2s)\}^{-1}\{(\kappa-s)(\kappa W-sm)\}^{-1/2}\alpha Z\begin{Bmatrix} f_{0j}(r) \\ g_{0j}(r) \end{Bmatrix} \tag{3.75}$$

† This equation defines $\eta \bmod 2\pi$, while the simpler equation

$$\mathrm{tg}(\eta) = \alpha ZT/\{(\kappa-s)p\}$$

contains less information, defining $\eta \bmod \pi$ only. The same is true for another expression,

$$\exp(2\mathrm{i}\eta) = (-\kappa p+\mathrm{i}\alpha Zm)/(\gamma p+\mathrm{i}\alpha ZW),$$

which is sometimes useful. This equation is inconvenient to derive: by squaring eqn (3.72) and simplifying we obtain

$$(\kappa m-\gamma W+\mathrm{i}\alpha Zp)/(\kappa W-\gamma m).$$

Multiplying numerator and denominator by $\gamma p+\mathrm{i}\alpha ZW$, we may again simplify the numerator and extract a factor $(\kappa W-\gamma m)$ which cancels the same factor in the denominator, so that the required result appears.

where $j=1$ or $j=2$ according to whether $s=\gamma$ or $s=-\gamma$,

$$\psi_3(r)=(2W)^{1/2}\mathrm{i}^{-s}\exp\{-(\pi/2)y\}\alpha Zp\{(\kappa-s)(\kappa W-sm)\}^{-1/2}$$
$$*\exp(\mathrm{i}\eta)\{\Gamma(s+\mathrm{i}y)\}^{-1}(2p)^{\mathrm{i}y}\begin{Bmatrix}f_{\infty 1}(r)\\ g_{\infty 1}(r)\end{Bmatrix}, \quad (3.76)$$

$$\psi_4(r)=(2W)^{1/2}(-\mathrm{i})^{-s}\exp\{-(\pi/2)y\}\alpha Zp\{(\kappa-s)(\kappa W-sm)\}^{-1/2}$$
$$*\exp(-\mathrm{i}\eta)\{\Gamma(s-\mathrm{i}y)\}^{-1}(2p)^{-\mathrm{i}y}\begin{Bmatrix}f_{\infty 2}(r)\\ g_{\infty 2}(r)\end{Bmatrix}. \quad (3.77)$$

Because $\psi_1=\psi_3+\psi_4$ according to eqn (3.62), it follows from eqns (3.75) to (3.77) that

$$2(2W)^{-1/2}\exp\{(\pi/2)y\}\{\Gamma(1+2s)\}^{-1}(2p)^{s}\begin{Bmatrix}f_{0j}(r)\\ g_{0j}(r)\end{Bmatrix}$$
$$=i^{-s}\exp(\mathrm{i}\eta)\{\Gamma(s+\mathrm{i}y)\}^{-1}(2p)^{\mathrm{i}y}\begin{Bmatrix}f_{\infty 1}(r)\\ g_{\infty 1}(r)\end{Bmatrix}$$
$$+(-i)^{-s}\exp(-\mathrm{i}\eta)\{\Gamma(s-\mathrm{i}y)\}^{-1}(2p)^{-\mathrm{i}y}\begin{Bmatrix}f_{\infty 2}(r)\\ g_{\infty 2}(r)\end{Bmatrix}, \quad (3.78)$$

where $j=1$ or $j=2$ according to whether $s=\gamma$ or $s=-\gamma$.

Equation (3.78) is the desired formula connecting the standard solutions of different type. We may rewrite it in a different form if we represent the complex factors by their moduli and phases. In this context it may be helpful to note that, while

$$\Gamma(s+\mathrm{i}y)=|\Gamma(s+\mathrm{i}y)|\exp\{\mathrm{i}\arg\Gamma(s+\mathrm{i}y)\},$$

we may write

$$\Gamma(s-\mathrm{i}y)=|\Gamma(s+\mathrm{i}y)|\exp\{-\mathrm{i}\arg\Gamma(s+\mathrm{i}y)\}.$$

Introducing the real phase

$$\delta=-\arg\Gamma(s+\mathrm{i}y)+\eta-(\pi/2)s, \quad (3.79)$$

it follows from eqn (3.78) that

$$2(2W)^{-1/2}\exp\{(\pi/2)y\}\,|\Gamma(s+\mathrm{i}y)|\,\{\Gamma(1+2s)\}^{-1}$$
$$*(2p)^{s}\begin{Bmatrix}f_{0j}(r)\\ g_{0j}(r)\end{Bmatrix}=\exp[\mathrm{i}\{\delta+y\ln(2p)\}]\begin{Bmatrix}f_{\infty 1}(r)\\ g_{\infty 1}(r)\end{Bmatrix}$$
$$+\exp[-\mathrm{i}\{\delta+y\ln(2p)\}]\begin{Bmatrix}f_{\infty 2}(r)\\ g_{\infty 2}(r)\end{Bmatrix}. \quad (3.80)$$

3.2.2.3.2. Coulomb radial wave functions. The right-hand side of eqn (3.80), which depends on the sign of s via the phase δ only, may be rewritten in real form. Using at infinity our real standard solutions, eqns

(3.43) and (3.44), rather than the complex ones, eqn (3.41), we obtain

$$\begin{Bmatrix} f^C(r) \\ g^C(r) \end{Bmatrix} = (2W)^{-1/2} \exp\{(\pi/2)y\} \, |\Gamma(s+iy)| \, \{\Gamma(1+2s)\}^{-1}$$

$$* (2p)^s \begin{Bmatrix} f_{0j}(r) \\ g_{0j}(r) \end{Bmatrix} = \cos(\Delta) \begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} + \sin(\Delta) \begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}, \quad (3.81)$$

where

$$\Delta = \delta + (\pi/2)\{l(\kappa)+1\} \quad (3.82)$$

is the Coulomb scattering phase shift, which measures the phase change as compared with the case of the identically vanishing potential.† The solutions $\{f^C(r), g^C(r)\}$ defined by the left-hand part of eqn (3.81) are usually called the 'regular' or irregular Coulomb radial wave functions according to whether $s = +\gamma$, $j = 1$ or $s = -\gamma$, $j = 2$, respectively. Since the leading terms of the solutions on the right-hand side of eqn (3.81), according to eqns (3.44) and (3.46), are

$$\begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} \sim r^{-1} \begin{Bmatrix} -(T/W)^{1/2} \sin(x) \\ \{(T+2m)/W\}^{1/2} \cos(x) \end{Bmatrix}, \quad (3.83)$$

$$\begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix} \sim r^{-1} \begin{Bmatrix} -(T/W)^{1/2} \cos(x) \\ -\{(T+2m)/W\}^{1/2} \sin(x) \end{Bmatrix}, \quad (3.84)$$

where $x = pr + y \ln(2pr) - (\pi/2)\{l(\kappa)+1\}$ according to eqn (3.42), the asymptotic behaviour, when $r \to \infty$, of the Coulomb radial wave functions is seen to be

$$\begin{Bmatrix} f^C(r) \\ g^C(r) \end{Bmatrix} \sim r^{-1} \begin{Bmatrix} -(T/W)^{1/2} \sin\{pr + y \ln(2pr) + \delta\} \\ \{(T+2m)/W\}^{1/2} \cos\{pr + y \ln(2pr) + \delta\} \end{Bmatrix}. \quad (3.85)$$

It should be noted again that δ depends on s. If the 'regular' and irregular Coulomb wave functions will be needed simultaneously, we will denote $\delta(s = \gamma)$ simply by δ and $\delta(s = -\gamma)$ by $\bar{\delta}$, and similarly for Δ, $f^C(r)$, and $g^C(r)$. We may find the Wronskian relation for the Coulomb wave functions

$$r^2\{f^C(r)\bar{g}^C(r) - g^C(r)\bar{f}^C(r)\} \equiv -(p/W)\sin(\delta - \bar{\delta}) \quad (3.86)$$

most easily by looking at the leading terms as $r \to \infty$ according to eqn (3.85). Investigating the behaviour when $r \to 0$, on the other hand, we may find from eqn (3.81) that the Wronskian, eqn (3.86), is equal to

$$(2W)^{-1} \exp(\pi y) \, |\Gamma(\gamma+iy)\Gamma(-\gamma+iy)| \, \{\Gamma(1+2\gamma)\Gamma(1-2\gamma)\}^{-1}$$

$$* r^2\{f_{01}(r)g_{02}(r) - g_{01}(r)f_{02}(r)\}. \quad (3.87)$$

† Apart from the pathological logarithmic contribution.

Here the products of gamma functions may be eliminated† by means of the reflection formula of the gamma function

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z), \tag{3.88}$$

so that

$$|\Gamma(\gamma+\mathrm{i}y)\Gamma(-\gamma+\mathrm{i}y)|\,\{\Gamma(1+2\gamma)\Gamma(1-2\gamma)\}^{-1} = \sin(2\pi\gamma)$$
$$* [2\gamma\,|\gamma+\mathrm{i}y|\,|\sin\{\pi(\gamma+\mathrm{i}y)\}|]^{-1} = (2\gamma)^{-1}\sin(2\pi\gamma)$$
$$* (\gamma^2+y^2)^{-1/2}[\{\sin(\pi\gamma)\}^2\{\cosh(\pi y)\}^2+\{\cos(\pi\gamma)\}^2\{\sinh(\pi y)\}^2]^{-1/2}.$$

The value of the remaining Wronskian in eqn (3.87) is known from eqn (3.24). After simplification by means of eqn (3.25) and the identity

$$p^2(\gamma^2+y^2) = (kp)^2+(\alpha Zm)^2$$

we then obtain finally

$$r^2\{f^{\mathrm{C}}(r)\bar{g}^{\mathrm{C}}(r) - g^{\mathrm{C}}(r)\bar{f}^{\mathrm{C}}(r)\} \equiv (p/W)\mathrm{Sign}(\alpha Z)\exp(\pi y)\cos(\pi\gamma)$$
$$* \sin(\pi\gamma)[\{\sin(\pi\gamma)\}^2\{\cosh(\pi y)\}^2+\{\cos(\pi\gamma)\}^2\{\sinh(\pi y)\}^2]^{-1/2}. \tag{3.89}$$

By comparison with eqn (3.86) we may conclude that

$$\sin(\delta-\bar{\delta}) = -\mathrm{Sign}(\alpha Z)\exp(\pi y)\cos(\pi\gamma)\sin(\pi\gamma)$$
$$* [\{\sin(\pi\gamma)\}^2\{\cosh(\pi y)\}^2+\{\cos(\pi\gamma)\}^2\{\sinh(\pi y)\}^2]^{-1/2}. \tag{3.90}$$

It should be possible to verify this result directly from the definition of δ. From eqn (3.79) we have

$$\delta-\bar{\delta} = -\arg\Gamma(\gamma+\mathrm{i}y)+\arg\Gamma(-\gamma+\mathrm{i}y)+\eta-\bar{\eta}-\pi\gamma \tag{3.91}$$

or

$$\exp\{\mathrm{i}(\delta-\bar{\delta})\} = \{|\Gamma(\gamma+\mathrm{i}y)|/\Gamma(\gamma+\mathrm{i}y)\}\{|\Gamma(-\gamma-\mathrm{i}y)|/\Gamma(-\gamma-\mathrm{i}y)\}$$
$$* \exp\{\mathrm{i}(\eta-\bar{\eta})\}\exp(-\mathrm{i}\pi\gamma) = \{-(\gamma+\mathrm{i}y)/|\gamma+\mathrm{i}y|\}[\sin\{\pi(\gamma+\mathrm{i}y)\}/$$
$$|\sin\{\pi(\gamma+\mathrm{i}y)\}|]\exp\{\mathrm{i}(\eta-\bar{\eta})\}\exp(-\mathrm{i}\pi\gamma). \tag{3.92}$$

In order to evaluate $\exp\{\mathrm{i}(\eta-\bar{\eta})\}$ conveniently we multiply

$$\exp(\mathrm{i}\eta) = \{-(\kappa-\gamma)p-\mathrm{i}\alpha ZT\}/\{2T(\kappa-\gamma)(\kappa W-\gamma m)\}^{1/2}$$

by

$$\exp(-\mathrm{i}\bar{\eta}) = \{-(\kappa+\gamma)p+\mathrm{i}\alpha ZT\}/\{2T(\kappa+\gamma)(\kappa W+\gamma m)\}^{1/2}$$

and obtain after simplification

$$\exp\{\mathrm{i}(\eta-\bar{\eta})\} = \mathrm{Sign}(\alpha Z)\{\alpha ZW+\mathrm{i}\gamma p\}/\{(kp)^2+(\alpha Zm)^2\}^{1/2}, \tag{3.93}$$

a formula which may be rewritten as

$$\exp\{\mathrm{i}(\eta-\bar{\eta})\} = \mathrm{Sign}(\alpha Z)\mathrm{i}(\gamma-\mathrm{i}y)/|\gamma-\mathrm{i}y|. \tag{3.94}$$

† In addition we use $|\Gamma(-\gamma+\mathrm{i}y)| = |\Gamma(-\gamma-\mathrm{i}y)|$ and the difference equation of the gamma function $\Gamma(z+1) = z\Gamma(z)$.

With this result our eqn (3.92) reduces to

$$\exp\{i(\delta-\bar{\delta})\} = -i\,\mathrm{Sign}(\alpha Z)\exp(-i\pi\gamma)\sin\{\pi(\gamma+iy)\}$$
$$/|\sin\{\pi(\gamma+iy)\}| = (-1/2)\mathrm{Sign}(\alpha Z)[\exp(-\pi y)-\exp(\pi y)$$
$$*\cos(2\pi\gamma)+i\exp(\pi y)\sin(2\pi\gamma)]/|\sin\{\pi(\gamma+iy)\}|. \quad (3.95)$$

Evaluating the denominator as above and looking at the imaginary part we may verify eqn (3.90). Looking at the real part we find the complementary formula

$$\cos(\delta-\bar{\delta}) = (1/2)\mathrm{Sign}(\alpha Z)\{\exp(\pi y)\cos(2\pi\gamma)-\exp(-\pi y)\}$$
$$*[\{\sin(\pi\gamma)\}^2\{\cosh(\pi y)\}^2+\{\cos(\pi\gamma)\}^2\{\sinh(\pi y)\}^2]^{-1/2}. \quad (3.96)$$

3.2.2.4. Power series expansion of the contour integral solutions

For obtaining the connection between the standard solutions of different type, it was sufficient to consider the leading terms of the expansion of the contour integral solutions. Nevertheless it is possible to derive the complete power series expansions of the solutions from the contour integrals. The expansion coefficients may then be obtained in a more explicit form, which is useful theoretically, while the recursive form of Sections 3.2.2.1 and 2 is quite satisfactory from a computational point of view. Devoting our attention again to the solution $\psi_3(r)$ of Section 3.2.2.3, we need the complete expansion in powers of $t-ip$ of the integrand in eqn (3.60). With

$$At+B = (B+ipA)[1+\{A/(B+ipA)\}(t-ip)],$$
$$t+ip = 2ip[1+\{(t-ip)/(2ip)\}],$$

and the binomial theorem in the form

$$(1-z)^{-a} = \sum_{n=0}^{\infty}\{\Gamma(a+n)/\Gamma(a)\}\{z^n/n!\} \quad (3.97)$$

we have to consider

$$(t+ip)^{-1-s+iy}(t-ip)^{-1-s-iy}(At+B) = (2ip)^{-1-s+iy}(B+ipA)$$
$$*(t-ip)^{-1-s-iy}[1+\{A/(B+ipA)\}(t-ip)]\sum_{n=0}^{\infty}\{\Gamma(1+s-iy+n)$$
$$/\Gamma(1+s-iy)\}\{(-2ip)^{-n}/n!\}(t-ip)^n.$$

Integration of the series term by term then leads, with $h=0$ or 1, to the integrals

$$\int_{-\infty+ip}^{(+ip+)}\exp(rt)(t-ip)^{-1-s-iy+n+h}\,dt$$
$$= \exp(ipr)r^{s+iy-n-h}\{(2\pi i)/\Gamma(1+s+iy-n-h)\},$$

which are related to the integrals of eqn (3.65) by the substitution $s \to s-n-h$. Collecting the results we find that the leading term of $\psi_3(r)$ as given by eqn (3.66) or eqn (3.74) has to be multiplied by the series

$$S_3 = \sum_{n=0}^{\infty} \{\Gamma(1+s-\mathrm{i}y+n)/\Gamma(1+s-\mathrm{i}y)\}\{(-2\mathrm{i}p)^{-n}/n!\}[\{\Gamma(1+s+\mathrm{i}y) /\Gamma(1+s+\mathrm{i}y-n)\}r^{-n}+\{A/(B+\mathrm{i}pA)\}\{\Gamma(1+s+\mathrm{i}y)/\Gamma(s+\mathrm{i}y-n)\}r^{-n-1}].$$

With the sum corresponding to the second term rewritten as

$$\{A/(B+\mathrm{i}pA\} \sum_{n=1}^{\infty} \{\Gamma(s-\mathrm{i}y+n)/\Gamma(1+s-\mathrm{i}y)\}\{(-2\mathrm{i}p)^{-n+1}/(n-1)!\} * \{\Gamma(1+s+\mathrm{i}y)/\Gamma(1+s+\mathrm{i}y-n)\}r^{-n},$$

we may write

$$S_3 = 1+\sum_{n=1}^{\infty} \{(-2\mathrm{i}pr)^{-n}/n!\}\{\Gamma(s-\mathrm{i}y+n)/\Gamma(s-\mathrm{i}y)\}\{\Gamma(1+s+\mathrm{i}y) /\Gamma(1+s+\mathrm{i}y-n)\}[\{(s-\mathrm{i}y+n)/(s-\mathrm{i}y)\}+\{A/(B+\mathrm{i}pA)\}\{-2\mathrm{i}p/(s-\mathrm{i}y)\}n].$$

Here the last factor can be simplified to give

$$1+\{B-\mathrm{i}pA)/(B+\mathrm{i}pA)\}\{1/(s-\mathrm{i}y)\}n.$$

Using

$$\Gamma(1+s+\mathrm{i}y)/\Gamma(1+s+\mathrm{i}y-n) = (-1)^n\Gamma(-s-\mathrm{i}y+n)/\Gamma(-s-\mathrm{i}y) \tag{3.98}$$

according to the reflection formula of the gamma function, eqn (3.88), and evaluating

$$(s-\mathrm{i}y)^{-1}\{(B-\mathrm{i}pA)/(B+\mathrm{i}pA)\} = \mp\mathrm{i}p(\alpha Zm-\mathrm{i}p\kappa)/\{(\kappa W-sm)(\kappa W+sm)\} = \mp\mathrm{i}p(\alpha Zm-\mathrm{i}p\kappa)/\{(kp)^2+(\alpha Zm)^2\} = \mp\mathrm{i}p/(\alpha Zm+\mathrm{i}p\kappa)$$

by a somewhat lengthy calculation, we obtain finally

$$S_3 = 1+\sum_{n=1}^{\infty} [1\mp\{\mathrm{i}p/(\alpha Zm+\mathrm{i}p\kappa)\}n]\{\Gamma(s-\mathrm{i}y+n)/\Gamma(s-\mathrm{i}y)\} * \{\Gamma(-s-\mathrm{i}y+n)/\Gamma(-s-\mathrm{i}y)\}\{(2\mathrm{i}pr)^{-n}/n!\} \tag{3.99}$$

where the upper sign is for $f(r)$ and the lower one for $g(r)$. This form of S_3 clearly shows that S_3 is independent of the sign of s, as it should be. The factor containing the gamma functions is, in fact, a polynomial in $s^2=\gamma^2$. By comparison of eqn (3.99) with

$$S_3 = \sum_{n=0}^{\infty} \begin{Bmatrix} a_n/a_0 \\ b_n/b_0 \end{Bmatrix} r^{-n}$$

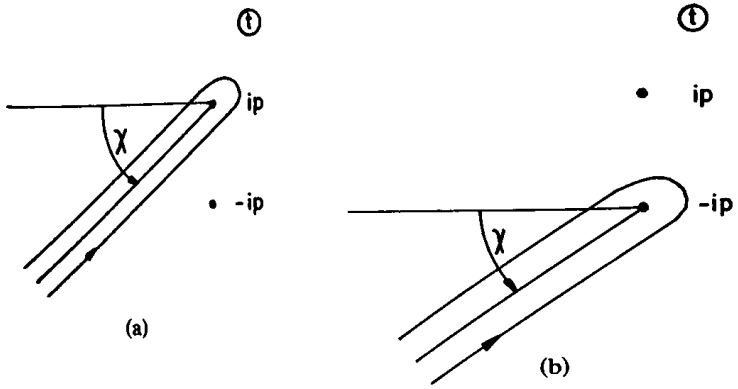


FIG. 3.2. Rotation of t-plane contours by an angle χ.

explicit formulae for the expansion coefficients a_n, b_n may be read off, which yield the same values as the recurrence relations, eqns (3.30) and (3.40), of course.

In a similar way the leading term of $\psi_4(r)$ has to be multiplied by a series S_4 which is simply the complex conjugate of S_3 and therefore need not be displayed here.

The contour integral solutions $\psi_3(r)$ and $\psi_4(r)$ are useful for another reason: although we want to use the asymptotic expansions, eqn (3.41) or (3.88), for real values of r only, it is nevertheless important theoretically to know their domain of validity. The integral representation for $\psi_3(r)$ is valid for $|\arg(r)|<\pi/2$, a condition which guarantees that the integral exists and that the expression (3.58) vanishes at the termini of the contour. Its asymptotic expansion therefore is valid in the same sector of the r plane, but the integral representation may be analytically continued into a larger sector by rotation of the contour around the relevant singular point, as shown in Fig. 3.2(a). The possible angles of rotation χ are restricted by the presence of the other singular point according to $-3\pi/2<\chi<\pi/2$. The integral representation then is valid for $|\arg(r)+\chi|<\pi/2$, but when expanded yields the same asymptotic expansion as before with $\chi=0$. The asymptotic expansion therefore is valid in the union of all the sectors corresponding to the permissible values of χ, so that the maximum sector of validity, in the case of asymptotic expansion of $\psi_3(r)$, is given by $-\pi<\arg(r)<2\pi$. There is a slight difference in the case of $\psi_4(r)$ since here, as may be seen from Fig. 3.2(b), the angle of rotation is restricted by $-\pi/2<\chi<3\pi/2$. Consequently, the maximum sector of validity for the asymptotic expansion of $\psi_4(r)$ is $-2\pi<\arg(r)<\pi$.

Thus the maximum sectors for $\psi_3(r)$ and $\psi_4(r)$ are overlapping but do not coincide, a behaviour which is well known from Hankel functions. Much smaller than these maximum sectors in which the asymptotic expansions are valid theoretically are the domains in which they are useful numerically, a fact which is well known too (Olver 1974).

If we want to obtain the power series expansion of $\psi_1(r)$ from the integral representation, eqn (3.60), we need the complete expansion in powers of $1/t$ of the integrand

$$t^h(t+\mathrm{i}p)^{-1-s+\mathrm{i}y}(t-\mathrm{i}p)^{-1-s-\mathrm{i}y} = t^{h-2s-2}\{1+(\mathrm{i}p/t)\}^{-1-s+\mathrm{i}y}\{1-(\mathrm{i}p/t)\}^{-1-s-\mathrm{i}y}$$

$$= t^{h-2s-2}\sum_{l=0}^{\infty}\{\Gamma(1+s-\mathrm{i}y+l)/\Gamma(1+s-\mathrm{i}y)\}\{(-\mathrm{i}p/t)^l/l!\}$$

$$*\sum_{j=0}^{\infty}\{\Gamma(1+s+\mathrm{i}y+j)/\Gamma(1+s+\mathrm{i}y)\}\{(\mathrm{i}p/t)^j/j!\}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}(-1)^j\frac{\Gamma(1+s-\mathrm{i}y+n-j)\Gamma(1+s+\mathrm{i}y+j)}{\Gamma(1+s-\mathrm{i}y)(n-j)!\,\Gamma(1+s+\mathrm{i}y)j!}\right\}(-\mathrm{i}p)^n t^{h-2s-2-n}.$$

We therefore have to evaluate, with $h=0$ or 1, the integrals

$$(2\pi\mathrm{i})^{-1}\int_{-\infty}^{(0+)}\exp(rt)t^{h-2s-2-n}\,\mathrm{d}t = r^{-h+2s+1+n}/\Gamma(-h+2s+2+n).$$

Collecting the results we may see that the leading term, eqn (3.64), of the expansion of $\psi_1(r)$ has to be multiplied by the series

$$S_1 = \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}(-1)^j\frac{\Gamma(1+s-\mathrm{i}y+n-j)\Gamma(1+s+\mathrm{i}y+j)}{\Gamma(1+s-\mathrm{i}y)(n-j)!\,\Gamma(1+s+\mathrm{i}y)j!}\right\}$$

$$*(-\mathrm{i}p)^n\left\{\frac{\Gamma(2s+1)}{\Gamma(2s+1+n)}r^n+(B/A)\frac{\Gamma(2s+1)}{\Gamma(2s+2+n)}r^{n+1}\right\}$$

or

$$S_1 = 1+\sum_{n=1}^{\infty}\left\{\sum_{j=0}^{n}(-1)^j\frac{\Gamma(1+s-\mathrm{i}y+n-j)\Gamma(1+s+\mathrm{i}y+j)}{\Gamma(1+s-\mathrm{i}y)(n-j)!\,\Gamma(1+s+\mathrm{i}y)j!}\right.$$

$$\left.-(B/A)(\mathrm{i}p)^{-1}\sum_{j=0}^{n-1}(-1)^j\frac{\Gamma(s-\mathrm{i}y+n-j)\Gamma(1+s+\mathrm{i}y+j)}{\Gamma(1+s-\mathrm{i}y)(n-1-j)!\,\Gamma(1+s+\mathrm{i}y)j!}\right\}$$

$$*\frac{\Gamma(2s+1)}{\Gamma(2s+1+n)}(-\mathrm{i}pr)^n \quad (3.100)$$

with

$$B/A = \begin{cases}-\alpha ZT/(\kappa+s) & \text{in the case of } f(r)\\ (\kappa+s)(T+2m)/(\alpha Z) & \text{in the case of } g(r)\end{cases}.$$

By comparison with

$$S_1 = \sum_{n=0}^{\infty} \begin{Bmatrix} a_n/a_0 \\ b_n/b_0 \end{Bmatrix} r^n,$$

explicit formulae for the expansion coefficients a_n or b_n, respectively, may be obtained, which generate the same coefficients as the recurrence relations (3.21), of course. That the coefficients are real may be seen as follows: for n even there is one j in the first sum over j such that $n-j=j$ and the corresponding terms is real. Of the remaining terms there are always pairs like those corresponding to $j=0$, n or $j=1$, $n-1$ etc. which are conjugate complex to each other and have the same sign so that their sum is real. For n odd the conjugate complex pairs of terms have a different sign and give an imaginary sum, which, when multiplied by $(-ip)^n$, again gives a real result. The second sum over j may be discussed in a similar way.

3.2.2.5. Integral representation of the standard solutions

Our standard solutions, which were originally defined by their power series expansions in Sections 3.2.2. 1 and 2, can alternatively be represented by contour integrals. Rather than to display such results, which may be obtained easily by comparison of eqn (3.60) with eqns (3.75) to (3.77), we want to obtain, if possible, for each of our standard solutions an ordinary integral representation with a straight line connecting two singular points as the path of integration. As shown in Section 3.2.2.3, such integral representations exist provided that we restrict ourselves to $s=-\gamma$. This actually means no restriction, neither for $j=1$ nor for $j=2$, in the case of the standard solutions $\{f_{\infty j}(r), g_{\infty j}(r)\}$, which in Section 3.2.2.4 have been shown to be independent of the sign of s. But of the standard solutions $\{f_{0j}(r), g_{0j}(r)\}$ only the 'regular' one corresponding to $j=1$ can be represented in this way, while for the other one ($j=2$) the contour integral cannot be replaced by an ordinary integral.

There are now two ways to proceed in the case of the solutions $\{f_{\infty j}(r), g_{\infty j}(r)\}$. We may either directly start from a straight line path integral which, according to Section 3.2.2.3, yields a solution, and find the leading term of its expansion. We here prefer the other way which consists of transforming the contour integral of $\psi_3(r)$ into an ordinary integral. For this purpose we may continuously deform the contour C_3 into an equivalent contour $\tilde{C}_3$ such that it consists of a small circle around the singular point $t=ip$ and two straight lines which coincide with the straight line from $-\infty+ip$ to $+ip$ as shown in Fig. 3.3. Finally the radius of the circle is to be made arbitrarily small. It is at this point that the restriction $s=-\gamma$ is needed to assure that the contribution to the integral from the circular part of the contour vanishes and the integrals over the

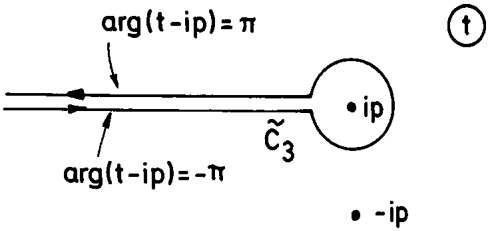


FIG. 3.3. The t-plane contour $\tilde{C}_3$. On the straight line parts, which have been drawn separately but may be assumed to coincide, the phases of the integrand are different.

straight line parts exist. According to eqn (3.60) with $s=-\gamma$, we then have to consider

$$\psi_3(r)=(2p)^{1-\gamma}r^{-1+\gamma}(2\pi i)^{-1}\int_{\tilde{C}_3}\exp(rt)\,*(t+ip)^{-1+\gamma+iy}(t-ip)^{-1+\gamma-iy}(At+B)\,dt.$$

On the straight line parts of the contour we have $\arg\{-(t-ip)\}=0$, while $\arg(t-ip)=-\pi$ or $+\pi$ according to whether we are concerned with the first or second straight line part of the contour. We may therefore write

$$(t-ip)^{-1+\gamma-iy}=\exp\{-i\pi(-1+\gamma-iy)\}(-t+ip)^{-1+\gamma-iy}$$

or

$$(t-ip)^{-1+\gamma-iy}=\exp\{i\pi(-1+\gamma-iy)\}(-t+ip)^{-1+\gamma-iy},$$

respectively, on the different straight line parts and obtain

$$\psi_3(r)=(2p)^{1-\gamma}r^{-1+\gamma}(2\pi i)^{-1}[\exp\{-i\pi(-1+\gamma-iy)\}-\exp\{i\pi(-1+\gamma-iy)\}]\int_{-\infty+ip}^{ip}\exp(rt)(t+ip)^{-1+\gamma+iy}\,*(-t+ip)^{-1+\gamma-iy}(At+B)\,dt.$$

Using a new variable of integration

$$z=r(-t+ip),$$

where we assume that r is real and not negative, we have

$$t+ip=2ip-(-t+ip)=2ip[1-\{z/(2ipr)\}]$$

and obtain

$$\psi_3(r)=(2p)^{iy}i^{-1+\gamma+iy}\pi^{-1}\sin\{\pi(\gamma-iy)\}(B+ipA)r^{iy-1}\,*\exp(ipr)\int_0^\infty\exp(-z)[1-\{z/(2ipr)\}]^{-1+\gamma+iy}z^{-1+\gamma-iy}\,*[1-\{A/(B+ipA)\}(z/r)]\,dz.$$

Because

$$B+\mathrm{i}pA=\begin{Bmatrix}-T^{1/2}\\ \mathrm{i}(T+2m)^{1/2}\end{Bmatrix}2^{1/2}\alpha Zp(s+\mathrm{i}y)\{(\kappa-s)(\kappa W-sm)\}^{-1/2}\exp(\mathrm{i}\eta)$$

according to eqns (3.70) and (3.73), and

$$\pi^{-1}(s+\mathrm{i}y)\Gamma(s+\mathrm{i}y)\sin\{\pi(-s-\mathrm{i}y)\}=1/\Gamma(-s-\mathrm{i}y)$$

according to the reflection formula of the gamma function, eqn (3.88), we obtain by comparison with eqn (3.76)

$$\begin{Bmatrix}f_{\infty 1}(r)\\ g_{\infty 1}(r)\end{Bmatrix}=\begin{Bmatrix}\mathrm{i}\{T/W\}^{1/2}\\ \{(T+2m)/W\}^{1/2}\end{Bmatrix}\{1/\Gamma(\gamma-\mathrm{i}y)\}r^{\mathrm{i}y-1}\exp(\mathrm{i}pr)\int_0^\infty\exp(-z)$$
$$*[1-\{z/(2\mathrm{i}pr)\}]^{-1+\gamma+\mathrm{i}y}z^{-1+\gamma-\mathrm{i}y}[1-\{A/(B+\mathrm{i}pA)\}(z/r)]\,\mathrm{d}z. \quad (3.101)$$

Here $A/(B+\mathrm{i}pA)$, which has to be evaluated for $s=-\gamma$, is equal to

$$(\kappa-\gamma)/\{-\alpha ZT+\mathrm{i}p(\kappa-\gamma)\}$$

in the case of $f_{\infty 1}(r)$, or equal to

$$\alpha Z/\{(\kappa-\gamma)(T+2m)+\mathrm{i}p\alpha Z\}$$

in the case of $g_{\infty 1}(r)$, according to eqn (3.61).

The integral representation for $\{f_{\infty 2}(r), g_{\infty 2}(r)\}$, which may be obtained in a similar way from $\psi_4(r)$, is simply given by the complex conjugate of the right-hand side of eqn (3.101) and therefore need not be displayed here.

Principally there are also two ways to proceed in the case of the standard solution $\{f_{01}(r), g_{01}(r)\}$. The desired ordinary integral may be derived from the contour integral with the contour C_2, but we did not investigate this contour integral in Section 3.2.2.3. We therefore now follow the other way by starting directly from the straight line path integral solution mentioned in Section 3.2.2.3. If $s=-\gamma$ we may consider the solution†

$$\psi_5(r)=(2p)^{1+s}r^{-1-s}\int_{-\mathrm{i}p}^{+\mathrm{i}p}\exp(rt)(t+\mathrm{i}p)^{-1-s+\mathrm{i}y}(t-\mathrm{i}p)^{-1-s-\mathrm{i}y}(At+B)\,\mathrm{d}t.$$

With a new variable of integration $x=t/(\mathrm{i}p)$ this may be written

$$\psi_5(r)=2^{1+s}p^{-s}r^{-1-s}\mathrm{i}\exp(-\pi y)\int_{-1}^{+1}\exp(\mathrm{i}prx)$$
$$*(1+x)^{-1-s+\mathrm{i}y}(1-x)^{-1-s-\mathrm{i}y}(\mathrm{i}pAx+B)\,\mathrm{d}x. \quad (3.102)$$

† It is convenient to include a factor $(2\pi\mathrm{i})^{-1}$ in the case of a contour integral but to omit it in the case of a straight line path of integration.

The power series expansion of this solution might be obtained by expansion of $\exp(ipr x)$ in powers of r and subsequent term by term integration of the resulting series. Since we already know the expansion from eqn (3.100) it suffices to restrict our attention to the leading term which determines the normalization factor. We then have to evaluate the integral

$$I_5 = \int_{-1}^{+1} (1+x)^{-1-s+iy}(1-x)^{-1-s-iy}(ipAx+B)\,dx.$$

With a new variable of integration, $z=(1+x)/2$, this becomes

$$I_5 = 2^{-1-2s}\int_0^1 z^{-1-s+iy}(1-z)^{-1-s-iy}\{B-ipA+2ipAz\}\,dz.$$

The integrals may now be evaluated by means of the integral representation of the beta function $B(a, b)$, which then may be expressed in terms of gamma functions, according to

$$\int_0^1 z^{a-1}(1-z)^{b-1}\,dz = B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b). \qquad (3.103)$$

The result is

$$I_5 = 2^{-1-2s}[\{(B-ipA)\Gamma(-s+iy)\Gamma(-s-iy)/\Gamma(-2s)\} + 2ipA\{\Gamma(1-s+iy)\Gamma(-s-iy)/\Gamma(1-2s)\}],$$

or, after simplification,

$$I_5 = -2^{-2s}(Bs+A\alpha ZW)\Gamma(-s+iy)\Gamma(-s-iy)/\Gamma(1-2s).$$

Evaluating $Bs+A\alpha ZW$ we obtain

$$I_5 = -2^{-2s}(\kappa W+sm)\begin{Bmatrix}\alpha Z\\ \kappa+s\end{Bmatrix}\Gamma(-s+iy)\Gamma(-s-iy)/\Gamma(1-2s),$$

where $s=-\gamma$, and finally

$$\psi_5(r) \sim -2i(2p)^{\gamma}r^{-1+\gamma}\exp(-\pi y)(\kappa W-\gamma m) * \begin{Bmatrix}\alpha Z\\ \kappa-\gamma\end{Bmatrix}\Gamma(\gamma+iy)\Gamma(\gamma-iy)/\Gamma(1+2\gamma). \qquad (3.104)$$

From Section 3.2.2.1 we know, on the other hand, that

$$\begin{Bmatrix} f_{01}(r)\\ g_{01}(r)\end{Bmatrix} \sim \{(\kappa-\gamma)(\kappa W-\gamma m)\}^{1/2}\begin{Bmatrix}\alpha Z/(\kappa-\gamma)\\ 1\end{Bmatrix} r^{-1+\gamma}. \qquad (3.105)$$

It follows from eqns (3.102), (3.104) and (3.105), by comparison of the

different normalization factors that

$$\begin{Bmatrix} f_{01}(r) \\ g_{01}(r) \end{Bmatrix} = -\Gamma(1+2\gamma)\{\Gamma(\gamma+\mathrm{i}y)\Gamma(\gamma-\mathrm{i}y)\}^{-1}\{(\kappa-\gamma)(\kappa W-\gamma m)\}^{-1/2}$$
$$* 2^{-2\gamma} r^{-1+\gamma} \int_{-1}^{+1} \exp(\mathrm{i}prx)(1+x)^{-1+\gamma+\mathrm{i}y}(1-x)^{-1+\gamma-\mathrm{i}y}$$
$$* (\mathrm{i}pAx+B)\,\mathrm{d}x, \quad (3.106)$$

where

$$\mathrm{i}pAx+B = \begin{cases} \mathrm{i}p(\kappa-\gamma)x-\alpha ZT & \text{in the case of } f(r) \\ \mathrm{i}p\alpha Zx+(\kappa-\gamma)(T+2m) & \text{in the case of } g(r) \end{cases} \quad (3.107)$$

according to eqn (3.61) with $s=-\gamma$. Equation (3.106) is the desired integral representation of our standard solution $\{f_{01}(r), g_{01}(r)\}$. The 'regular' one of the Coulomb wave functions, eqn (3.81), then has the integral representation

$$\begin{Bmatrix} f^{\mathrm{C}}_{\mathrm{reg}}(r) \\ g^{\mathrm{C}}_{\mathrm{reg}}(r) \end{Bmatrix} = -\exp\{(\pi/2)y\}\{|\Gamma(\gamma+\mathrm{i}y)|\}^{-1}\{2W(\kappa-\gamma)(\kappa W-\gamma m)\}^{-1/2}$$
$$*(p/2)^{\gamma} r^{-1+\gamma} \int_{-1}^{+1} \exp(\mathrm{i}prx)(1+x)^{-1+\gamma+\mathrm{i}y}(1-x)^{-1+\gamma-\mathrm{i}y}$$
$$*(\mathrm{i}pAx+B)\,\mathrm{d}x \quad (3.108)$$

with $\mathrm{i}pAx+B$ from eqn (3.107) as before.

3.2.2.6. Power series solutions relative to an ordinary point

Although we have only considered until now solutions defined by their behaviour at a singular point, local solutions relative to an ordinary point are useful too and will be treated in this section. Let r_0, with $r_0 \neq 0, \infty$, denote any ordinary point. Local power series solutions relative to the point $r=r_0$ may then be written

$$\begin{Bmatrix} rf(r) \\ rg(r) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} (r-r_0)^n. \quad (3.109)$$

Here we have already used the fact that r_0 is an ordinary point with the consequence that the characteristic exponents are equal to zero. Also, since the solutions, when r is sufficiently large, behave like waves whose amplitude decreases as r^{-1}, it is advantageous to expand $\{rf, rg\}$ rather than $\{f, g\}$. The differential equations, with

$$z = r - r_0, \quad (3.110)$$

then read

$$(z+r_0)(rf)'-\kappa(rf)+\{T(z+r_0)+\alpha Z\}(rg)=0, \qquad (3.111a)$$

$$(z+r_0)(rg)'+\kappa(rg)-\{(T+2m)(z+r_0)+\alpha Z\}(rf)=0. \qquad (3.111b)$$

Inserting the power series, eqn (3.109), and rearranging the terms according to equal powers of z we have, with $a_{-2}=b_{-2}=a_{-1}=b_{-1}=0$,

$$\sum_{n=0}^{\infty} z^{n-1}\begin{Bmatrix} nr_0a_n+(n-1-\kappa)a_{n-1}+\{Tr_0+\alpha Z\}b_{n-1}+Tb_{n-2} \\ nr_0b_n+(n-1+\kappa)b_{n-1}-\{(T+2m)r_0+\alpha Z\}a_{n-1}-(T+2m)a_{n-2}\end{Bmatrix} = 0.$$

Since this is to hold identically in z, each term of the sum over n has to vanish separately. For the coefficients a_n, b_n we therefore obtain, with $n=0, 1, 2, \ldots$, the recursive system of linear equations

$$nr_0a_n=-(n-1-\kappa)a_{n-1}-\{Tr_0+\alpha Z\}b_{n-1}-Tb_{n-2},$$

$$nr_0b_n=-(n-1+\kappa)b_{n-1}+\{(T+2m)r_0+\alpha Z\}a_{n-1}+(T+2m)a_{n-2}.$$

For $n=0$ the right-hand side of these equations is zero, but the left-hand side is zero too, irrespective of the values of a_0 or b_0. Therefore, as expected for an ordinary point, $a_0=r_0f(r_0)$ and $b_0=r_0g(r_0)$ are the constants of integration, to which, independently of each other, arbitrary values may be assigned. The equations for $n\geq 1$, with $a_{-1}=b_{-1}=0$, then allow evaluation of the other coefficients recursively according to

$$a_n=[-(n-1-\kappa)a_{n-1}-\{Tr_0+\alpha Z\}b_{n-1}-Tb_{n-2}]/(nr_0), \qquad (3.112a)$$

$$b_n=[-(n-1+\kappa)b_{n-1}+\{(T+2m)r_0+\alpha Z\}a_{n-1}+(T+2m)a_{n-2}]/(nr_0). \qquad (3.112b)$$

Since there are no singular points other than $r=0$ or $r=\infty$, it is the singular point $r=0$ which limits the domain of convergence of the power series. The expansion (3.109) therefore converges for $|r-r_0|<r_0$.

The solution (3.109) we just have constructed contains two constants of integration and therefore is the general solution. A set of standard solutions relative to the ordinary point r_0 might be defined, for example by choosing $a_0=1$, $b_0=0$ and $a_0=0$, $b_0=1$, respectively, but we will not make use of this possibility.

3.2.2.7. A method of numerical integration or analytic continuation

The local power series solutions relative to an ordinary point form the basis of a very accurate algorithm for integrating the differential equations or, stated otherwise, for analytically continuing their solution. Let the solution $\{f(R_1), g(R_1)\}$ be given at a point R_1 and be wanted at a

point R_2 far away. We may then choose a finite sequence of points r_{0k} such that $r_{01} = R_1$, $r_{0n} = R_2$, and the r_{0k} for $k = 2, 3, \ldots, n-1$ are intermediate points between R_1 and R_2. Starting with $r_0 = r_{01}$, $a_0 = r_{01} f(r_{01})$, $b_0 = r_{01} g(r_{01})$ we may use the power series, eqn (3.109), in order to evaluate the solution $\{r_{02} f(r_{02}), r_{02} g(r_{02})\}$ at the intermediate point r_{02}, which then immediately becomes the starting point for the next step. More gnerally we use the power series (3.109) with $r_0 = r_{0k}$, $a_0 = r_{0k} f(r_{0k})$, $b_0 = r_{0k} g(r_{0k})$ in order to evaluate the solution $\{r_{0k+1} f(r_{0k+1}), r_{0k+1} g(r_{0k+1})\}$ at the point r_{0k+1}. By repeated use of this procedure for $k = 1, 2, \ldots, n-1$ we finally arrive at the desired point R_2.

The result $\{f(R_2), g(R_2)\}$ is independent of the choice of the intermediate points as long as sufficiently rapid convergence of the power series is guaranteed. This means that in each step the step length $h = r - r_0$ should not exceed half the convergence radius, i.e. should obey the condition

$$|h| \leqslant h_1 = r_0/2. \tag{3.113}$$

The remainder of the series truncated after a sufficiently large number of terms then is bounded in magnitude by the magnitude of the last term included and therefore can easily be controlled. An additional restriction on the step length comes from the requirement that possible loss of significant digits by cancellation of large terms should be avoided. The step length therefore has to be chosen small enough so that the low-order terms of the series do not increase too much. Appropriate conditions of this type, found by inspection of the recurrence relations (3.112) or (3.119), are

$$|h| \leqslant h_2 = r_0/k \tag{3.114}$$

and

$$|h| \leqslant h_3 = (\pi/2)\{T(T+2m)\}^{-1/2} = (\pi/2)/p. \tag{3.115}$$

Condition (3.115), which becomes effective when r_0 is large, means that the step length should not seriously exceed a quarter of the 'asymptotic wave length' $2\pi/p$ with which, approximately, the solution oscillates.

Rather than work with the three separate conditions, eqns (3.113) to (3.115), we may combine them to obtain one condition

$$h \leqslant h_4 \tag{3.116a}$$

where

$$h_4 = h_1 h_2 h_3/(h_1 h_2 + h_2 h_3 + h_3 h_1). \tag{3.116b}$$

In this way the transition between situations where different criteria are important becomes smoother. The advantage is that when ever any two of the criteria, eqns (3.113) to (3.115), are nearly equally important, the step length, which then would tend to become too large, is reduced.

Finally, it is desirable to allow for the possibility of an easy change of the step length within reasonable limits. For this purpose we may introduce positive parameters l_1, l_2, l_3 of the order of unity and compute the step length from

$$h = H_1H_2H_3/(H_1H_2+H_2H_3+H_3H_1) \tag{3.117a}$$

where

$$H_j = l_jh_j. \tag{3.117b}$$

Here the computational parameters l_j (with $j = 1, 2, 3$) serve for changing the step length strategy. While l_1 should not exceed 1, larger values up to 2 for l_2 or l_3 still give satisfactory performance.

For realizing the algorithm it is advantageous to work directly with the products of the coefficients and the appropriate powers of the step length

$$A_n = a_nh^n, \tag{3.118a}$$

$$B_n = b_nh^n, \tag{3.118b}$$

which obey the recurrence relations

$$A_n = [-(n-1-\kappa)(h/r_0)A_{n-1} - \{Th+\alpha Z(h/r_0)\}B_{n-1} - Th(h/r_0)B_{n-2}]/n, \tag{3.119a}$$

$$B_n = [-(n-1+\kappa)(h/r_0)B_{n-1} + \{(T+2m)h+\alpha Z(h/r_0)\}A_{n-1} + (T+2m)h(h/r_0)A_{n-2}]/n, \tag{3.119b}$$

following easily from eqn (3.112). We then simply have

$$\begin{Bmatrix} rf(r) \\ rg(r) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} A_n \\ B_n \end{Bmatrix}. \tag{3.120}$$

The index N at which the series are terminated numerically is determinated for each step individually so that

$$|A_N| < eps \left| \sum_{n=0}^{N} A_n \right| \tag{3.121a}$$

and

$$|B_N| < eps \left| \sum_{n=0}^{N} B_n \right| \tag{3.121b}$$

where *eps* is related to the relative machine precision of the computer, such that $eps = 5\times10^{-11}$ for instance if the computer uses a mantissa of ten decimal places. With this choice of *eps* the inclusion of further terms with $n > N$ and, because of condition (3.113), even the inclusion of the complete tail of the series would not change the computed sum, which therefore is exact apart from round-off errors.

As to the round-off errors it should be mentioned that their influence, although fairly small, can be reduced further if, after N has been

determined, the required sums are recomputed by addition in reverse order according to† $n = N(-1)0$ rather than $n = 0(1)N$.

The advantage of the method just described, compared to the usual numerical integration methods, is that the solution of the original differential equation is computed rather than the solution of a related difference equation obtained by discretization of the derivatives. The computed solution therefore is independent of the chosen step lengths within wide limits. This fact offers a sensitive and convenient possibility for checking the algorithm as well as the underlying equations.

Despite our rather extended knowledge about the Coulomb wave functions, the method is useful in the Coulomb case chosen here for demonstration. For while there is no problem evaluating the Coulomb wave functions for sufficiently small or sufficiently large values of r, an intermediate region remains which otherwise would be numerically inaccessible.‡

Moreover, the method is advantageous in the case of other, more complicated potentials, as far as the Taylor expansion coefficients of the potential relative to an arbitrary expansion point can be computed easily and fast enough. Depending on the particular potential under consideration, it may then be necessary to include additional criteria in the step length control if the potential has singular points in the finite part of the complex plane or if its Taylor expansion coefficients can become comparatively large.

3.2.2.8. The limit of vanishing coupling constant

The methods used so far for the Coulomb potential apply, with considerable simplification, also to the case of the identically vanishing potential. In this way we might find the connection between Bessel and Hankel functions and the various power series expansions of these functions, results which in Section 3.2.1 were assumed to be known. On the other hand we might take the results for the Coulomb potential and consider the limit of $\alpha Z \to 0$. This procedure yields the expected results in the case of the solutions defined relative to the singular point $r = \infty$ or to any ordinary point of the differential equations. For the solutions defined relative to the singular point $r = 0$, however, the behaviour of the limit might appear unexpected at first sight: while the 'regular' Coulomb wave function components essentially tend to (spherical) Bessel functions as expected, the irregular ones tend to the same Bessel functions. This

† This is the shorthand notation for $n = N$ step -1 until 0.

‡ It should be mentioned that if the irregular Coulomb wave function is to be computed, integration in the direction of increasing r can become numerically unstable. In this case one should start with the asymptotic expansion at infinity and integrate in the opposite direction. Another, although less safe, possibility has been considered by Bühring (1965*b*).

strange behaviour is related to the fact that the difference of the characteristic exponents relative to $r=0$, which is (provided that Z is an integer) always different from an integer for the Coulomb case, becomes equal to an integer when $\alpha Z=0$.

In order to discuss this matter in more detail we observe that, when $\alpha Z \to 0$, we have

$$\gamma = k + O\{(\alpha Z)^2\},$$
$$y = O(\alpha Z),$$

so that $\gamma - k$ vanishes faster than y. We therefore may obtain the limit $\alpha Z \to 0$ of our various formulas by taking first $\gamma = k$ and later $y = 0$. Our standard solution $\{f_{02}(r), g_{02}(r)\}$ may then be seen to become undefined or proportional to the other standard solution $\{f_{01}(r), g_{01}(r)\}$, but with an infinite constant of proportionality. As a consequence, the Wronskian (3.24) becomes undefined as signalized by the factor Sign(αZ). The irregular Coulomb wave function $\{\bar{f}^C(r), \bar{g}^C(r)\}$ differs from $\{f_{02}(r), g_{02}(r)\}$ by a normalization factor only, which because

$$|\Gamma(-\gamma + iy)|/\Gamma(1-2\gamma) = O(\alpha Z)$$

vanishes, and therefore becomes equal to the limit of the 'regular' solution. That the 'regular' and the irregular Coulomb wave functions become linearly depenent in the limit $\alpha Z \to 0$ may be seen also from the Wronskian eqn (3.86) which then vanishes. For, when $\alpha Z \to \pm 0$, we have†

$$\arg\Gamma(\gamma + iy) \to \arg\Gamma(k + iy) \to \arg\Gamma(k) = 0,$$
$$\arg\Gamma(-\gamma + iy) \to \arg\Gamma(-k + iy) \to \arg\Gamma(-k \pm i0) = k\pi \pm (\pi/2) \bmod 2\pi.$$

It follows from eqn (3.93) that

$$\eta - \bar{\eta} \to \pm\pi/2.$$

Collecting the results we may see from eqn (3.91) that

$$\delta - \bar{\delta} \to 0 \bmod 2\pi$$

or

$$\sin(\delta - \bar{\delta}) \to 0$$

when $\alpha Z \to 0$. The same result may be obtained from eqn (3.90), which yields

$$\sin(\delta - \bar{\delta}) \to -(-1)^k \sin(\pi\gamma)/\sinh(\pi y) \to \sin\{\pi(k-\gamma)\}/\sinh(\pi y)$$
$$= O\{(\alpha Z)^2\}/O(\alpha Z) = O(\alpha Z) \to 0.$$

† The notation $a \to b$ here is an abbreviation for $\lim_{\alpha Z \to 0} a = \lim_{\alpha Z \to 0} b$.

3.2.3. Potential of a uniform nuclear charge distribution

We here consider the electron radial wave functions for the potential

$$V(r)=\begin{cases}-\{\alpha Z/(2R)\}\{3-(r/R)^2\} & \text{if} \quad 0\leqslant r\leqslant R\\ -\alpha Z/r & \text{if} \quad R\leqslant r\leqslant\infty\end{cases} \tag{3.122}$$

corresponding to a uniformly charged spherical nucleus of radius R. For $r\geqslant R$ this is the Coulomb potential so that the solution outside the nucleus can be represented by a suitable linear combination of any two linearly independent Coulomb solutions presented in Section 3.2.2.

3.2.3.1. Local solutions relative to the point $r=0$

Inside the nucleus the potential is simply a polynomial in r, and $r=0$ is a regular singular point of the differential equations (3.1). We therefore consider power series solutions

$$\begin{Bmatrix} f^{\mathrm{I}}(r)\\ g^{\mathrm{I}}(r)\end{Bmatrix}=N\sum_{n=0}^{\infty}\begin{Bmatrix} a_n\\ b_n\end{Bmatrix}r^{\nu+n}, \tag{3.123}$$

where N is an arbitrary normalization factor and the coefficients a_n, b_n and the exponent ν have to be determined from the differential equations. Inserting the power series into the differential equations, collecting the terms with equal powers of r, and agreeing that

$$a_{-1}=b_{-1}=a_{-2}=b_{-2}=a_{-3}=b_{-3}=0,$$

we have

$$\sum_{n=0}^{\infty}\begin{Bmatrix}(\nu+n-\kappa+1)a_n+\{T+3\alpha Z/(2R)\}b_{n-1}-\{\alpha Z/(2R^3)\}b_{n-3}\\ (\nu+n+\kappa+1)b_n-\{T+2m+3\alpha Z/(2R)\}a_{n-1}+\{\alpha Z/(2R^3)\}a_{n-3}\end{Bmatrix}r^{\nu+n}=\begin{Bmatrix}0\\0\end{Bmatrix},$$

and since this has to hold identically in r we obtain, for $n=0, 1, 2, \ldots$, the recursive system of linear equations

$$(\nu+n-\kappa+1)a_n=-\{T+3\alpha Z/(2R)\}b_{n-1}+\{\alpha Z/(2R^3)\}b_{n-3}, \tag{3.124a}$$

$$(\nu+n+\kappa+1)b_n=\{T+2m+3\alpha Z/(2R)\}a_{n-1}-\{\alpha Z/(2R^3)\}a_{n-3}. \tag{3.124b}$$

For $n=0$ the equations reduce to

$$(\nu-\kappa+1)a_0=0,$$
$$(\nu+\kappa+1)b_0=0.$$

Assuming without loss of generality that of the initial coefficients a_0, b_0 at

least one is different from zero, compatibility requires that either $\nu = \kappa - 1$, $b_0 = 0$ if $a_0 \neq 0$ or $\nu = -\kappa - 1$, $a_0 = 0$ if $b_0 \neq 0$, so that we have two possible values for the exponent ν as expected. We notice, however, that since κ is an integer, the difference of the exponents is an integer, so that only the larger exponent safely gives a solution of the assumed type, eqn (3.123), while the second solution might be expected to have, but in fact does not have, a more complicated structure containing terms logarithmic in r. We will not investigate this second solution, which is not interesting from a physical point of view, further. We may choose the initial coefficient which is different from zero equal to 1. With $k = |\kappa|$ we then have

$$\nu = k - 1, \tag{3.125}$$

$$a_0 = 1, b_0 = 0 \quad \text{if} \quad \kappa = k, \tag{3.126a}$$

$$a_0 = 0, b_0 = 1 \quad \text{if} \quad \kappa = -k. \tag{3.126b}$$

For $n \geqslant 1$ the factors of a_n, b_n are now different from zero so that we may solve eqn (3.124) for a_n, b_n, obtaining the result

$$a_n = [-\{T + 3\alpha Z/(2R)\}b_{n-1} + \{\alpha Z/(2R^3)\}b_{n-3}]/(n + k - \kappa), \tag{3.127a}$$

$$b_n = [\{T + 2m + 3\alpha Z/(2R)\}a_{n-1} - \{\alpha Z/(2R^3)\}a_{n-3}]/(n + k + \kappa), \tag{3.127b}$$

where $a_{-2} = b_{-2} = a_{-1} = b_{-1} = 0$.

For later convenience we may dispose of the normalization factor N by choosing

$$N = \{(2k-1)!!\}^{-1} p^{k-1}. \tag{3.128}$$

This completes the construction of the solution for the region $0 \leqslant r \leqslant R$ inside the nucleus,

$$\begin{Bmatrix} f^{\mathrm{I}}(r) \\ g^{\mathrm{I}}(r) \end{Bmatrix} = \{(2k-1)!!\}^{-1}(pr)^{k-1} \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^n, \tag{3.129}$$

with the coefficients a_n, b_n from eqns (3.126) and (3.127).

3.2.3.2. *The global solution*

The global solution we want to construct is the solution $\{f(r), g(r)\}$ which is continuous at $r = R$, is proportional to our standard solution (3.129) inside the nucleus, and is properly normalized at infinity. We may globally characterize this solution by two real constants† $\alpha_\kappa > 0$ and Δ

† The sign of α_κ is undetermined by the physical problem and irrelevant since the observables do not depend on it. To make α_κ well-defined we may choose $\alpha_\kappa > 0$ by convention.

according to

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \alpha_\kappa \begin{Bmatrix} f^{\mathrm{I}}(r) \\ g^{\mathrm{I}}(r) \end{Bmatrix} \quad \text{for} \quad 0 \leqslant r \leqslant R, \tag{3.130}$$

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} \sim r^{-1} * \begin{Bmatrix} -(T/W)^{1/2} \sin[pr + (\alpha ZW/p)\ln(2pr) - \pi\{l(\kappa)+1\}/2 + \Delta] \\ \{(T+2m)/W\}^{1/2} \cos[pr + (\alpha ZW/p)\ln(2pr) - \pi\{l(\kappa)+1\}/2 + \Delta] \end{Bmatrix} \quad \text{when} \quad r \to \infty, \tag{3.131}$$

and it is in fact the amplitude α_κ and the scattering phase shift Δ which we want to evaluate.† To do so is not as trivial as it might seem at first sight. We therefore are going to describe two methods; one simple but time consuming on the computer, the other fast but complicated.

3.2.3.2.1. The simple but slow method. There are two reasons why we want to describe this method: the first is that it is very useful for checking the other method; the second is that it is in principle suitable for the case of more complicated potentials, such as will be needed in later sections.

The evaluation of the scattering phase shift Δ becomes most simple if we use the Coulomb solutions (3.44) defined relative to $r = \infty$ as a basis for representing our solution for $r \geqslant R$, outside the nucleus, where we then have

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \alpha_\kappa A \begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} + \alpha_\kappa B \begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}. \tag{3.132}$$

By looking at the asymptotic behaviour, eqns (3.83) and (3.84), of these Coulomb solutions and comparing with eqn (3.131) we may immediately see that simply

$$\cos(\Delta) = \alpha_\kappa A, \tag{3.133a}$$

$$\sin(\Delta) = \alpha_\kappa B. \tag{3.133b}$$

The constants A, B, α_κ have to be determined by continuously matching the different representations, eqns (3.130) and (3.132), of the solution at $r = R$. For this purpose we have to evaluate the local solutions at $r = R$. This is no problem for $\{f^{\mathrm{I}}(R), g^{\mathrm{I}}(R)\}$, but the series representations, eqn (3.44), of the Coulomb solutions on the right-hand side of eqn (3.132) are semiconvergent asymptotic expansions rather than convergent series and therefore are capable of reasonable accuracy only for values of r much

† It should be noted that here Δ is the scattering phase shift for the potential, eqn (3.122), different from the Coulomb phase shift, eqn (3.82), which will be denoted by Δ^{C} if needed in this section.

larger than the small nuclear radius R, say for $r \geq R_2 \gg R$. As may be estimated from the recurrence relations (3.47) or (3.40) for the coefficients of the asymptotic expansions, a value of R_2 is needed such that pR_2 is not smaller than about 10, a more detailed choice depending on the accuracy required and to some extent also on the parameters of the differential equations.

To fill the gap we need for the middle region $R \leq r \leq R_2$ the special Coulomb solution $\{f^M(r), g^M(r)\}$ which obeys the initial condition

$$\begin{Bmatrix} f^M(R) \\ g^M(R) \end{Bmatrix} = \begin{Bmatrix} f^I(R) \\ g^I(R) \end{Bmatrix}. \tag{3.134}$$

A method to construct such a solution, which may be looked at as consisting of a sequence of local solutions relative to ordinary points of the differential equation, has been described in Sections 3.2.2.4 and 3.2.2.5. Since the middle region is large and the sequence correspondingly long, the construction of this solution is comparatively time consuming.

After we have evaluated $\{f^M(R_2), g^M(R_2)\}$ in this way we may do the matching at R_2 according to

$$\begin{Bmatrix} f^M(R_2) \\ g^M(R_2) \end{Bmatrix} = A \begin{Bmatrix} f_{\infty 3}(R_2) \\ g_{\infty 3}(R_2) \end{Bmatrix} + B \begin{Bmatrix} f_{\infty 4}(R_2) \\ g_{\infty 4}(R_2) \end{Bmatrix}, \tag{3.135}$$

which yields

$$A = \{f^M(R_2) g_{\infty 4}(R_2) - g^M(R_2) f_{\infty 4}(R_2)\} / \{f_{\infty 3}(R_2) g_{\infty 4}(R_2) - g_{\infty 3}(R_2) f_{\infty 4}(R_2)\}, \tag{3.136a}$$

$$B = \{f_{\infty 3}(R_2) g^M(R_2) - g_{\infty 3}(R_2) f^M(R_2)\} / \{f_{\infty 3}(R_2) g_{\infty 4}(R_2) - g_{\infty 3}(R_2) f_{\infty 4}(R_2)\}, \tag{3.136b}$$

or, if the denominator is simplified via the Wronskian relation (3.47b),

$$A = (R_2^2 W/p)\{f^M(R_2) g_{\infty 4}(R_2) - g^M(R_2) f_{\infty 4}(R_2)\}, \tag{3.137a}$$

$$B = (R_2^2 W/p)\{f_{\infty 3}(R_2) g^M(R_2) - g_{\infty 3}(R_2) f^M(R_2)\}. \tag{3.137b}$$

Since

$$(\alpha_\kappa A)^2 + (\alpha_\kappa B)^2 = 1$$

follows from eqn (3.133), we may now evaluate

$$\alpha_\kappa = (A^2 + B^2)^{-1/2} \tag{3.138}$$

and then may also evaluate Δ from eqn (3.133), thereby finishing the task.

3.2.3.2.2. The fast but complicated method. As a basis for representing the solution outside the nucleus the fast method uses the Coulomb

solutions (3.23) defined relative to $r = 0$, that is, apart from the normalization, the 'regular' and irregular Coulomb wave functions, which may accurately be evaluated at $r = R$. For $r \geqslant R$ we then have

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \alpha_\kappa E \begin{Bmatrix} f_{01}(r) \\ g_{01}(r) \end{Bmatrix} + \alpha_\kappa H \begin{Bmatrix} f_{02}(r) \\ g_{02}(r) \end{Bmatrix}. \tag{3.139}$$

Continuity at $r = R$ determines the coefficients

$$E = D_1/D, \tag{3.140a}$$

$$H = D_2/D, \tag{3.140b}$$

where

$$D_1 = f^1(R)g_{02}(R) - g^1(R)f_{02}(R), \tag{3.141}$$

$$D_2 = f_{01}(R)g^1(R) - g_{01}(R)f^1(R), \tag{3.142}$$

$$D = f_{01}(R)g_{02}(R) - g_{01}(R)f_{02}(R), \tag{3.143}$$

or, using the Wronskian relation (3.24),

$$D = \mathrm{Sign}(\alpha Z)2\gamma\{(kp)^2 + (\alpha Z)^2 m^2\}^{1/2} R^{-2}. \tag{3.144}$$

In order to exhibit the asymptotic behaviour of the solution (3.139) we may also write

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \alpha_\kappa A \begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} + \alpha_\kappa B \begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}, \tag{3.145}$$

where the coefficients A, B can be found if the basis solutions of eqh (3.139) are expressed in terms of the basis solutions of eqn (3.145) by means of eqn (3.81). The result is

$$\begin{Bmatrix} A \\ B \end{Bmatrix} = (2W)^{1/2} \exp\{-(\pi/2)y\} \Big[E\Gamma(1+2\gamma)\{|\Gamma(\gamma+iy)|\}^{-1}(2p)^{-\gamma} * \begin{Bmatrix} \cos(\Delta^C) \\ \sin(\Delta^C) \end{Bmatrix} + H\Gamma(1-2\gamma)\{|\Gamma(-\gamma+iy)|\}^{-1}(2p)^{+\gamma} \begin{Bmatrix} \cos(\bar{\Delta}^C) \\ \sin(\bar{\Delta}^C) \end{Bmatrix} \Big]. \tag{3.146}$$

Here Δ^C or $\bar{\Delta}^C$ denote the Coulomb phase shift of eqns (3.82) and (3.79) according to whether $s = \gamma$ or $s = -\gamma$, respectively. In the same way as in the other method we now may evaluate the desired amplitude α_κ and the scattering phase shift Δ by

$$\alpha_\kappa = (A^2 + B^2)^{-1/2}, \tag{3.147}$$

$$\cos(\Delta) = \alpha_\kappa A, \tag{3.148a}$$

$$\sin(\Delta) = \alpha_\kappa B. \tag{3.148b}$$

The eqn (3.146) may be cast into a form which is more suitable for numerical evaluation as well as for discussion in another context. Con-

cerning the gamma functions all that is needed is $\ln\Gamma(1+2\gamma)$ and $\ln\Gamma(\gamma+\mathrm{i}y)$, since the gamma functions of other variables may be eliminated by means of the reflection formula

$$\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z) \tag{3.149}$$

and the difference equation

$$\Gamma(z+1)=z\Gamma(z). \tag{3.150}$$

Other relations which are useful in this context are

$$|\Gamma(z)|=\exp(\mathrm{Re}\ln\Gamma(z)), \tag{3.151}$$

$$\arg\Gamma(z)=\mathrm{Im}\ln\Gamma(z), \tag{3.152}$$

$$\arg\Gamma(z^*)=-\arg\Gamma(z), \tag{3.153}$$

where Re or Im means the real or imaginary part, respectively, and z^* is to denote the conjugate complex of z. The Coulomb phases may be split,

$$\Delta^{\mathrm{C}}=\chi+\eta, \tag{3.154a}$$

$$\bar{\Delta}^{\mathrm{C}}=\bar{\chi}+\bar{\eta}, \tag{3.154b}$$

where

$$\chi=(\pi/2)\{l(\kappa)+1-\gamma\}-\mathrm{Im}\ln\Gamma(\gamma+\mathrm{i}y) \tag{3.155}$$

and

$$\exp(\mathrm{i}\eta)=\{-(\kappa-\gamma)p-\mathrm{i}\alpha ZT\}/\{2T(\kappa-\gamma)(\kappa W-\gamma m)\}^{1/2} \tag{3.156}$$

according to eqns (3.72), (3.79) and (3.82) with $s=\gamma$. The quantities $\bar{\chi}$ and $\bar{\eta}$ are the corresponding quantities with γ replaced by $-\gamma$. By means of eqns (3.149) and (3.150), $\bar{\chi}$ may be cast into the form

$$\bar{\chi}=(\pi/2)\{l(\kappa)+1+\gamma\}-\mathrm{Im}\ln\Gamma(\gamma+\mathrm{i}y)+\arg[\sin\{\pi(-\gamma+\mathrm{i}y)\}]-\arg(\gamma+\mathrm{i}y). \tag{3.157}$$

We then have

$$\begin{Bmatrix}\cos(\Delta^{\mathrm{C}})\\ \sin(\Delta^{\mathrm{C}})\end{Bmatrix}=\{2T(\kappa-\gamma)(\kappa W-\gamma m)\}^{-1/2}\begin{Bmatrix}-(\kappa-\gamma)p\cos(\chi)+\alpha ZT\sin(\chi)\\ -\alpha ZT\cos(\chi)-(\kappa-\gamma)p\sin(\chi)\end{Bmatrix}, \tag{3.158}$$

$$\begin{Bmatrix}\cos(\bar{\Delta}^{\mathrm{C}})\\ \sin(\bar{\Delta}^{\mathrm{C}})\end{Bmatrix}=\{2T(\kappa+\gamma)(\kappa W+\gamma m)\}^{-1/2} * \begin{Bmatrix}-(\kappa+\gamma)p\cos(\bar{\chi})+\alpha ZT\sin(\bar{\chi})\\ -\alpha ZT\cos(\bar{\chi})-(\kappa+\gamma)p\sin(\bar{\chi})\end{Bmatrix}. \tag{3.159}$$

Furthermore we may write, in place of eqn (3.146),

$$\begin{Bmatrix}A\\ B\end{Bmatrix}=N\left[E\begin{Bmatrix}\cos(\Delta^{\mathrm{C}})\\ \sin(\Delta^{\mathrm{C}})\end{Bmatrix}+UH\begin{Bmatrix}\cos(\bar{\Delta}^{\mathrm{C}})\\ \sin(\bar{\Delta}^{\mathrm{C}})\end{Bmatrix}\right] \tag{3.160}$$

with

$$N = (2W)^{1/2}(2p)^{-\gamma}\exp\{-(\pi/2)y + \ln\Gamma(1+2\gamma) - \operatorname{Re}\ln\Gamma(\gamma+iy)\}, \tag{3.161}$$

$$U = 2\gamma(2p)^{2\gamma}(\gamma^2+y^2)^{1/2}\{\sin(2\pi\gamma)\}^{-1} * [\{\sin(\pi\gamma)\}^2 + \{\sinh(\pi y)\}^2]^{1/2}\exp\{2\operatorname{Re}\ln\Gamma(\gamma+iy) - 2\ln\Gamma(1+2\gamma)\}. \tag{3.162}$$

The eqns (3.140) to (3.144), (3.147), (3.148), (3.155), and (3.157) to (3.162) may now be used for writing a computer program which evaluates the desired amplitude α_κ and the scattering phase shift Δ. The method just described is in principle the method† of Bhalla and Rose (1960, 1962), but differs in some important details.

For later use we may write the relevant quantities, by applying eqns (3.147), (3.148), and (3.160), even more explicitly in the form

$$\alpha_\kappa = N^{-1}|E^{-1}|\{1 + 2U(H/E)\cos(\Delta^C - \bar{\Delta}^C) + U^2(H/E)^2\}^{-1/2}, \tag{3.163}$$

with

$$N^{-1} = (2W)^{-1/2}(2p)^{\gamma}\exp\{(\pi/2)y\}\,|\Gamma(\gamma+iy)|\,\{\Gamma(1+2\gamma)\}^{-1}, \tag{3.164a}$$

$$E^{-1} = \{f_{01}(R)g_{02}(R) - g_{01}(R)f_{02}(R)\}/\{f^I(R)g_{02}(R) - g^I(R)f_{02}(R)\}, \tag{3.164b}$$

$$\cos(\Delta) = \operatorname{Sign}(E)\cos(\Delta^C)\{1 + U(H/E)\cos(\bar{\Delta}^C)/\cos(\Delta^C)\} * \{1 + 2U(H/E)\cos(\Delta^C - \bar{\Delta}^C) + U^2(H/E)^2\}^{-1/2}, \tag{3.165}$$

$$\sin(\Delta) = \operatorname{Sign}(E)\sin(\Delta^C)\{1 + U(H/E)\sin(\bar{\Delta}^C)/\sin(\Delta^C)\} * \{1 + 2U(H/E)\cos(\Delta^C - \bar{\Delta}^C) + U^2(H/E)^2\}^{-1/2}, \tag{3.166}$$

where

$$H/E = -\{f^I(R)g_{01}(R) - g^I(R)f_{01}(R)\}/\{f^I(R)g_{02}(R) - g^I(R)f_{02}(R)\} \tag{3.167}$$

according to eqns (3.140) to (3.142).

3.2.4. General potential of a screened extended nuclear charge

3.2.4.1. The most general type of potential considered

As to the potential $V(r)$ it is reasonable for our purpose to assume that $V(r)$ and its first order derivative $V'(r)$ are bounded continuous functions of r on the closed interval $[0, \infty]$. If the potential is due to a charge

† Earlier work of this type has been done by Sliv and Volchok (1956).

distribution (or may thought of as being generated from a 'charge distribution') with radial charge density† $\rho(r)$ according to

$$V(r) = (1/r)\int_0^r \rho(x)x^2\,dx + \int_r^\infty \rho(x)x\,dx \tag{3.168}$$

or

$$\Delta V(r) = -\rho(r) \tag{3.169}$$

where

$$\Delta V(r) = (1/r)\{rV(r)\}'' = V''(r) + (2/r)V'(r), \tag{3.170}$$

our assumption means that $r\rho(r)$ is bounded on $[0,\infty]$ but not necessarily continuous. Our assumptions until now are therefore not stronger than necessary to avoid point charges. As to the representation of $rV(r)$ we now make the additional assumption that it is piecewise analytic and in each of a finite number of subintervals given as an analytic function from which the Taylor series expansions may be obtained in the form

$$V(r) = \sum_{m=0}^{\infty} v_m(0)r^m \tag{3.171}$$

at the origin,

$$V(r) = \sum_{m=1}^{\infty} v_m(\infty)r^{-m} \tag{3.172}$$

at infinity, and

$$rV(r) = \sum_{m=0}^{\infty} \bar{v}_m(r_0)(r-r_0)^m \tag{3.173}$$

at any point r_0 different from 0 and ∞. In the same way as for the simpler potentials in the previous sections we may now obtain local solutions of the differential eqns (3.1).

3.2.4.2. Power series solution relative to $r=0$

The appropriate local power series solution relative to $r=0$ is

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \{(2k-1)!!\}^{-1}(pr)^{k-1}\sum_{n=0}^{\infty}\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^n \tag{3.174}$$

where

$$a_0 = 1, \qquad b_0 = 0, \quad \text{if} \quad \kappa = k, \tag{3.175a}$$

$$a_0 = 0, \qquad b_0 = 1 \quad \text{if} \quad \kappa = -k, \tag{3.175b}$$

$$a_n = \left\{-(T-v_0)b_{n-1} + \sum_{m=1}^{n-1} v_m b_{n-m-1}\right\} \Big/ (n+k-\kappa), \tag{3.175c}$$

$$b_n = \left\{(T+2m-v_0)a_{n-1} - \sum_{m=1}^{n-1} v_m a_{n-m-1}\right\} \Big/ (n+k+\kappa). \tag{3.175d}$$

† Strictly speaking in the sense of electrodynamics our $V(r)$ and $\rho(r)$ are the potential and the radial charge density, respectively, multiplied by the elementary charge of the electron including the sign.

Here the symbol v_m has been used for the potential coefficients $v_m(0)$ of eqn (3.171). These formulas may be obtained in the same way as the corresponding formulas of Section 3.2.3.1, to which they reduce in the special case where $v=-\frac{3}{2}\alpha Z/R$, $v_2=\frac{1}{2}\alpha Z/R^3$, $v_m=0$ if $m\neq 0, 2$.

3.2.4.3. Power series solutions relative to an ordinary point

The general local power series solution realtive to an ordinary point $r_0\neq 0, \infty$ is

$$\begin{Bmatrix} rf(r) \\ rg(r) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} (r-r_0)^n \tag{3.176}$$

where

$$a_0 = r_0 f(r_0), \tag{3.177a}$$

$$b_0 = r_0 g(r_0), \tag{3.177b}$$

and

$$a_n = \left[-(n-1-\kappa)a_{n-1} - \{Tr_0 - v_0\}b_{n-1} - (T-v_1)b_{n-2} + \sum_{m=2}^{n-1} v_m b_{n-m-1} \right] \Big/ (nr_0) \tag{3.177c}$$

$$b_n = \left[-(n-1+\kappa)b_{n-1} + \{(T+2m)r_0 - v_0\}a_{n-1} + (T+2m-v_1)a_{n-2} - \sum_{m=2}^{n-1} v_m a_{n-m-1} \right] \Big/ (nr_0). \tag{3.177d}$$

Here the symbol v_m has been used for the potential coefficients $\tilde{v}_m(r_0)$ of eqn (3.173). The expansion (3.176) normally converges for $|r-r_0|<r_0$. The convergence domain might be smaller, however, if $V(r)$ has singular points in the complex plane. If one of these singular points, say r_1, of $V(r)$ has a smaller distance from r_0 than the origin has, the convergence domain is given by $|r-r_0|<|r_1-r_0|<r_0$. The eqns (3.177) may be obtained in the same way as the corresponding ones of Section 3.2.2.6, to which they reduce in the special case where $v_0=-\alpha Z$, $v_m=0$ if $m\neq 0$.

3.2.4.4. Power series solutions relative to $r=\infty$

It is not the general form, eqn (3.172), of the potential in a neighbourhood of infinity which is needed for nuclear beta-decay, but the simple Coulomb potential

$$V(r) = v_1(\infty) r^{-1}$$

which suffices. The corresponding solutions therefore are the standard Coulomb solutions (3.41) and (3.44) with $-\alpha Z$ replaced by $v_1(\infty)$. They will be needed in particular with $v_1(\infty)=-\alpha$ corresponding to $Z=1$ if screening is allowed for or with its original value of Z if a model without screening is considered.

In another context, however, potentials with a long-range tail other than the Coulomb tail may become important. Although treatment of the general case, eqn (3.172), is possible, we here want to restrict our attention to the simple case

$$V(r) = v_4(\infty) r^{-4} \tag{3.178}$$

which may be of interest for electron scattering at atomic physics energies. By the usual method, as explained in Section 3.2.2.2, we may find the asymptotic expansion

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} \simeq r^{-1} \exp(t_0 r) \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^{-n}, \tag{3.179}$$

where

$$a_0 = t_0, \tag{3.180a}$$

$$b_0 = T + 2m, \tag{3.180b}$$

$$a_n = \{(n-\kappa)(n-1+\kappa)a_{n-1} - T v_4 a_{n-3} - t_0 v_4 b_{n-3} + (n-\kappa) v_4 b_{n-4}\}/(2n t_0), \tag{3.180c}$$

$$b_n = \{(n+\kappa)(n-1-\kappa)b_{n-1} - (T+2m) v_4 b_{n-3} + t_0 v_4 a_{n-3} - (n-\kappa) v_4 a_{n-4}\}/(2n t_0) \tag{3.180d}$$

with v_4 written simply for $v_4(\infty)$ of eqn (3.178). The two possible values of t_0, which yield two linearly independent solutions, are $t_0 = ip$ or $t_0 = -ip$. By taking suitable linear combinations in analogy to eqns (3.43) and (3.44) we may obtain a fundamental set of two standard solutions which are real (when r is).

3.2.4.5. *A specific model of a potential for a screened extended nuclear charge*

In order to give a concrete example of a potential for which the results of the preceding subsections can be applied, we are going to construct a potential which is suitable for nuclear beta-decay, yet relatively simple. It is based on the observation that the potential of a neutral atom, which is essentially a screened Coulomb potential, can with reasonable accuracy be represented by an expression of the form

$$V(r) = -(\alpha Z/r) \sum_{n=1}^{N} a_n \exp(-\beta_n r) \tag{3.181}$$

where

$$\sum_{n=1}^{N} a_n = 1, \tag{3.182}$$

if Z is to remain the nuclear charge number, but otherwise a_n and also β_n are free parameters to be adjusted for each atom under consideration. Although with $N = 3$ potentials suitable for many applications have been

obtained (Rozental 1936; Ruark 1940; Moliere 1947; Byatt 1956), eqn (3.181) is flexible enough to simulate an even more refined structure† if N is increased and some of the parameters a_n are allowed to be negative (Bonham and Strand 1963; Tietz 1965; Strand and Tietz 1966). By means of eqns (3.169) and (3.170) the radial charge density corresponding to the potential, eqn (3.181), may be seen to be

$$\rho(r) = (\alpha Z/r) \sum_{n=1}^{N} a_n \beta_n^2 \exp(-\beta_n r). \tag{3.183}$$

This is only the electronic part of the charge distribution to which the singular point charge of the nucleus has to be added. We here need an extended nuclear charge, however, for which we adopt the uniform model with constant charge density $-3\alpha Z/R^3$ for $0 \leqslant r \leqslant R$. In order to retain the simple parabolic behaviour of the potential inside the nucleus we use the electronic charge density, eqn (3.183), for $r \geqslant R$ only. Furthermore, it is advantageous from a computational point of view to use an electronic charge distribution which is strictly zero outside some 'atomic radius' R_2. We therefore use eqn (3.183) for the interval $R \leqslant r \leqslant R_3$ with some $R_3 < R_2$ only. The remaining charge is distributed in the interval $R_3 \leqslant r \leqslant R_2$, where the sum over exponentials is replaced simply by a polynomial of the second degree. The charge distribution then is

$$\rho(r) = \begin{cases} -3\alpha Z/R^3 & \text{for } 0 \leqslant r \leqslant R \\ (\alpha \tilde{Z}/r) \sum_{n=1}^{N} a_n \beta_n^2 \exp(-\beta_n r) & \text{for } R \leqslant r \leqslant R_3 \\ (\alpha \tilde{Z}/r)(R_3 - R_2)^{-2}[2\sigma + 6\mu\{(r-R_2)/(R_3-R_2)\} & \\ \quad + 12\nu\{(r-R_2)/(R_3-R_2)\}^2] & \text{for } R_3 \leqslant r \leqslant R_2 \\ 0 & \text{for } R_2 \leqslant r \leqslant \infty \end{cases}$$

Here the constants

$$\sigma = \tfrac{1}{2}D_2 - 3D_1 + 6D_0, \tag{3.185a}$$

$$\mu = -D_2 + 5D_1 - 8D_0, \tag{3.185b}$$

$$\nu = \tfrac{1}{2}D_2 - 2D_1 + 3D_0, \tag{3.185c}$$

† Another method of refinement is to use an expression

$$V(r) = -(\alpha Z/r)\left\{\sum_{n=1}^{N} a_n \exp(-\beta_n r) + \sum_{n=1}^{M} b_n r \exp(-\tilde{\beta}_n r)\right\}$$

with suitable parameters $a_n, b_n, \beta_n, \tilde{\beta}_n$, again with $\sum_{n=1}^{N} a_n = 1$ while the b_n are not restricted in such a way. This model with $N = 3, M = 2$ has been introduced by Strand and Bonham (1964).

$$D_0 = \sum_n a_n \exp(-\beta_n R_3) + \sum_n a_n\{1-(1+\beta_n R)\exp(-\beta_n R)\}, \tag{3.186a}$$

$$D_1 = -(R_3 - R_2)\sum_n a_n\beta_n \exp(-\beta_n R_3), \tag{3.186b}$$

$$D_2 = (R_3 - R_2)^2 \sum_n a_n\beta_n^2 \exp(-\beta_n R_3) \tag{3.186c}$$

have been determined in such a way that the total electronic charge is correctly equal to $\tilde{Z}$, that the potential has the desired form, eqn (3.181) (apart from the very small modification due to the finite extension of the nuclear charge) in the interval $R \leqslant r \leqslant R_3$, and that $\rho(r)$ is continuous at $r = R_3$. The electronic charge number $\tilde{Z}$ may be different from the nuclear charge number Z, for in our model we need the daughter nucleus with the electronic cloud of the parent atom. The radii R_2 and R_3, reasonably of the order of one or a few Bohr radii, can be chosen so as to satisfy another desirable condition, namely that the charge density should not change its sign in the interval $R_3 \leqslant r \leqslant R_2$. In the case of one exponential term only corresponding to $N = 1$, this condition can be evaluated explicitly and then is found to be satisfied if $1{\cdot}4 < \beta_1(R_2 - R_3) < 4$. With

$$v_0(0) = -\tfrac{3}{2}(\alpha Z/R) + \alpha\tilde{Z}\sum_n a_n\beta_n \exp(-\beta_n R), \tag{3.187a}$$

$$v_2(0) = \tfrac{1}{2}\alpha Z/R^3 \tag{3.187b}$$

the potential may, by means of eqn (3.168), be found to be

$$V(r) = \begin{cases} v_0(0) + v_2(0)r^2 & \text{for } 0 \leqslant r \leqslant R \\ -\{\alpha(Z-\tilde{Z})/r\} - (\alpha\tilde{Z}/r)\left[\sum_n a_n \exp(-\beta_n r)\right. & \\ \quad \left. + \sum_n a_n\{1-(1+\beta_n R)\exp(-\beta_n R)\}\right] & \text{for } R \leqslant r \leqslant R_3 \\ -\{\alpha(Z-\tilde{Z})/r\} - (\alpha\tilde{Z}/r)[\sigma\{(r-R_2)/(R_3-R_2)\}^2 & \\ \quad + \mu\{(r-R_2)/(R_3-R_2)\}^3 + \nu\{(r-R_2)/ & \\ \quad (R_3-R_2)\}^4] & \text{for } R_3 \leqslant r \leqslant R_2 \\ -\alpha(Z-\tilde{Z})/r & \text{for } R_2 \leqslant r \leqslant \infty. \end{cases} \tag{3.188}$$

Since we have for each of the exponential terms

$$\exp(-\beta r) = \exp(-\beta r_0)\sum_{m=0}^{\infty}(1/m!)(-\beta)^m(r-r_0)^m,$$

it is straightforward to obtain the required expansion coefficients of this potential.

There is a reason that we have constructed this potential so as to retain the simple strict parabolic behaviour inside the nucleus. At the time this potential had been used for computing beta-decay Coulomb functions (Bühring 1965; Behrens and Jänecke 1969), a more complicated behaviour, in particular with odd powers of r, would have modified the formulas for the beta-decay observables in an unpleasant way as compared to the case without screening. In the light of more recent developments (Behrens and Bühring 1971), on which this book is based, however, such a restriction is no longer necessary. It then may look more reasonable to use the following potential

$$V(r)=\begin{cases} -\frac{3}{2}(\alpha Z/R)+\frac{1}{2}(\alpha Z/R)(r/R^2) & \\ \quad +(\alpha\tilde{Z}/r)\sum_{n=1}^{N} a_n\{1-\exp(-\beta_n r)\} & \text{for } 0\leqslant r\leqslant R \\ -\{\alpha(Z-\tilde{Z})/r\}-(\alpha\tilde{Z}/r)\sum_{n=1}^{N} a_n \exp(-\beta_n r) & \text{for } R\leqslant r\leqslant R_3 \\ -\{\alpha(Z-\tilde{Z})/r\}-(\alpha\tilde{Z}/r)[\sigma\{(r-R_2)/(R_3-R_2)\}^2 & \\ \quad +\mu\{(r-R_2)/(R_3-R_2)\}^3+\nu\{(r-R_2)/ & \\ \quad (R_3-R_2)\}^4] & \text{for } R_3\leqslant r\leqslant R_2 \\ -\alpha(Z-\tilde{Z})/r & \text{for } R_2\leqslant r\leqslant\infty \end{cases} \tag{3.189}$$

with

$$D_0=\sum_n a_n \exp(-\beta_n R_3) \tag{3.190}$$

redefined. Numerically, however, the difference between the two potentials is very small and unimportant, since we have $\beta_n R\ll 1$ for reasonable values of β_n and R.

3.2.4.6. Specific models for an extended nuclear charge without screening

If it is the charge distribution for which a model is given, rather than the potential itself, the required potential coefficients may conveniently be obtained directly from the coefficients $D_n(r_0)$ of the expansion of the charge density, which we may write as†

$$\rho(r)=\sum_{n=0}^{\infty} D_n(r_0)(r-r_0)^n \quad \text{if} \quad r_0\neq\infty. \tag{3.191}$$

† We here assume that $\rho(r)$ remains bounded at $r=0$, in addition to the assumptions of Section 3.2.4.1.

The potential coefficients, defined by eqns (3.171), (3.172), or (3.173), respectively, may then be found by means of eqn (3.168) to be

$$v_0(0) = \int_0^\infty \rho(x) x \, dx, \tag{3.192a}$$

$$v_1(0) = 0, \tag{3.192b}$$

$$v_{n+2}(0) = -D_n(0)/\{(n+2)(n+3)\} \tag{3.192c}$$

in the case of $r_0 = 0$ or

$$\tilde{v}_0(r_0) = \int_0^{r_0} \rho(x) x^2 \, dx + r_0 \int_{r_0}^\infty \rho(x) x \, dx = \int_0^{r_0} \rho(x) x^2 \, dx - r_0 \int_0^{r_0} \rho(x) x \, dx + r_0 v_0(0), \tag{3.193a}$$

$$\tilde{v}_1(r_0) = \int_{r_0}^\infty \rho(x) x \, dx = -\int_0^{r_0} \rho(x) x \, dx + v_0(0), \tag{3.193b}$$

$$\tilde{v}_2(r_0) = -\tfrac{1}{2} r_0 \rho(r_0) = -\tfrac{1}{2} r_0 D_0(r_0), \tag{3.193c}$$

$$\tilde{v}_{n+2}(r_0) = -\{r_0 D_n(r_0) + D_{n-1}(r_0)\}/\{(n+1)(n+2)\} \tag{3.193d}$$

in the case of $r_0 \neq 0$. Furthermore, if an $R_1 < \infty$ exists such that $\rho(r) \equiv 0$ for $r > R_1$, we have

$$v_0(\infty) = 0, \tag{3.194a}$$

$$v_1(\infty) = \int_0^\infty \rho(x) x^2 \, dx, \tag{3.194b}$$

$$v_n(\infty) = 0 \quad \text{for} \quad n = 2, 3, 4, \ldots. \tag{3.194c}$$

The required integrals over the charge distribution, if not elementary, may be computed by means of the continuation method, in a similar way as the solution of the differential equations. The contribution to the integral from the subinterval $[u, w]$ where $u \leq r_0 \leq w$ may be found to be

$$\int_u^w \rho(x) x^2 \, dx = \sum_{n=0}^\infty D_n(r_0)[\{(w-r_0)^{n+3} - (u-r_0)^{n+3}\}/(n+3) + 2r_0\{(w-r_0)^{n+2} - (u-r_0)^{n+2}\}/(n+2) + r_0^2\{(w-r_0)^{n+1} - (u-r_0)^{n+1}\}/(n+1)], \tag{3.195}$$

or

$$\int_u^w \rho(x) x \, dx = \sum_{n=0}^\infty D_n(r_0)[\{(w-r_0)^{n+2} - (u-r_0)^{n+2}\}/(n+2) + r_0\{(w-r_0)^{n+1} - (u-r_0)^{n+1}\}/(n+1)], \tag{3.196}$$

respectively. These expressions simplify if u, w, r_0 are such that either

$w - r_0 = r_0 - u$ or $u = r_0$ or $w = r_0$. The first choice may be advantageous for the definite integrals which have to be computed before the continuation of the solution of the differential equations can start. The second choice is appropriate for the integrals depending on r_0, which have to be accumulated during the continuation process of the solution. The same is true for the third choice if the continuation process is to be performed in the direction of decreasing r, as is required in bound state problems, to be considered later in Section 3.3.5.

We are now prepared to consider two more refined models of an extended nuclear charge (without screening), which have been used to investigate the sensitivity to the detailed shape of the nuclear charge distribution for nuclear beta-decay (Behrens and Bühring 1970).

3.2.4.6.1. Gaussian distribution. The (truncated) shell-model or modified Gaussian distribution is

$$\rho(r) = \begin{cases} N\{1 + A(r/a)^2\}\exp\{-(r/a)^2\} & \text{for} \quad 0 \leqslant r \leqslant R_1 \\ 0 & \text{for} \quad R_1 \leqslant r \leqslant \infty \end{cases} \tag{3.197}$$

with the normalization factor $N = N(R_1)$ such that

$$\int_0^\infty \rho(x) x^2 \, \mathrm{d}x = -\alpha Z. \tag{3.198}$$

In the case of $R_1 = \infty$ we have

$$N(\infty) = -8\alpha Z(2 + 3A)^{-1} a^{-3} \pi^{-1/2}. \tag{3.199}$$

This model contains two parameters a and A, besides the cut-off radius R_1, which, however, usually is chosen large enough so as to ensure that $\rho(R_1) \simeq 0$ to any prescribed accuracy.

The expansion coefficients of the charge density

$$D_{2n}(0) = (-1)^n N(1 - nA) a^{-2n} / \{(2n)!\}, \tag{3.200a}$$

$$D_{2n+1}(0) = 0 \tag{3.200b}$$

are straightforward to obtain in the case of $r_0 = 0$. For $r_0 \neq 0$ they may be computed recursively. It is not more complicated to consider here the more general case

$$\rho(r) = g(r)\exp\{f(r)\} \tag{3.201}$$

where $g(r)$ and $f(r)$ are sufficiently simple analytic functions like polynomials. We then are concerned with a sequence of functions $\{F_n(r)\}_{n=0,1,2,\ldots}$ defined by

$$(n!)^{-1}(\mathrm{d}^n/\mathrm{d}r^n)[g(r)\exp\{f(r)\}] \equiv F_n(r)\exp\{f(r)\}. \tag{3.202}$$

With

$$f^{(l)}(r) = \mathrm{d}^l f/\mathrm{d}r^l, \qquad f^{(0)}(r) = f(r),$$
$$g^{(l)}(r) = \mathrm{d}^l g/\mathrm{d}r^l, \qquad g^{(0)}(r) = g(r),$$
$$F_n^{(0)}(r) = F_n(r),$$
$$F_n^{(l)}(r) = \mathrm{d}^l F_n/\mathrm{d}r^l$$

we then obtain

$$F_0^{(l)} = g^{(l)}, \tag{3.203}$$
$$F_{n+1} = \{F_n^{(1)} + f^{(1)}F_n\}/(n+1), \tag{3.204}$$
$$F_{n+1}^{(l)} = \left\{F_n^{(l+1)} + \sum_{j=0}^{l}\binom{l}{j} f^{(1+j)}F_n^{(l-j)}\right\}\Big/(n+1), \tag{3.205}$$

and the required expansion coefficients of the charge density are

$$D_n(r_0) = F_n(r_0)\exp\{f(r_0)\}. \tag{3.206}$$

In the specific case here under consideration, the recurrence relations (3.203)–(3.205) have to be run with the initial values

$$f = -(r_0/a)^2,$$
$$f^{(1)} = -2r_0/a^2,$$
$$f^{(2)} = -2/a^2,$$
$$f^{(l)} = 0 \quad \text{for} \quad l = 3, 4, 5, \ldots,$$
$$g = N\{1 + A(r_0/a)^2\},$$
$$g^{(1)} = 2NAr_0/a^2,$$
$$g^{(2)} = 2NA/a^2,$$
$$g^{(l)} = 0 \quad \text{for} \quad l = 3, 4, 5, \ldots.$$

From a computational point of view, initial values $g^{(l)}\exp\{f(r_0)\}$ should be used in place of $g^{(l)}$ so that the recurrence relations directly generate the desired $D_n(r_0)$ in place of F_n.

3.2.4.6.2. Fermi distribution. The (truncated) Fermi distribution is

$$\rho(r) = \begin{cases} N[1+\exp\{\beta(r-c)\}]^{-1} & \text{for} \quad 0 \leqslant r \leqslant R_1 \\ 0 & \text{for} \quad R_1 \leqslant r \leqslant \infty \end{cases} \tag{3.207}$$

with the normalization factor $N = N(R_1)$ such that eqn (3.198) holds. This model contains two parameters c and β, besides the cut-off radius R_1. Again the expansion coefficients of the charge density may be obtained recursively. Introducing the sequence of functions $\{Z_m(r)\}_{m=0,1,2,\ldots}$ by

$$Z_m(r) = (-1)^m[\exp\{\beta(r-c)\}]^m/[1+\exp\{\beta(r-c)\}]^{m+1}, \tag{3.208}$$

we may find for their first order derivative

$$Z_m'(r) = (m+1)\beta Z_{m+1}(r) + m\beta Z_m(r). \tag{3.209}$$

It follows that the derivatives $\rho^{(n)}(r) = \mathrm{d}^n\rho/\mathrm{d}r^n$ of the charge density may be written in the form

$$\rho^{(n)}(r) = N\beta^n n! \sum_{m=0}^{n} C_m^{(n)} Z_m(r) \tag{3.210}$$

with certain constant coefficients $C_m^{(n)}$. Because $\rho^{(n+1)}(r) = \{\rho^{(n)}(r)\}'$ and eqns (3.209)–(3.210), the coefficients may be found to satisfy the recurrence relation

$$C_m^{(n+1)} = m(C_{m-1}^{(n)} + C_m^{(n)})/(n+1) \tag{3.211a}$$

with

$$C_0^{(0)} = 1, \qquad C_0^{(k)} = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots, \tag{3.211b}$$

$$C_n^{(n)} = 1, \qquad C_{n+k}^{(n)} = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots. \tag{3.211c}$$

Evaluating the coefficients $C_m^{(n)}$ from the recurrence relation (3.211) and also the functions $Z_m(r)$ for $r = r_0$ recursively from

$$Z_0 = [1 + \exp\{\beta(r_0 - c)\}]^{-1}, \tag{3.212a}$$

$$Z_m = -Z_{m-1}\exp\{\beta(r_0 - c)\}/[1 + \exp\{\beta(r_0 - c)\}], \tag{3.212b}$$

we may obtain the desired expansion coefficients $D_n(r_0)$ of the charge density by

$$D_n(r_0) = N\beta^n \sum_{m=0}^{n} C_m^{(n)} Z_m. \tag{3.213}$$

It should be noted, however, that numerical application of eqn (3.213) is not recommended if βr_0 is significantly larger than βc, since in this case strong cancellation of the terms in eqn (3.213) would occur for the larger values of n. We may easily avoid this by using eqn (3.213) for $\beta(r_0 - c) \leqslant 0$ only and taking advantage of the symmetry relation

$$D_n(c + x) = (-1)^{n+1} D_n(c - x), \tag{3.214}$$

valid for $n = 1, 2, 3, \ldots$, which follows from the fact that

$$\rho(r) - \tfrac{1}{2}N = \tfrac{1}{2}N[1 - \exp\{\beta(r - c)\}]/[1 + \exp\{\beta(r - c)\}]$$

is an odd function of $(r - c)$.

Finally, it should be noted that the Fermi charge distribution and the associated potential have poles in the complex plane. In particular those at $r_1 = c + \mathrm{i}(\pi/\beta)$ and $r_1^* = c - \mathrm{i}(\pi/\beta)$ determine the convergence domain

$$|r - r_0| < |r_1 - r_0| = \{(c - r_0)^2 + (\pi/\beta)^2\}^{1/2} \tag{3.215}$$

for the expansions of $\rho(r)$ or $V(r)$ around any point $r_0 \neq \infty$ on the real axis. This is relevant for a reasonable choice of the step lengths $w - r_0$ or $r_0 - u$, which should not exceed half the convergence radius, if the integrals are evaluated by means of eqn (3.195) or (3.196), respectively. For the same reason the choice of step lengths may be influenced in the case of the differential equations, as has already been mentioned in the context of eqn (3.176).

3.2.4.7. Computation of phase shifts and amplitudes

With refined potentials such like those just introduced in Sections 3.2.4.5 and 3.2.4.6, the computation of the amplitudes (normalization factors of the electron radial wave functions) and scattering phase shifts may essentially proceed along the same lines as described in detail for the simple potential of the uniform nuclear charge distribution in Section 3.2.3.2.1. The local solution relative to the origin, eqn (3.174), is continued by repeated use of the appropriate local solutions relative to an ordinary point, eqn (3.176), until the point R_2 has been reached where the amplitude and the phase shift may be determined by comparison with the standard solutions relative to infinity.

If the potential to be used is even more complicated, such that its expansion coefficients cannot be computed easily and fast enough, the required outwards integration of the differential equations may be performed by means of a standard numerical integration method in place of our continuation technique. The same comment applies to the case when the potential is given numerically on a mesh of a finite number of discrete points only. It may be advantageous, however, to apply the numerical integration method to suitably transformed, non-linear equations (Lin *et al.* 1963) rather than to the original differential equations (3.1).

Another possibility for such more complicated cases is to consider the potential on a mesh of a finite number of discrete points and to approximate it in each subinterval, formed by two adjacent mesh points, by an interpolation polynomial of relatively low degree, such that at the mesh points the required continuity conditions are satisfied. If third degree polynomials, for instance, are used for approximating $rV(r)$ on the subintervals, it is possible to achieve continuity even for $\{rV(r)\}'' = r\rho(r)$. With such a spline representation of the potential, application of the continuation method is again advantageous, since in each subinterval only the first four potential coefficients are different from zero. What is computed then is the solution of the differential equations, but with a slightly modified potential. By means of one of the usual numerical integration methods, on the other hand, the solution with the exact potential would be computed, but for certain difference equations in place of the differential equations.

3.3. Bound state solutions

3.3.1. Definition of bound state solutions

Bound state solutions might occur when $|W|<m$ or $-2m<T<0$ so that p becomes imaginary. We then may write†

$$p=\mathrm{i}\lambda \tag{3.216}$$

where

$$\lambda=\{(2m+T)(-T)\}^{1/2}=\{(m+W)(m-W)\}^{1/2}>0. \tag{3.217}$$

Consequently the two local solutions relative to the point $r=\infty$ now become exponentially decaying or increasing, respectively, rather than being oscillatory for r sufficiently large, while their behaviour near $r=0$ is not significantly changed. Bound state solutions then are defined by some integrability condition which selects one of the standard solutions relative to $r=0$ and requires that it be exponentially decaying at infinity. Since generally both the exponentially decaying and increasing solutions contribute, the bound state condition can be met only for particular discrete values of W, the bound state energies $W_{n'}$. The usual integrability and normalization condition for the bound state solutions is

$$\int_0^\infty [\{f(r)\}^2+\{g(r)\}^2]r^2\,\mathrm{d}r=1. \tag{3.218}$$

Apart from a sign, the bound state solutions then have been defined uniquely.

3.3.2. A generalized Wronskian relation

For obtaining the appropriate normalization of the bound state solutions a certain generalized Wronskian relation is useful. We proceed in a similar way as in Section 3.1.1, but rather than consider two different solutions of the differential equations we consider one solution and its derivative with respect to T. Denoting this derivative by a dot,

$$\dot{f}(r)=\partial f(r)/\partial T, \qquad \dot{f}'(r)=\partial f'(r)/\partial T,$$

etc., we have

$$\begin{aligned}
rf'-(\kappa-1)f+(T-V)rg&=0,\\
rg'+(\kappa+1)g-(T+2m-V)rf&=0,\\
r\dot{f}'-(\kappa-1)\dot{f}+(T-V)r\dot{g}+rg&=0,\\
r\dot{g}'+(\kappa+1)\dot{g}-(T+2m-V)r\dot{f}-rf&=0.
\end{aligned}$$

† The choice of the sign in eqn (3.216) is quite arbitrary.

Multiplying the first of these equations by $-r\dot{g}$, the second by $r\dot{f}$, the third by rg, the fourth by $-rf$, and adding, we obtain

$$r^2(\dot{f}g-f\dot{g})'+2r(\dot{f}g-f\dot{g})+r^2(g^2+f^2)=0$$

or

$$\{r^2(\dot{f}g-f\dot{g})\}'+r^2(g^2+f^2)=0. \tag{3.219}$$

3.3.3. Bound state solutions for the Coulomb potential

Our starting point for the bound state problem is eqn (3.78), but with $j=1$ corresponding to $s=\gamma$ only because of the bound state condition, eqn (3.218), which would otherwise be violated at the origin. Since now p is imaginary according to eqn (3.216), the Coulomb parameter y becomes imaginary too, so that

$$\mathrm{i}y=\alpha ZW/\lambda \tag{3.220}$$

is real. For evaluating $\exp(\mathrm{i}\eta)$, which now becomes real too, we note that eqns (3.216) and (3.217) imply

$$T^{1/2}=\mathrm{i}(-T)^{1/2}. \tag{3.221}$$

From eqn (3.72) we then have

$$\exp(\mathrm{i}\eta)=\{-(\kappa-s)\lambda-\alpha ZT\}/\{2(-T)(\kappa-s)(\kappa W-sm)\}^{1/2},$$
$$\exp(-\mathrm{i}\eta)=\{-(\kappa-s)\lambda+\alpha ZT\}/\{2(-T)(\kappa-s)(\kappa W-sm)\}^{1/2}.$$

Since the product of these two expressions is equal to unity, we also have

$$\exp(\mathrm{i}\eta)=\{2(-T)(\kappa-s)(\kappa W-sm)\}^{1/2}/\{-(\kappa-s)\lambda+\alpha ZT\}, \tag{3.222}$$

$$\exp(-\mathrm{i}\eta)=\{2(-T)(\kappa-s)(\kappa W-sm)\}^{1/2}/\{-(\kappa-s)\lambda-\alpha ZT\}, \tag{3.223}$$

a representation which is more convenient for our purpose. With these modifications, eqn (3.78) multiplied by appropriate factors now reads

$$N\begin{Bmatrix}f(r)\\g(r)\end{Bmatrix}=(-TW)^{-1/2}\{(\kappa-\gamma)(\kappa W-\gamma m)\}^{-1/2}\{\Gamma(1+2\gamma)\}^{-1}\begin{Bmatrix}f_{01}(r)\\g_{01}(r)\end{Bmatrix}$$
$$=\exp\{\mathrm{i}\pi(-\gamma+\alpha ZW/\lambda)\}(2\lambda)^{-\gamma+\alpha ZW/\lambda}[\{-(\kappa-\gamma)\lambda+\alpha ZT\}\Gamma(\gamma+\alpha ZW/\lambda)]^{-1}$$
$$*\begin{Bmatrix}f_{\infty 1}(r)\\g_{\infty 1}(r)\end{Bmatrix}+(2\lambda)^{-\gamma-\alpha ZW/\lambda}[\{-(\kappa-\gamma)\lambda-\alpha ZT\}\Gamma(\gamma-\alpha ZW/\lambda)]^{-1}$$
$$*\begin{Bmatrix}f_{\infty 2}(r)\\g_{\infty 2}(r)\end{Bmatrix}. \tag{3.224}$$

For later use we have denoted this solution by $\{Nf(r), Ng(r)\}$, where $\{f(r), g(r)\}$ means the desired normalized solution and $N>0$ is the

normalization factor in question. The asymptotic behaviour,† as $r \to \infty$, of the standard solutions on the right-hand side of eqn (3.224) is, according to eqns (3.31) and (3.41), given by

$$\begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} \sim \{W(W+m)\}^{-1/2} \begin{Bmatrix} -\lambda \\ W+m \end{Bmatrix} r^{-1+\alpha ZW/\lambda} \exp(-\lambda r) \quad (3.225)$$

if $-3\pi/2 < \arg(r) < 3\pi/2$,

$$\begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix} \sim \{W(W+m)\}^{-1/2} \begin{Bmatrix} \lambda \\ W+m \end{Bmatrix} r^{-1-\alpha ZW/\lambda} \exp(\lambda r) \quad (3.226)$$

if $-5\pi/2 < \arg(r) < \pi/2$, while the left-hand side is well-defined, its asymptotic behaviour, as $r \to 0$, being

$$\sim (-TW)^{-1/2} \{\Gamma(1+2\gamma)\}^{-1} \begin{Bmatrix} \alpha Z/(\kappa-\gamma) \\ 1 \end{Bmatrix} r^{-1+\gamma} \quad (3.227)$$

according to eqns (3.20), (3.22), and (3.23).

3.3.3.1. Bound state energies

Because of the bound state condition, eqn (3.218), which otherwise could not be satisfied at infinity, bound states may occur only for particular values of W such that the coefficient of the exponentially increasing solution on the right-hand side of eqn (3.224) vanishes. This is the case, because of the poles of the gamma function if

$$\gamma - \alpha ZW/\lambda = -n' \quad (3.228)$$

where $n' = 0, 1, 2, \ldots$ is a positive integer or zero.‡ We are going to consider further this eigenvalue equation for the bound state energies assuming§ that $\alpha Z > 0$. With $\lambda^2 = m^2 - W^2$ according to eqn (3.217), eqn (3.228) yields

$$\{(\gamma + n')/(\alpha Z)\}^{-2} = (m/W)^2 - 1,$$

† The maximum sectors of validity of the asymptotic expansions here are different from those obtained in Section 3.2.2.4 for the continuum state solutions. The reason is that the two singular points lying on the imaginary axis of the t-plane in Fig. 3.2 now have been moved onto the real axis according to $ip \to -\lambda$, $-ip \to \lambda$. As a consequence the possible angles of rotation χ for the contour of the integral representation are now restricted in a different way according to $-\pi < \chi < \pi$ in the case of $\psi_3(r)$ or $0 < \chi < 2\pi$ in the case of $\psi_4(r)$, respectively.

‡ Following other authors, we denote this integer by n' rather than n since an additional quantum number $n = n' + k$ is commonly in use.

§ Since λ is always >0 by definition and the eigenvalue equation, eqn (3.228), contains the product αZW, eigen solutions do exist for $\alpha Z < 0$ too, but with energy eigenvalues $W < 0$. In the case of $n' = 0$ it then is for $\kappa > 0$ only but not for $\kappa < 0$ that an eigen solution occurs. Otherwise, the eigenvalues are, apart from the sign, the same as for $\alpha Z > 0$. This is as expected since a positron with negative energy W may be interpreted as an electron with positive energy.

where the sign of W, which has been lost by squaring, has to be positive according to eqn (3.228). We then immediately obtain the energy eigenvalues

$$W_{n'} = m[1+\{\alpha Z/(\gamma+n')\}^2]^{-1/2}, \tag{3.229a}$$

or

$$W_{n'} = m(\gamma+n')(k^2+2\gamma n'+n'^2)^{-1/2}, \tag{3.229b}$$

which are identified by the index $n' = 0, 1, 2, \ldots$ and are independent of the sign of κ. It remains, however, to look at the other factor,

$$\{-(\kappa-\gamma)\lambda - \alpha ZT\}, \tag{3.230}$$

which might have a zero compensating one of the poles of the gamma function. If $\kappa = -k$, this factor always remains positive, but it has a zero if $\kappa = +k$. The equation to be satisfied then becomes

$$\lambda/(m-W) = \alpha Z/(k-\gamma)$$

or, after squaring,

$$(m+W)/(m-W) = (k+\gamma)/(k-\gamma)$$

or finally

$$W = m\gamma/k. \tag{3.231}$$

This value of W is contained in the set $\{W_{n'}\}$ of eqn (3.229) and corresponds to the label $n' = 0$. The factor (3.230) therefore compensates one pole of the gamma function if $\kappa = +k$, so that there is no bound state solution† corresponding to $n' = 0$ in the case of $\kappa = +k$. The values of λ associated with the energy eigenvalues are found from eqns (3.228) and (3.229*b*) to be

$$\lambda_{n'} = \alpha Zm(k^2+2\gamma n'+n'^2)^{-1/2}, \tag{3.232a}$$

which reduces to

$$\lambda_0 = \alpha Zm/k \tag{3.232b}$$

in the case of $n' = 0$.

3.3.3.2. *Normalization of the bound state solutions*

It remains to find the appropriate normalization of the eigen functions. Integration from zero to infinity of eqn (3.219) yields, for an arbitrarily normalized bound state solution,

$$\{r^2(\dot{f}g - f\dot{g})\}\Big|_0^\infty = -\int_0^\infty (g^2+f^2)r^2\,\mathrm{d}r. \tag{3.233}$$

† This statement might seem to be in disagreement with the discussion by Rose (1961). But his eigen functions, if used with $n' = 0$ while $\kappa = +k$, vanish identically so that there is, in fact, no eigen solution.

The solution which is properly normalized according to eqn (3.218) then obeys the relation

$$\{r^2(\dot{f}g - f\dot{g})\}\Big|_0^\infty = -1 \tag{3.234}$$

or

$$\{r^2 g^2 \partial(f/g)/\partial T\}\Big|_0^\infty = -1 \tag{3.235}$$

or

$$\{r^2 f^2 \partial(g/f)/\partial T\}\Big|_0^\infty = 1. \tag{3.236}$$

We are going to evaluate the last of these equations using the solution (3.224).

As $r \to 0$ we have

$$g(r)/f(r) = g_{01}(r)/f_{01}(r) \sim \alpha Z/(\kappa + \gamma)$$

and therefore

$$[\partial\{g(r)/f(r)\}/\partial T]_{r=0} = 0.$$

Equation (3.236) then reduces to

$$\{r^2 f^2 \partial(g/f)/\partial T\}_{r=\infty} = 1. \tag{3.237}$$

To study the behaviour, when $r \to \infty$, of $g(r)/f(r)$ we first rewrite the right-hand side of eqn (3.224) in an abbreviated form as

$$N\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = E_1 \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + E_2 \begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}. \tag{3.238}$$

We then have to evaluate

$$\partial\{g(r)/f(r)\}/\partial T = \partial\{(E_1 g_{\infty 1} + E_2 g_{\infty 2})/(E_1 f_{\infty 1} + E_2 f_{\infty 2})\}/\partial T$$

at $r = \infty$. After differentiation with respect to T, which again is denoted by a dot, we may set T equal to an eigenvalue $T_{n'} = W_{n'} - m$. Because $E_2(T_{n'}) = 0$, only the terms not containing the factor E_2 need to be kept so that

$$\partial\{g(r)/f(r)\}/\partial T = \{E_1^2(f_{\infty 1}\dot{g}_{\infty 1} - \dot{f}_{\infty 1} g_{\infty 1}) + E_1 \dot{E}_2 (f_{\infty 1} g_{\infty 2} - g_{\infty 1} f_{\infty 2})\}\{E_1 f_{\infty 1}\}^{-2} \tag{3.239}$$

and

$$N\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = E_1 \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} \tag{3.240}$$

if $T = T_{n'}$.

Using the Wronskian relation

$$r^2\{f_{\infty 1}(r)g_{\infty 2}(r)-g_{\infty 1}(r)f_{\infty 2}(r)\}\equiv -2\lambda/W$$

we have from eqns (3.239) and (3.240)

$$r^2f^2\partial(g/f)/\partial T=N^{-2}\{E_1^2r^2(f_{\infty 1}\dot{g}_{\infty 1}-g_{\infty 1}\dot{f}_{\infty 1})-2(\lambda/W)E_1\dot{E}_2\}. \tag{3.241}$$

We may now take the limit $r\to\infty$ of eqn (3.241). Using eqn (3.237) we then obtain

$$N^2=-2(\lambda/W)E_1\dot{E}_2 \tag{3.242}$$

where all the quantities have to be evaluated for an eigenvalue. Factors of E_2 which do not vanish for an eigenvalue therefore need not be differentiated and may immediately be simplified by means of the eigenvalue eqn (3.228). Also E_1 may be simplified in this way. We therefore obtain

$$E_1=(-2\lambda)^{n'}[\{-(\kappa-\gamma)\lambda+\alpha ZT\}\Gamma(2\gamma+n')]^{-1}, \tag{3.243}$$

$$E_2=(2\lambda)^{-2\gamma-n'}[\{-(\kappa-\gamma)\lambda-\alpha ZT\}\Gamma(\gamma-\alpha ZW/\lambda)]^{-1}, \tag{3.244}$$

by comparison of eqn (3.238) with eqn (3.224) and subsequent use of eqn (3.228). In order to evaluate $\dot{E}_2$ we use

$$\mathrm{d}\{1/\Gamma(z)\}/\mathrm{d}z=-\psi(z)/\Gamma(z),$$

$$\lim_{z\to -n'}\{\psi(z)/\Gamma(z)\}=(-1)^{n'+1}n'! \quad \text{for} \quad n'=0,1,2,\ldots,$$

$$\mathrm{d}\lambda/\mathrm{d}T=-W/\lambda,$$

$$\mathrm{d}W/\mathrm{d}T=1,$$

$$\mathrm{d}(W/\lambda)/\mathrm{d}T=m^2/\lambda^3,$$

and obtain

$$\dot{E}_2=(2\lambda)^{-2\gamma-n'}\{-(\kappa-\gamma)\lambda-\alpha ZT\}^{-1}(-1)^{n'}(-\alpha Z)m^2\lambda^{-3}n'!. \tag{3.245}$$

We then have

$$\dot{E}_2/E_1=-\alpha Zm^2\lambda^{-3}(2\lambda)^{-2\gamma-2n'}\Gamma(2\gamma+n')n'!\,R \tag{3.426}$$

where we have introduced the ratio

$$R=\{-(\kappa-\gamma)\lambda+\alpha ZT\}/\{-(\kappa-\gamma)\lambda-\alpha ZT\}, \tag{3.247}$$

which may be rewritten† as

$$R=-(\kappa\lambda+\alpha Zm)/(\gamma\lambda-\alpha ZW)$$

† From eqns (3.222) and (3.223) we may write $R=\exp(-\mathrm{i}\eta)/\exp(\mathrm{i}\eta)$ or $R=\exp(-2\mathrm{i}\eta)$. A formula mentioned in the footnote concerning eqn (3.72) then yields $R=(-\kappa p-\mathrm{i}\alpha Zm)/(\gamma p-\mathrm{i}\alpha ZW)$. With $p=\mathrm{i}\lambda$ the required result appears.

or

$$R = (\gamma\lambda + \alpha ZW)/(\alpha Zm - \kappa\lambda) \tag{3.248}$$

by means of the identity

$$(\alpha Zm + \kappa\lambda)(\alpha Zm - \kappa\lambda) = -(\gamma\lambda + \alpha ZW)(\gamma\lambda - \alpha ZW). \tag{3.249}$$

Evaluated for an eigenvalue R becomes

$$R = (2\gamma + n')\lambda/(\alpha Zm - \kappa\lambda). \tag{3.250}$$

From eqns (3.242) and (3.246) we then obtain†

$$N/E_1 = (-1)^{n'}\,\mathrm{sign}(-\kappa)[(2/W)\alpha Zm^2\lambda^{-1}(2\lambda)^{-2\gamma-2n'}n'! \\ *\Gamma(1+2\gamma+n')/(\alpha Zm - \kappa\lambda)]^{1/2}. \tag{3.251}$$

We may also evaluate N from eqns (3.242), (3.243) and (3.245), which after simplification by means of the identity

$$\{-(\kappa-\gamma)\lambda + \alpha ZT\}\{-(\kappa-\gamma)\lambda - \alpha ZT\} = -2T(\kappa-\gamma)(\kappa W - \gamma m) \tag{3.252}$$

becomes

$$N = [\{\alpha Z/(-TW)\}(m/\lambda)^2(2\lambda)^{-2\gamma}\{n'!/\Gamma(2\gamma+n')\}/\{(\kappa-\gamma)(\kappa W - \gamma m)\}]^{1/2}. \tag{3.253}$$

3.3.3.3. Bound state solutions for $n' \neq 0$

Using eqn (3.253) and the left-hand part of eqn (3.224), which is equal to the leading term given by eqn (3.227) multiplied by the series S_1 of eqn (3.100) with $s = \gamma$, we obtain the normalized bound state solutions in the form

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \{(\kappa-\gamma)(\kappa W - \gamma m)\Gamma(2\gamma+n')\}^{1/2}(\alpha Zn'!)^{-1/2} \\ *\{\Gamma(1+2\gamma)\}^{-1}(\lambda/m)\begin{Bmatrix} \alpha Z/(\kappa-\gamma) \\ 1 \end{Bmatrix} r^{-1}(2\lambda r)^{\gamma} S_1 \tag{3.254}$$

† Since $N > 0$ by definition, the sign of N/E_1 is given by the sign of E_1. We have

$$-(\kappa-\gamma)\lambda + \alpha ZT = -\kappa\lambda + \gamma\lambda + \alpha ZW - \alpha Zm = -\kappa\lambda + 2\gamma\lambda + n'\lambda - \alpha Zm \\ = \lambda\{-\kappa + 2\gamma + n' - (k^2 + 2\gamma n' + n'^2)^{1/2}\}.$$

Here use has been made of eqns (3.228) and (3.232*a*). Because

$$\gamma + n' < (k^2 + 2\gamma n' + n'^2)^{1/2} < k + n'$$

we may conclude that the sign of the expression under consideration is given by $\mathrm{sign}(-\kappa)$. The sign of N/E_1 then is $(-1)^{n'}\,\mathrm{sign}(-\kappa)$.

where

$$S_1 = 1 + \sum_{n=1}^{\infty} \left[\sum_{j=n+1-n'}^{n} \frac{(n'-1)!\,\Gamma(1+2\gamma+n'+j)}{(n'-1-n+j)!\,(n-j)!\,j!\,\Gamma(1+2\gamma+n')} \right.$$
$$\left. -\lambda^{-1} \begin{Bmatrix} -\alpha Z T/(\kappa+\gamma) \\ (\kappa+\gamma)(T+2m)/(\alpha Z) \end{Bmatrix} \sum_{j=n-n'}^{n-1} \frac{(n'-1)!\,\Gamma(1+2\gamma+n'+j)}{(n'-n+j)!\,(n-1-j)!\,j!\,\Gamma(1+2\gamma+n')} \right]$$
$$* \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+n)} (-\lambda r)^n. \quad (3.255)$$

Here we had to evaluate, in the case of the eigenvalue condition (3.228), the ratio of gamma functions

$$\Gamma\{1+\gamma-\alpha Z(W/\lambda)+n-j\}/\Gamma\{1+\gamma-\alpha Z(W/\lambda)\}$$
$$\to (-1)^{n-j}(n'-1)!/(n'-1-n+j)!$$

either by taking the limit of this expression when the arguments of the gamma functions become equal to a negative integer or zero, or by using the reflection formula (3.88), for each of the gamma functions before applying the eigenvalue condition. Finally, the sum over j has explicitly been restricted to those terms which do not vanish. The presentation of the bound state solutions in the form of eqns (3.254) and (3.255) is useful in showing their behaviour near the origin.

Using eqns (3.240), (3.251), and (3.225) we may obtain the normalized bound state solutions in the form

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = (-1)^{n'} \operatorname{sign}(-\kappa)\{\lambda(\alpha Z m - \kappa\lambda)\}^{1/2}\{2\alpha Z(W+m)n'!\,\Gamma(1+2\gamma$$
$$+n')\}^{-1/2} m^{-1} \begin{Bmatrix} -\lambda \\ m+W \end{Bmatrix} r^{-1}(2\lambda r)^{\gamma+n'} \exp(-\lambda r) S_3. \quad (3.256)$$

Looking more closely at the asymptotic series, eqn (3.99), which now becomes

$$S_3 = 1 + \sum_{j=1}^{\infty} [1 \pm \{\lambda/(\alpha Z m - \lambda\kappa)\}j][\Gamma\{\gamma-\alpha Z(W/\lambda)+j\}$$
$$/\Gamma\{\gamma-\alpha Z(W/\lambda)\}][\Gamma\{-\gamma-\alpha Z(W/\lambda)+j\}/\Gamma\{-\gamma-\alpha Z(W/\lambda)\}](-2\lambda r)^{-j}/j!$$

we find that, because of the eigenvalue condition (3.228), the series terminates at $j = n'$ and therefore

$$S_3 = 1 + \sum_{j=1}^{n'} [1 \pm \{\lambda/(\alpha Z m - \lambda\kappa)\}j]\{\Gamma(-2\gamma-n'+j)$$
$$/\Gamma(-2\gamma-n')\}n'!\,\{(n'-j)!\,j!\}^{-1}(2\lambda r)^{-j} \quad (3.257)$$

is an exact representation which is valid everywhere. Here again a ratio of

gamma functions has been evaluated by means of

$$\lim_{z\to -n'} \{\Gamma(z+j)/\Gamma(z)\} = (-1)^j n'!/(n'-j)!.$$

The presentation of the bound state solutions in the form of eqns (3.256) and (3.257) is useful in showing their behaviour near infinity.

A third representation may be obtained if, after extraction of an appropriate power of r, the series S_3 is looked at as a polynomial in r rather than $1/r$. By backwards summation and use of the reflection formula for the gamma function we have

$$S_3 = [1 \pm \{n'\lambda/(\alpha Zm - \kappa\lambda)\}]\{\Gamma(1+2\gamma+n')/\Gamma(1+2\gamma)\}(-2\lambda r)^{-n'}\tilde{S}_3$$

where

$$\tilde{S}_3 = 1 + \sum_{j=1}^{n'} \left\{1 - \frac{\pm j\lambda}{(\alpha Zm - \kappa\lambda) \pm n'\lambda}\right\} \frac{\Gamma(1+2\gamma)n'!}{\Gamma(1+2\gamma+j)j!\,(n'-j)!}(-2\lambda r)^j. \quad (3.259)$$

From eqn (3.256) we then find that the bound state solutions may be represented in the form

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \text{sign}(-\kappa)\{\Gamma(1+2\gamma)\}^{-1}\{\lambda\Gamma(1+2\gamma+n')\}^{1/2}\{2\alpha Z(\alpha Zm - \kappa\lambda)$$

$$* (m+W)n'!\}^{-1/2} m^{-1} \begin{Bmatrix} -\lambda\{\alpha Zm - \kappa\lambda + n'\lambda\} \\ (m+W)\{\alpha Zm - \kappa\lambda - n'\lambda\} \end{Bmatrix} r^{-1}(2\lambda r)^{\gamma} \exp(-\lambda r)\tilde{S}_3. \quad (3.260)$$

This form is suitable for comparison with the results of Rose (1961), and agreement may be found,† apart from the irrelevant sign factor $\text{sign}(-\kappa)$ corresponding to a different convention.

On the other hand, our eqns (3.254) and (3.260), which represent the same functions, have to agree. Looking at the radial dependence we may, from the behaviour when $r \to 0$, immediately conclude that

$$S_1 = \tilde{S}_3 \exp(-\lambda r). \quad (3.261)$$

To verify this in full by expanding the exponential function requires a

† Rewriting factors such like

$$n'!/(n'-j)! = (-1)^j \Gamma(-n'+j)/\Gamma(-n'),$$

we may discover that, expressed in terms of confluent hypergeometric series

$${}_1F_1(a; c; z) = \sum_{j=0}^{\infty} \{\Gamma(a+j)/\Gamma(a)\}\{\Gamma(c)/\Gamma(c+j)\}z^j/j!,$$

the polynomial $\tilde{S}_3$ becomes

$$\tilde{S}_3 = (\alpha Zm - \kappa\lambda \pm n'\lambda)^{-1}\{(\alpha Zm - \kappa\lambda)_1F_1(-n'; 1+2\gamma; 2\lambda r) \pm n'\lambda_1F_1(1-n'; 1+2\gamma; 2\lambda r)\}.$$

little more effort. To show† that the apparently different normalization factors agree is somewhat inconvenient.

3.3.3.4. Bound state solutions for $n'=0$

In this case we have, from eqn (3.100) and the eigenvalue condition (3.228),

$$S_1 = 1 + \sum_{n=1}^{\infty} \left[\sum_{j=0}^{n} (-1)^j \frac{\Gamma(1+2\gamma+j)}{\Gamma(1+2\gamma)j!} + \lambda^{-1}B/A \sum_{j=0}^{n-1} (-1)^j * \frac{\Gamma(1+2\gamma+j)}{\Gamma(1+2\gamma)j!} \right] \frac{\Gamma(2\gamma+1)}{\Gamma(2\gamma+1+n)} (\lambda r)^n,$$

where $\lambda^{-1}B/A$, by means of eqns (3.231) and (3.232*b*), because $\kappa = -k$, reduces to

$$\lambda^{-1}B/A = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix}.$$

Of the sums over j therefore, while all the terms for $j=0$ until $n-1$ cancel, only the term with $j=n$ of the first sum survives, so that S_1 reduces to

$$S_1 = 1 + \sum_{n=1}^{\infty} (-\lambda r)^n / n! = \exp(-\lambda r).$$

From eqn (3.254) with $n'=0$ we then obtain after simplification by means of eqns (3.231) and (3.232*b*) the normalized bound state solution‡

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = [\{(k+\gamma)/k\}\lambda/\Gamma(1+2\gamma)]^{1/2} \begin{Bmatrix} -\alpha Z/(k+\gamma) \\ 1 \end{Bmatrix} r^{-1}(2\lambda r)^{\gamma} \exp(-\lambda r). \tag{3.262}$$

This result may also be obtained from eqn (3.256) where S_3, as given by eqn (3.257), reduces to $S_3 \equiv 1$, for $\alpha Zm - \lambda\kappa$, which now becomes equal to $2\alpha Zm$, remains different from zero.

† For this purpose it is helpful to use, according to eqns (3.247) and (3.250),

$$\lambda(2\gamma+n')/(\alpha Zm - \kappa\lambda) = \{-(\kappa-\gamma)\lambda + \alpha ZT\}/\{-(\kappa-\gamma)\lambda - \alpha ZT\}$$

in the square root factors of eqn (3.260) and to rewrite $n'\lambda$ by means of the eigenvalue equation (3.228). Furthermore, eqns (3.217) and (3.252) are needed. The sign of the expression $\alpha Zm - \kappa\lambda - n'\lambda$ may be shown to be sign$(-\kappa)$.

‡ In this simple case it would be easy to check the normalization condition, eqn (3.218), directly by performing the integral

$$\int_0^{\infty} (2\lambda r)^{2\gamma} \exp(-2\lambda r)\, dr = (2\lambda)^{-1}\Gamma(1+2\gamma).$$

3.3.4. Bound state solutions for the potential of a uniform nuclear charge distribution

A solution which satisfies the boundary condition at the origin is

$$N\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \begin{Bmatrix} f^{\mathrm{I}}(r) \\ g^{\mathrm{I}}(r) \end{Bmatrix} \quad \text{for} \quad 0 \leqslant r \leqslant R \tag{3.263a}$$

$$N\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = E\begin{Bmatrix} f_{01}(r) \\ g_{01}(r) \end{Bmatrix} + H\begin{Bmatrix} f_{02}(r) \\ g_{02}(r) \end{Bmatrix} \quad \text{for} \quad R \leqslant r \tag{3.263b}$$

Here $\{f(r), g(r)\}$ denotes the normalized solution we finally want to construct, and N is the normalization factor in question. Inside the nucleus the solution is given by the local solutions $\{f^{\mathrm{I}}(r), g^{\mathrm{I}}(r)\}$, introduced by eqn (3.129), corresponding to the potential of the uniformly extended nuclear charge distribution of radius R. Outside the nucleus we need an appropriate linear combination of standard Coulomb solutions like $\{f_{01}(r), g_{01}(r)\}$, $\{f_{02}(r), g_{02}(r)\}$. The constants E and H are determined by the requirement that the solution be continuous at $r = R$ and may be found to be

$$E = D_1/D, \tag{3.264}$$

$$H = D_2/D, \tag{3.265}$$

where

$$D_1 = f^{\mathrm{I}}(R)g_{02}(R) - g^{\mathrm{I}}(R)f_{02}(R), \tag{3.266}$$

$$D_2 = f_{01}(R)g^{\mathrm{I}}(R) - g_{01}(R)f^{\mathrm{I}}(R), \tag{3.267}$$

$$D = f_{01}(R)g_{02}(R) - g_{01}(R)f_{02}(R) \tag{3.268}$$

or

$$D = 2\gamma R^{-2}\{(\alpha Zm + \kappa\lambda)(\alpha Zm - \kappa\lambda)\}^{1/2} \tag{3.269}$$

by means of eqn (3.24). We now have to study the behaviour at infinity of the solution (3.263). From eqn (3.224) we have

$$\begin{aligned}\begin{Bmatrix} f_{0j}(r) \\ g_{0j}(r) \end{Bmatrix} &= (-TW)^{1/2}\{(\kappa - s)(\kappa W - sm)\}^{1/2}\Gamma(1+2s) \\ &* \Bigg[\exp\{i\pi(-s + \alpha ZW/\lambda)\}(2\lambda)^{-s+\alpha ZW/\lambda}\{-(\kappa - s)\lambda + \alpha ZT\}^{-1} \\ &* \{\Gamma(s + \alpha ZW/\lambda)\}^{-1}\begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + (2\lambda)^{-s-\alpha ZW/\lambda}\{-(\kappa - s)\lambda - \alpha ZT\}^{-1} \\ &* \{\Gamma(s - \alpha ZW/\lambda)\}^{-1}\begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}\Bigg],\end{aligned} \tag{3.270}$$

where we have written more generally j in place of 1 and s in place of γ. This equation is valid also for the other solution corresponding to $j = 2$ and $s = -\gamma$, which was not needed above in the pure Coulomb case.

Writing our solution, eqn (3.263*b*), in the form

$$N\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = A\begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + B\begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix} \tag{3.271}$$

we may find, by means of eqn (3.270), the coefficients

$$\begin{aligned} A = {} & (-TW)^{1/2}[E\{(\kappa-\gamma)(\kappa W-\gamma m)\}^{1/2}\Gamma(1+2\gamma)\exp\{i\pi(-\gamma+\alpha ZW/\lambda)\} \\ & * (2\lambda)^{-\gamma+\alpha ZW/\lambda}\{-(\kappa-\gamma)\lambda+\alpha ZT\}^{-1}\{\Gamma(\gamma+\alpha ZW/\lambda)\}^{-1} \\ & \qquad + H\{(\kappa+\gamma)(\kappa W+\gamma m)\}^{1/2} \\ & * \Gamma(1-2\gamma)\exp\{i\pi(\gamma+\alpha ZW/\lambda)\}(2\lambda)^{\gamma+\alpha ZW/\lambda}\{-(\kappa+\gamma)\lambda+\alpha ZT\}^{-1} \\ & * \{\Gamma(-\gamma+\alpha ZW/\lambda)\}^{-1}], \end{aligned} \tag{3.272}$$

$$\begin{aligned} B = {} & (-TW)^{1/2}[E\{(\kappa-\gamma)(\kappa W-\gamma m)\}^{1/2}\Gamma(1+2\gamma)(2\lambda)^{-\gamma-\alpha ZW/\lambda} \\ & * \{-(\kappa-\gamma)\lambda-\alpha ZT\}^{-1}\{\Gamma(\gamma-\alpha ZW/\lambda)\}^{-1}+H\{(\kappa+\gamma)(\kappa W+\gamma m)\}^{1/2} \\ & * \Gamma(1-2\gamma)(2\lambda)^{\gamma-\alpha ZW/\lambda}\{-(\kappa+\gamma)\lambda \\ & \qquad -\alpha ZT\}^{-1}\{\Gamma(-\gamma-\alpha ZW/\lambda)\}^{-1}]. \end{aligned} \tag{3.273}$$

The boundary condition at infinity requires that the coefficient B be zero. We therefore have to look for the zeros, with respect to W, of B. By means of the eigenvalue condition

$$B = 0 \tag{3.274}$$

the coefficient A may be simplified considerably. By somewhat lengthy manipulations† we obtain

$$\begin{aligned} A = {} & E(-TW)^{1/2}(2\lambda)^{-\gamma+\alpha ZW/\lambda}\{(\kappa-\gamma)(\kappa W-\gamma m)\}^{1/2}\Gamma(1+2\gamma) \\ & * \{-(\kappa-\gamma)+\alpha ZT\}^{-1}\{\Gamma(\gamma+\alpha ZW/\lambda)\}^{-1}\sin(2\pi\gamma)/\sin\{\pi(\gamma \\ & \qquad +\alpha ZW/\lambda)\}. \end{aligned} \tag{3.275}$$

This result shows that A becomes real for every eigenvalue, a fact which looks reasonable but was not at all evident from eqn (3.272).

It seems to be advantageous to further treat the eigenvalue condition in close analogy to the pure Coulomb case.‡ Guided by eqn (3.228) we

† Using the identities

$$\frac{\{-(\kappa+\gamma)\lambda-\alpha ZT\}\{-(\kappa-\gamma)\lambda+\alpha ZT\}}{\{-(\kappa+\gamma)\lambda+\alpha ZT\}\{-(\kappa-\gamma)\lambda-\alpha ZT\}} = \frac{\alpha ZW+\gamma\lambda}{\alpha ZW-\gamma\lambda},$$

$$\frac{\Gamma(-\gamma-\alpha ZW/\lambda)\Gamma(\gamma+\alpha ZW/\lambda)}{\Gamma(-\gamma+\alpha ZW/\lambda)\Gamma(\gamma-\alpha ZW/\lambda)}\frac{\alpha ZW+\gamma\lambda}{\alpha ZW-\gamma\lambda} = -\frac{\sin\{\pi(\gamma-\alpha ZW/\lambda)\}}{\sin\{\pi(\gamma+\alpha ZW/\lambda)\}},$$

$$\exp(-i\pi\gamma)+\exp(i\pi\gamma)\sin\{\pi(\gamma-\alpha ZW/\lambda)\}/\sin\{\pi(\gamma+\alpha ZW/\lambda)\} = \exp(-i\pi\alpha ZW/\lambda)\sin(2\pi\gamma)/\sin\{\pi(\gamma+\alpha ZW/\lambda)\}.$$

‡ At least as long as Z is small enough so that γ remains real.

therefore write

$$\gamma - \alpha ZW/\lambda = -Q. \tag{3.276}$$

Here Q is no longer an integer, but we may write

$$Q = n' + \varepsilon, \tag{3.277}$$

where n' is the integer which labels the different bound states as before and $\varepsilon = \varepsilon(n')$ becomes zero in the limit of a vanishing nuclear radius. As a consequence of eqn (3.276), several of the equations for the pure Coulomb case remain formally the same except for the replacement of n' by Q. Therefore, we now have

$$W_{n'} = m[1 + \{\alpha Z/(\gamma + Q)\}^2]^{-1/2} \tag{3.278}$$

or

$$W_{n'} = m(\gamma + Q)(k^2 + 2\gamma Q + Q^2)^{-1/2} \tag{3.279}$$

and

$$\lambda_{n'} = \alpha Zm(k^2 + 2\gamma Q + Q^2)^{-1/2}. \tag{3.280}$$

Another advantage of eqns (3.276) and (3.277) is that ε is expected to be confined to the interval $0 < \varepsilon < 1$, for since the electron is less strongly bound in the field of the extended nucleus, ε has to be positive. On the other hand, $\varepsilon = 1$ can never satisfy the eigenvalue condition, eqn (3.274), since then the first term of eqn (3.273) would vanish because of the pole of the gamma function corresponding to the next pure Coulomb bound state. However, the bound state energies depend continuously on the radius of the charge distribution, and since ε cannot reach the value 1, it cannot attain any larger value.

If some non-vanishing factors are omitted, the eigenvalue condition may now be written

$$\Gamma(1+2\gamma)\{-(\kappa-\gamma)\lambda - \alpha ZT\}^{-1}\{\Gamma(-Q)\}^{-1} + (D_2/D_1)[\{(\kappa+\gamma)(\kappa W + \gamma m)\}$$
$$/\{(\kappa-\gamma)(\kappa W - \gamma m)\}]^{1/2}(2\lambda)^{2\gamma}\Gamma(1-2\gamma)$$
$$* \{-(\kappa+\gamma)\lambda - \alpha ZT\}^{-1}\{\Gamma(-2\gamma - Q)\}^{-1} = 0 \tag{3.281}$$

or finally

$$\Gamma(1+2\gamma)\{-(\kappa-\gamma)\lambda - \alpha ZT\}^{-1}\Gamma(1+Q)\sin(\pi\varepsilon) + (D_2/D_1)$$
$$* [\{(\kappa+\gamma)(\kappa W + \gamma m)\}$$
$$/\{(\kappa-\gamma)(\kappa W - \gamma m)\}]^{1/2}(2\lambda)^{2\gamma}\{-(\kappa+\gamma)\lambda$$
$$- \alpha ZT\}^{-1} 2\pi\gamma\{\Gamma(1+2\gamma+Q)/\Gamma(1+2\gamma)\}$$
$$* \{\sin(2\pi\gamma + \pi\varepsilon)/\sin(2\pi\gamma)\} = 0. \tag{3.282}$$

Here all the quantities $D_2, D_1, \lambda, T, W, Q$ depend on ε via eqns (3.277), (3.279), and (3.280). The eigenvalue condition in the form of eqn (3.282) then, for every n', may be looked at as a transcendental equation for ε,

the root of which in the interval $(0, 1)$ may safely be found by any numerical iterative method.

Concerning the normalization of the bound state wave functions, we may proceed essentially in the same way as for the pure Coulomb case. In analogy to eqn (3.242) we obtain

$$N^2 = -2(\lambda/W)A\dot{B}, \tag{3.283}$$

evaluated for the eigenvalue under consideration. The normalized solution then is

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = N^{-1} \begin{Bmatrix} f^{\mathrm{I}}(r) \\ g^{\mathrm{I}}(r) \end{Bmatrix} \quad \text{for} \quad 0 \leqslant r \leqslant R, \tag{3.284a}$$

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = (E/N) \begin{Bmatrix} f_{01}(r) \\ g_{01}(r) \end{Bmatrix} + (H/N) \begin{Bmatrix} f_{02}(r) \\ g_{02}(r) \end{Bmatrix} \quad \text{for} \quad R \leqslant r, \tag{3.284b}$$

or

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = (A/N) \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} \quad \text{for} \quad R \leqslant r, \tag{3.284c}$$

where the last representation is appropriate to show the asymptotic behaviour when $r \to \infty$. It is because of the complexity of B that we here do not want to further evaluate $\dot{B}$ and N.

3.3.5. Bound state solutions for a general potential

3.3.5.1. Eigenvalue condition and normalization of the bound state solutions

Since we have to satisfy two boundary conditions, it is useful (Rose 1961) to introduce 'left' or L-solutions $\{f_{\mathrm{L}}(r), g_{\mathrm{L}}(r)\}$ which satisfy the boundary condition at $r = 0$ and 'right' or R-solutions $\{f_{\mathrm{R}}(r), g_{\mathrm{R}}(r)\}$ which satisfy the boundary condition at $r = \infty$. The normalization of these solutions is still arbitrary, but the ratios

$$\rho = f/g \tag{3.285}$$

are uniquely defined. The requirement that the two ratios for the L- and R- solution are to agree at any one† point r then gives the eigenvalue condition.

Assuming for the moment that the eigenvalues are already known, we want to consider the normalization problem. We may rewrite eqn (3.219) as

$$(r^2 g^2 \dot{\rho})' + r^2(g^2 + f^2) = 0. \tag{3.286}$$

† If they agree at one point r, then they agree for all r.

Integrating from 0 to r and taking an L-solution, which is bounded† at $r=0$, we have

$$r^2\{g_L(r)\}^2\dot{\rho}_L(r)+\int_0^r[\{g_L(x)\}^2+\{f_L(x)\}^2]x^2\,dx=0. \tag{3.287}$$

Integrating eqn (3.286) from r to ∞ and taking an R-solution, which vanishes exponentially at $r=\infty$, we have

$$-r^2\{g_R(r)\}^2\dot{\rho}_R(r)+\int_r^\infty[\{g_R(x)\}^2+\{f_R(x)\}^2]x^2\,dx=0. \tag{3.288}$$

We now further consider eqns (3.287) and (3.288) assuming that the eigenvalue condition is satisfied. The L-solution and R-solution then are proportional to each other and to the normalized eigen solution $\{f(r), g(r)\}$. We therefore may write

$$M\{f_R(r), g_R(r)\}=N\{f_L(r), g_L(r)\}=\{f(r), g(r)\}, \tag{3.289}$$

where M and N are the normalization constants in question. Multiplying eqn (3.287) by N^2 and eqn (3.288) by M^2 and adding we obtain

$$r^2\{g(r)\}^2\{\dot{\rho}_L(r)-\dot{\rho}_R(r)\}+\int_0^\infty[\{g(x)\}^2+\{f(x)\}^2]x^2\,dx=0 \tag{3.290}$$

or, because of the normalization condition (3.218)

$$r^2\{g(r)\}^2\{\dot{\rho}_R(r)-\dot{\rho}_L(r)\}=1. \tag{3.291}$$

From eqns (3.289) and (3.291), the normalization constants may be found to be

$$M=\{rg_R(r)\}^{-1}\{\dot{\rho}_R(r)-\dot{\rho}_L(r)\}^{-1/2}, \tag{3.292}$$

$$N=\{rg_L(r)\}^{-1}\{\dot{\rho}_R(r)-\dot{\rho}_L(r)\}^{-1/2}, \tag{3.293}$$

where

$$\dot{\rho}_R(r)-\dot{\rho}_L(r)=(\dot{f}_R/g_R)-(f_R/g_R)(\dot{g}_R/g_R)-(\dot{f}_L/g_L)+(f_L/g_L)(\dot{g}_L/g_L). \tag{3.294}$$

Here all the functions are needed for the eigenvalue under consideration. The right-hand side of each of the eqns (3.292) and (3.293) then is independent of r and therefore may be evaluated for any convenient value of r. An appropriate choice of this matching point which is reasonable from a computational point of view will be made later.

† In the case of the pure Coulomb potential and $k=1$ it is only the L-solution multiplied by r, rather than the L-solution itself, which is bounded and vanishes at $r=0$.

3.3.5.2. Numerical procedure

The computation of each energy eigenvalue is an iterative process which needs a starting value not too far away from the true value. The well-known bound state energies of the pure Coulomb field from eqn (3.229) or those of the unscreened uniformly extended nuclear charge model, which can be computed safely and relatively fast from eqns (3.276) to (3.280), and eqn (3.282), may serve this purpose. From the observed mismatch of the L- and R-solutions at the matching point r_M, an improved energy may be computed and the whole procedure repeated. In more detail this goes as follows. Since we may write the eigenvalue condition

$$f_L(r_M)/g_L(r_M) = f_R(r_M)/g_R(r_M) \tag{3.295a}$$

in the form

$$g_R(r_M)f_L(r_M) - g_L(r_M)f_R(r_M) = 0, \tag{3.295b}$$

we consider, as a function of the energy T, the quantity

$$D(T) = g_R(r_M)f_L(r_M) - g_L(r_M)f_R(r_M) \tag{3.296}$$

so that the eigenvalues are the zeros of $D(T)$. If T_0 denotes the appropriate starting value for the particular eigenvalue under consideration, we may find the corresponding zero by (damped) Newton iteration, for instance, as the limit of the sequence $\{T_n\}_{n=0,1,2,\ldots}$ defined by the starting value T_0 and the recurrence relation

$$T_{n+1} = T_n - \mu D(T_n)/\dot{D}(T_n). \tag{3.297}$$

Here $\dot{D}$ means $\mathrm{d}D/\mathrm{d}T$, and μ with $0 < \mu \leq 1$ is a damping factor, depending on n, which may be chosen <1 for some of the steps if necessary to avoid divergence (or convergence to an unwanted eigenvalue) of the sequence in the case of too bad a starting value. The use of Newton iteration might seem at first sight not recommendable because it needs the first order derivative with respect to T of the wave functions, but this derivative is needed anyway for computing the normalization factors of the eigen solution, after the eigenvalue has been found, and we will show how it can be computed conveniently and relatively fast.

An appropriate (unnormalized) L-solution, valid in a neighbourhood of the origin, is

$$\begin{Bmatrix} f_L(r) \\ g_L(r) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^{n+k-1} \tag{3.298}$$

with

$$a_0 = 1, \quad b_0 = 0 \quad \text{if} \quad \kappa = k,$$
$$a_0 = 0, \quad b_0 = 1 \quad \text{if} \quad \kappa = -k,$$

and the other a_n, b_n given by the recurrence relations (3.175c–d). We also need the derivative with respect to T of this solution, again denoted by a dot,

$$\begin{Bmatrix} \dot{f}_L(r) \\ \dot{g}_L(r) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} \dot{a}_n \\ \dot{b}_n \end{Bmatrix} r^{n+k-1}. \tag{3.299}$$

Differentiating the initial coefficients and the recurrence relations (3.175c–d) with respect to T, we find that

$$\dot{a}_0 = \dot{b}_0 = 0, \tag{3.300}$$

$$\dot{a}_n = \left\{ -b_{n-1} - (T - v_0)\dot{b}_{n-1} + \sum_{m=1}^{n-1} v_m \dot{b}_{n-m-1} \right\} \Big/ (n + k - \kappa), \tag{3.301a}$$

$$\dot{b}_n = \left\{ a_{n-1} + (T + 2m - v_0)\dot{a}_{n-1} - \sum_{m=1}^{n-1} v_m \dot{a}_{n-m-1} \right\} \Big/ (n + k + \kappa). \tag{3.301b}$$

By means of these recurrence relations, the coefficients $\dot{a}_n, \dot{b}_n$ may be evaluated concurrently with the a_n, b_n.

An appropriate (unnormalized) R-solution is

$$\begin{Bmatrix} f_R(r) \\ g_R(r) \end{Bmatrix} \sim \exp(-\lambda r) r^{-1-v_{-1}W/\lambda} \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^{-n} \tag{3.302}$$

with

$$a_0 = -\lambda, \tag{3.303a}$$

$$b_0 = T + 2m, \tag{3.303b}$$

$$a_n = \{(n - 1 + \kappa + v_{-1}W/\lambda)a_{n-1} + v_{-1}b_{n-1}\}\{(n - \kappa) - v_{-1}m/\lambda\}/(-2n\lambda), \tag{3.304a}$$

$$b_n = \{-v_{-1}a_{n-1} + (n - 1 - \kappa + v_{-1}W/\lambda)b_{n-1}\}\{(n + \kappa) + v_{-1}m/\lambda\}/(-2n\lambda). \tag{3.304b}$$

Apart from a normalization factor, this is one of the standard Coulomb solutions relative to $r = \infty$ written here more generally† with $v_{-1} = v_{-1}(0) = v_1(\infty)$ in place of $-\alpha Z$. Using

$$\begin{aligned} dW/dT &= 1, \\ d\lambda/dT &= -W/\lambda, \\ d(W/\lambda)/dT &= m^2/\lambda^3 \end{aligned}$$

† It is needed here with $v_{-1} = -\alpha$ corresponding to $Z = 1$.

we obtain for the derivative with respect to T of the R-solution

$$\begin{Bmatrix} \dot{f}_R(r) \\ \dot{g}_R(r) \end{Bmatrix} \sim \exp(-\lambda r) r^{-1-v_{-1}W/\lambda} \left[\sum_{n=0}^{\infty} \begin{Bmatrix} \dot{a}_n \\ \dot{b}_n \end{Bmatrix} r^{-n} + \{(W/\lambda)r - v_{-1}(m^2/\lambda^3)\ln(r)\} \sum_{n=0}^{\infty} \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} r^{-n} \right] \quad (3.305)$$

where

$$\dot{a}_0 = W/\lambda, \quad (3.306a)$$

$$\dot{b}_0 = 1, \quad (3.306b)$$

$$\dot{a}_n = [\{v_{-1}(m^2/\lambda^3)a_{n-1} + (n-1+\kappa+v_{-1}W/\lambda)\dot{a}_{n-1} + v_{-1}\dot{b}_{n-1}\} * \{(n-\kappa) - v_{-1}m/\lambda\} + \{(n-1+\kappa+v_{-1}W/\lambda)a_{n-1} + v_{-1}b_{n-1}\} * \{(n-\kappa) - 2v_{-1}m/\lambda\}(W/\lambda^2)]/(-2n\lambda), \quad (3.307a)$$

$$\dot{b}_n = [\{v_{-1}(m^2/\lambda^3)b_{n-1} - v_{-1}\dot{a}_{n-1} + (n-1-\kappa+v_{-1}W/\lambda)\dot{b}_{n-1}\} * \{(n+\kappa) + v_{-1}m/\lambda\} + \{-v_{-1}a_{n-1} + (n-1-\kappa+v_{-1}W/\lambda)b_{n-1}\} * \{(n+\kappa) + 2v_{-1}m/\lambda\}\{W/\lambda^2\}]/(-2n\lambda). \quad (3.307b)$$

The L-solution, eqns (3.298) and (3.299), may be evaluated for a sufficiently small value of r and has to be continued outwards up to the matching point r_M. Similarly, the R-solution, eqns (3.302) and (3.305), may be evaluated for a sufficiently large value of r and has to be continued inwards down to the matching point. In both cases this can be done by repeated use of the general solution relative to an ordinary point r_0, as given by eqns (3.176) and (3.177), and its derivative with respect to T, which may be found to be

$$\begin{Bmatrix} r\dot{f}(r) \\ r\dot{g}(r) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} \dot{a}_n \\ \dot{b}_n \end{Bmatrix} (r-r_0)^n \quad (3.308)$$

where

$$\dot{a}_0 = r_0\dot{f}(r_0), \qquad \dot{b}_0 = r_0\dot{g}(r_0), \quad (3.309)$$

and

$$\dot{a}_n = \left[-r_0 b_{n-1} - b_{n-2} - (n-1-\kappa)\dot{a}_{n-1} - \{Tr_0 - v_0\}\dot{b}_{n-1} - (T-v_1)\dot{b}_{n-2} + \sum_{m=2}^{n-1} v_m \dot{b}_{n-m-1} \right] \Big/ (nr_0), \quad (3.310a)$$

$$\dot{b}_n = \left[r_0 a_{n-1} + a_{n-2} - (n-1+\kappa)\dot{b}_{n-1} + \{(T+2m)r_0 - v_0\}\dot{a}_{n-1} + (T+2m-v_1)\dot{a}_{n-2} - \sum_{m=2}^{n-1} v_m \dot{a}_{n-m-1} \right] \Big/ (nr_0). \quad (3.310b)$$

Here in each continuation step the appropriate values of the potential

coefficients $v_m = \tilde{v}_m(r_0)$ for the model under consideration have to be used. Again the coefficients $\dot{a}_n$, $\dot{b}_n$ are to be computed concurrently with the a_n, b_n. This computation is performed within the loops already present for the computation of the a_n, b_n and uses intermediate results like the (sometimes expensive) potential coefficients v_m which are already available. It is therefore comparatively fast.

The choice of the matching point r_M is governed by the requirement that the continuation (or integration) of both the L- and R-solutions be numerically stable. This means that, in the direction of continuation, the complementary, unwanted solution of the differential equations should not increase significantly faster than the desired solution. The essential behaviour of the wanted solution may roughly be characterized by $(\lambda r)^\gamma \exp(-\lambda r)$ as compared with the complementary solution which roughly behaves as $(\lambda r)^\gamma \exp(\lambda r)$ or $(\lambda r)^{-\gamma} \exp(-\lambda r)$, respectively, depending on which direction of continuation is being considered. The point $r = \gamma/\lambda$ where the expression representing the wanted solution has its maximum then seems to be an appropriate choice for the matching point r_M. Using for λ the corresponding Coulomb eigenvalue from eqn (3.232) we obtain

$$r_M = \{\gamma/(\alpha Zm)\}\{k^2 + 2\gamma n' + n'^2\}^{1/2} \simeq \gamma(\gamma + n')/(\alpha Zm). \qquad (3.311)$$

A more accurate discussion of this matter seems to be difficult as well as unnecessary since the choice of the matching point r_M is not expected to be very critical.

3.4. The case when $|\alpha Z| > k$

3.4.1. Coulomb radial wave functions

Although for nuclear beta-decay values of Z greater than 137 do not seem to be of interest, we are going to comment on this case which recently received attention in another context (Pieper and Greiner 1969; Müller, Rafelski, and Greiner 1973). However, we restrict ourselves to a few points which fit into the present context. The reader who is interested in the many aspects which are beyond the scope of our treatment should consult the recent review article by Rafelski, Fulcher, and Klein (1978).

Modifications do occur for the Coulomb radial wave functions in so far as the exponent

$$\gamma = \{k^2 - (\alpha Z)^2\}^{1/2},$$

until now tacitly assumed to be real, may become imaginary, in particular if $k = 1$. The two standard wave functions relative to the origin then

become conjugate complex to each other, so that they are of the same type and there is no longer any reason for calling them differently and preferring one of them as the 'regular' one.

As a consequence the (point charge) Coulomb wave functions have no longer any physical significance. At $r=0$ the usual integrability requirement does not select one of the standard solutions, so that in the case of $W>m$ the physical scattering phase shift remains undefined and in the case of $-m<W<m$ the bound state condition, eqn (3.218), can be satisfied for every value of W, corresponding to a continuous bound state energy spectrum.

Nevertheless (point charge) Coulomb wave functions are of interest, for they occur as mathematical objects in physically significant models like the uniform model of an extended nuclear charge. In such a model there is no drastic change as Z increases from the region where $\alpha Z<k$ into the region where $\alpha Z>k$. No change at all occurs, in fact, with our simple but slow method (Section 3.2.3.2.1) for computing the amplitude of the wave function and the scattering phase shift, since the exponent γ does not enter. Our fast but complicated method (Section 3.2.3.2.2), although still applicable in principle, becomes inconvenient since now many complex quantities are involved. It will therefore be reformulated below in terms of real quantities. Similarly, the bound state condition, eqns (3.273) and (3.274), which becomes a complex equation, will be restated using real quantities only.

3.4.1.1. Coulomb solutions relative to the origin

In view of these applications it is reasonable to introduce, relative to $r=0$, yet another set of standard Coulomb wave functions which are real (when r is real and positive). It is convenient to define them in a slightly different way for the continuum or bound state case, respectively. In order to treat both cases simultaneously, we want to introduce the parameter μ where

$$\mu=\tfrac{1}{2} \quad \text{if} \quad -m<W<m \quad \text{(bound state case)} \tag{3.312a}$$

or

$$\mu=p \quad \text{if} \quad m<W \quad \text{(continuum state case).} \tag{3.312b}$$

In a first step we may introduce the solutions

$$\begin{Bmatrix}\bar{f}_{01}(r)\\ \bar{g}_{01}(r)\end{Bmatrix}=\{(\kappa-\gamma)(\kappa W-\gamma m)\}^{-1/2}(2\mu)^{\gamma}\begin{Bmatrix}f_{01}(r)\\ g_{01}(r)\end{Bmatrix}, \tag{3.313a}$$

$$\begin{Bmatrix}\bar{f}_{02}(r)\\ \bar{g}_{02}(r)\end{Bmatrix}=\{(\kappa+\gamma)(\kappa W+\gamma m)\}^{-1/2}(2\mu)^{-\gamma}\begin{Bmatrix}f_{02}(r)\\ g_{02}(r)\end{Bmatrix}, \tag{3.313b}$$

which are just the earlier ones, but with a different normalization. Their

behaviour at the origin is simply given by

$$r\begin{Bmatrix}\tilde{f}_{01}(r)\\ \tilde{g}_{01}(r)\end{Bmatrix}\sim(2\mu r)^{\gamma}\begin{Bmatrix}\alpha Z/(\kappa-\gamma)\\ 1\end{Bmatrix}, \tag{3.314a}$$

$$r\begin{Bmatrix}\tilde{f}_{02}(r)\\ \tilde{g}_{02}(r)\end{Bmatrix}\sim(2\mu r)^{-\gamma}\begin{Bmatrix}\alpha Z/(\kappa+\gamma)\\ 1\end{Bmatrix}. \tag{3.314b}$$

In order to assign the indexes 1 or 2 of these solutions in a unique way, we may arbitrarily choose the phase of γ by defining

$$\gamma = ic \tag{3.315a}$$

where

$$c = |\gamma| = \{(\alpha Z)^2 - k^2\}^{1/2} > 0. \tag{3.315b}$$

We are now prepared to define our new standard solutions

$$\begin{Bmatrix}f_{03}\\ g_{03}\end{Bmatrix} = \tfrac{1}{2}\begin{Bmatrix}\tilde{f}_{01}+\tilde{f}_{02}\\ \tilde{g}_{01}+\tilde{g}_{02}\end{Bmatrix}, \tag{3.316a}$$

$$\begin{Bmatrix}f_{04}\\ g_{04}\end{Bmatrix} = -\tfrac{1}{2}\mathrm{i}\begin{Bmatrix}\tilde{f}_{01}-\tilde{f}_{02}\\ \tilde{g}_{01}-\tilde{g}_{02}\end{Bmatrix}, \tag{3.316b}$$

the leading terms of which, when $r \to 0$, may be found from eqns (3.314) to (3.316) to be

$$\begin{Bmatrix}f_{03}(r)\\ g_{03}(r)\end{Bmatrix}\sim\begin{Bmatrix}(\alpha Z)^{-1}\kappa\cos\{c\ln(2\mu r)\}-(\alpha Z)^{-1}c\sin\{c\ln(2\mu r)\}\\ \cos\{c\ln(2\mu r)\}\end{Bmatrix}, \tag{3.317a}$$

$$\begin{Bmatrix}f_{04}(r)\\ g_{04}(r)\end{Bmatrix}\sim\begin{Bmatrix}(\alpha Z)^{-1}\kappa\sin\{c\ln(2\mu r)\}+(\alpha Z)^{-1}c\cos\{c\ln(2\mu r)\}\\ \sin\{c\ln(2\mu r)\}\end{Bmatrix}. \tag{3.317b}$$

In a similar way the complete expansion of these solutions may be written

$$r\begin{Bmatrix}f_{03}(r)\\ g_{03}(r)\end{Bmatrix} = \cos\{c\ln(2\mu r)\}\sum_{n=0}^{\infty}\begin{Bmatrix}A_n^{\mathrm{I}}\\ B_n^{\mathrm{I}}\end{Bmatrix}r^n - \sin\{c\ln(2\mu r)\}\sum_{n=0}^{\infty}\begin{Bmatrix}A_n^{\mathrm{II}}\\ B_n^{\mathrm{II}}\end{Bmatrix}r^n, \tag{3.318a}$$

$$r\begin{Bmatrix}f_{04}(r)\\ g_{04}(r)\end{Bmatrix} = \sin\{c\ln(2\mu r)\}\sum_{n=0}^{\infty}\begin{Bmatrix}A_n^{\mathrm{I}}\\ B_n^{\mathrm{I}}\end{Bmatrix}r^n + \cos\{c\ln(2\mu r)\}\sum_{n=0}^{\infty}\begin{Bmatrix}A_n^{\mathrm{II}}\\ B_n^{\mathrm{II}}\end{Bmatrix}r^n, \tag{3.318b}$$

where the initial coefficients are

$$A_0^{\mathrm{I}}=\kappa/(\alpha Z), \tag{3.319a}$$

$$B_0^{\mathrm{I}}=1, \tag{3.319b}$$

$$A_0^{\mathrm{II}}=c/(\alpha Z), \tag{3.319c}$$

$$B_0^{\mathrm{II}}=0. \tag{3.319d}$$

The coefficients with $n>0$ may be evaluated from the recurrence relations

$$A_n^{\mathrm{I}}=\{n(n^2+4c^2)\}^{-1}\{-(n^2+\kappa n+2c^2)TB_{n-1}^{\mathrm{I}}-c(n+2\kappa)TB_{n-1}^{\mathrm{II}} - n\alpha Z(T+2m)A_{n-1}^{\mathrm{I}}-2c\alpha Z(T+2m)A_{n-1}^{\mathrm{II}}\}, \tag{3.320a}$$

$$A_n^{\mathrm{II}}=\{n(n^2+4c^2)\}^{-1}\{-(n^2+\kappa n+2c^2)TB_{n-1}^{\mathrm{II}}+c(n+2\kappa)TB_{n-1}^{\mathrm{I}} - n\alpha Z(T+2m)A_{n-1}^{\mathrm{II}}+2c\alpha Z(T+2m)A_{n-1}^{\mathrm{I}}\}, \tag{3.320b}$$

$$B_n^{\mathrm{I}}=\{n(n^2+4c^2)\}^{-1}\{(n^2-\kappa n+2c^2)(T+2m)A_{n-1}^{\mathrm{I}} + c(n-2\kappa)(T+2m)A_{n-1}^{\mathrm{II}}-n\alpha ZTB_{n-1}^{\mathrm{I}}-2c\alpha ZTB_{n-1}^{\mathrm{II}}\}, \tag{3.320c}$$

$$B_n^{\mathrm{II}}=\{n(n^2+4c^2)\}^{-1}\{(n^2-\kappa n+2c^2)(T+2m)A_{n-1}^{\mathrm{II}} - c(n-2\kappa)(T+2m)A_{n-1}^{\mathrm{I}}-n\alpha ZTB_{n-1}^{\mathrm{II}}+2c\alpha ZTB_{n-1}^{\mathrm{I}}\}, \tag{3.320d}$$

which are not difficult to obtain from eqn (3.21). It follows from eqn (3.317) that the new standard solutions (3.318) obey the Wronskian relation

$$r^2\{f_{03}(r)g_{04}(r)-g_{03}(r)f_{04}(r)\}\equiv -c/(\alpha Z). \tag{3.321}$$

3.4.1.2. Connection between the solutions of different type

We are going to investigate the asymptotic behaviour, when $r\to\infty$, of the new standard solutions (3.318) by expressing them in terms of the appropriate standard solutions relative to $r=\infty$.

In the bound state case we want to write for brevity

$$\begin{Bmatrix}\tilde{f}_{01}(r)\\ \tilde{g}_{01}(r)\end{Bmatrix}=K(\gamma)\begin{Bmatrix}f_{\infty 1}(r)\\ g_{\infty 1}(r)\end{Bmatrix}+L(\gamma)\begin{Bmatrix}f_{\infty 2}(r)\\ g_{\infty 2}(r)\end{Bmatrix}, \tag{3.22a}$$

where

$$K(\gamma)=(-TW)^{1/2}\Gamma(1+2\gamma)\exp\{i\pi(-\gamma+\alpha ZW/\lambda)\}(2\lambda)^{-\gamma}(2\lambda)^{\alpha ZW/\lambda} *\{-(\kappa-\gamma)\lambda+\alpha ZT\}^{-1}\{\Gamma(\gamma+\alpha ZW/\lambda)\}^{-1}, \tag{3.323}$$

$$L(\gamma)=(-TW)^{1/2}\Gamma(1+2\gamma)(2\lambda)^{-\gamma}(2\lambda)^{-\alpha ZW/\lambda}\{-(\kappa-\gamma)\lambda-\alpha ZT\}^{-1} *\{\Gamma(\gamma-\alpha ZW/\lambda)\}^{-1} \tag{3.324}$$

may be found from eqns (3.270) and (3.313). We then have

$$\begin{Bmatrix} \tilde{f}_{02}(r) \\ \tilde{g}_{02}(r) \end{Bmatrix} = K(-\gamma) \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + L(-\gamma) \begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix} \tag{3.322b}$$

and therefore

$$\begin{Bmatrix} f_{03}(r) \\ g_{03}(r) \end{Bmatrix} = \tfrac{1}{2}\{K(\mathrm{ic}) + K(-\mathrm{ic})\} \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + \tfrac{1}{2}\{L(\mathrm{ic}) + L(-\mathrm{ic})\} \begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}, \tag{3.325a}$$

$$\begin{Bmatrix} f_{04}(r) \\ g_{04}(r) \end{Bmatrix} = -\tfrac{1}{2}\mathrm{i}\{K(\mathrm{ic}) - K(-\mathrm{ic})\} \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} - \tfrac{1}{2}\mathrm{i}\{L(\mathrm{ic}) - L(-\mathrm{ic})\} \begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}. \tag{3.325b}$$

Here $\{f_{\infty 1}(r), g_{\infty 1}(r)\}$ is the exponentially decreasing and $\{f_{\infty 2}(r), g_{\infty 2}(r)\}$ the exponentially increasing solution according to eqns (3.225) and (3.226).

In the case of the continuum state solutions we write for brevity

$$\begin{Bmatrix} \tilde{f}_{01}(r) \\ \tilde{g}_{01}(r) \end{Bmatrix} = M(\gamma)(2p)^{\mathrm{i}y} \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + N(\gamma)(2p)^{-\mathrm{i}y} \begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}, \tag{3.326a}$$

where we have from eqns (3.72),†(3.78), and (3.313)

$$M(\gamma) = (TW)^{1/2} \exp\{-\tfrac{1}{2}\pi y\}\Gamma(1+2\gamma)(i)^{-\gamma} * \{-(\kappa-\gamma)p + \mathrm{i}\alpha ZT\}^{-1}\{\Gamma(\gamma+\mathrm{i}y)\}^{-1}, \tag{3.327}$$

$$N(\gamma) = (TW)^{1/2} \exp\{-\tfrac{1}{2}\pi y\}\Gamma(1+2\gamma)(-\mathrm{i})^{-\gamma} * \{-(\kappa-\gamma)p - \mathrm{i}\alpha ZT\}^{-1}\{\Gamma(\gamma-\mathrm{i}y)\}^{-1}. \tag{3.328}$$

We then have

$$\begin{Bmatrix} \tilde{f}_{02}(r) \\ \tilde{g}_{02}(r) \end{Bmatrix} M(-\gamma)(2p)^{\mathrm{i}y} \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + N(-\gamma)(2p)^{-\mathrm{i}y} \begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix} \tag{3.326b}$$

and therefore

$$\begin{Bmatrix} f_{03}(r) \\ g_{03}(r) \end{Bmatrix} = \tfrac{1}{2}\{M(\mathrm{ic}) + M(-\mathrm{ic})\}(2p)^{\mathrm{i}y} \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + \tfrac{1}{2}\{N(\mathrm{ic}) + N(-\mathrm{ic})\}(2p)^{-\mathrm{i}y} \begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}, \tag{3.329a}$$

$$\begin{Bmatrix} f_{04}(r) \\ g_{04}(r) \end{Bmatrix} = -\tfrac{1}{2}\mathrm{i}\{M(\mathrm{ic}) - M(-\mathrm{ic})\}(2p)^{\mathrm{i}y} \begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} - \tfrac{1}{2}\mathrm{i}\{N(\mathrm{ic}) - N(-\mathrm{ic})\}(2p)^{-\mathrm{i}y} \begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix}. \tag{3.329b}$$

† With eqn (3.72) rewritten in the form

$$\exp(\mathrm{i}\eta) = \{2T(\kappa - s)(\kappa W - sm)\}^{1/2}/\{-(\kappa - s)p + \mathrm{i}\alpha ZT\}.$$

We want to rewrite eqn (3.329) using real quantities. From eqn (3.43) we may obtain

$$(2p)^{iy}\begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} = \exp[i(\pi/2)\{l(\kappa)+1\}]\left[\begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} - i\begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}\right], \tag{3.330a}$$

$$(2p)^{-iy}\begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix} = \exp[-i(\pi/2)\{l(\kappa)+1\}]\left[\begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} + i\begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}\right]. \tag{3.330b}$$

By inspection of eqns (3.327) and (3.328) we notice that, if $\gamma = ic$ is imaginary, $N(\gamma)$ is the conjugate complex of $M(-\gamma)$. It therefore suffices to introduce four (rather than eight) real quantities Q, S, ζ, ξ by writing

$$M(ic) + M(-ic) = Q\exp(i\zeta), \tag{3.331a}$$

$$N(ic) + N(-ic) = Q\exp(-i\zeta), \tag{3.331b}$$

$$-i\{M(ic) - M(-ic)\} = S\exp(i\xi), \tag{3.332a}$$

$$-i\{N(ic) - N(-ic)\} = -S\exp(-i\xi). \tag{3.332b}$$

The eqn (3.329) may now be rewritten

$$Q^{-1}\begin{Bmatrix} f_{03}(r) \\ g_{03}(r) \end{Bmatrix} = \cos[\zeta + \tfrac{1}{2}\pi\{l(\kappa)+1\}]\begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} + \sin[\zeta + \tfrac{1}{2}\pi\{l(\kappa)+1\}]\begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}, \tag{3.333a}$$

$$S^{-1}\begin{Bmatrix} f_{04}(r) \\ g_{04}(r) \end{Bmatrix} = \cos[\xi + \tfrac{1}{2}\pi\{l(\kappa)+1\}]\begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} + \sin[\xi + \tfrac{1}{2}\pi\{l(\kappa)+1\}]\begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}. \tag{3.333b}$$

It follows, by means of eqns (3.42), (3.83), (3.84) or (3.42), (3.44), and (3.46), that the asymptotic behaviour, when $r \to \infty$, of the new standard solutions is given by

$$\begin{Bmatrix} f_{03}(r) \\ g_{03}(r) \end{Bmatrix} \sim Qr^{-1}\begin{Bmatrix} -(T/W)^{1/2}\sin\{pr + y\ln(2pr) + \zeta\} \\ \{(T+2m)/W\}^{1/2}\cos\{pr + y\ln(2pr) + \zeta\} \end{Bmatrix} \tag{3.334a}$$

$$\begin{Bmatrix} f_{04}(r) \\ g_{04}(r) \end{Bmatrix} \sim Sr^{-1}\begin{Bmatrix} -(T/W)^{1/2}\sin\{pr + y\ln(2pr) + \xi\} \\ \{(T+2m)/W\}^{1/2}\cos\{pr + y\ln(2pr) + \xi\} \end{Bmatrix}. \tag{3.334b}$$

Here the real quantities Q, S, ζ, ξ have to be evaluated from their defining equations, eqns (3.327), (3.328), (3.331) and (3.332). This is somewhat inconvenient to do, but we do not see any way for simplification, except that $1/\Gamma\{i(c+y)\}$ should be replaced by $i(1/\pi)\sinh\{\pi(c+y)\}\Gamma\{1-i(c+y)\}$,

and similarly for the other gamma functions appearing as a denominator. Finally we should like to mention that the identity

$$\alpha ZpQS \sin(\zeta - \xi) = cW \tag{3.335}$$

holds, for if the Wronskian of the new standard solutions is evaluated by means of eqn (3.334), the result is

$$r^2\{f_{03}(r)g_{04}(r) - g_{03}(r)f_{04}(r)\} \equiv -(p/W)QS \sin(\zeta - \xi). \tag{3.336}$$

By comparison with eqn (3.321) the identity (3.335) is verified.

3.4.2. The uniform model of an extended nuclear charge

3.4.2.1. Bound states for the uniform model

In the case of the uniform model, eqn (3.122), we may proceed essentially in the same way as in Section 3.3.4, but using the new standard Coulomb solutions $\{f_{0j}(r), g_{0j}(r)\}$ with $j = 3, 4$ rather than those with $j = 1, 2$. Denoting again the normalized bound state solution by $\{f(r), g(r)\}$ we have

$$N\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \begin{Bmatrix} f^{\mathrm{I}}(r) \\ g^{\mathrm{I}}(r) \end{Bmatrix} \quad \text{for} \quad 0 \leqslant r \leqslant R, \tag{3.337a}$$

$$N\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = E\begin{Bmatrix} f_{03}(r) \\ g_{03}(r) \end{Bmatrix} + H\begin{Bmatrix} f_{04}(r) \\ g_{04}(r) \end{Bmatrix} \quad \text{for} \quad R \leqslant r. \tag{3.337b}$$

Here $\{f^{\mathrm{I}}(r), g^{\mathrm{I}}(r)\}$ is the local solution, eqn (3.129), for the uniform model inside the nucleus, and N is a normalization factor still to be determined. Because of the required continuity at $r = R$ we have

$$E = D_1/D, \tag{3.338}$$

$$H = D_2/D, \tag{3.339}$$

with

$$D_1 = f^{\mathrm{I}}(R)g_{04}(R) - g^{\mathrm{I}}(R)f_{04}(R), \tag{3.340}$$

$$D_2 = f_{03}(R)g^{\mathrm{I}}(R) - g_{03}(R)f^{\mathrm{I}}(R), \tag{3.341}$$

$$D = f_{03}(R)g_{04}(R) - g_{03}(R)f_{04}(R), \tag{3.342}$$

or

$$D = -c/(\alpha ZR^2) \tag{3.343}$$

by means of the Wronskian relation (3.321). In order to discuss the asymptotic behaviour, when $r \to \infty$, of the solution we write in place of eqn (3.337*b*)

$$N\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = A\begin{Bmatrix} f_{\infty 1}(r) \\ g_{\infty 1}(r) \end{Bmatrix} + B\begin{Bmatrix} f_{\infty 2}(r) \\ g_{\infty 2}(r) \end{Bmatrix} \tag{3.344}$$

where the coefficients A, B, which may be found by means of eqn (3.325), are

$$A=\tfrac{1}{2}\{K(\mathrm{ic})+K(-\mathrm{ic})\}E-\tfrac{1}{2}\mathrm{i}\{K(\mathrm{ic})-K(-\mathrm{ic})\}H, \tag{3.345}$$

$$B=\tfrac{1}{2}\{L(\mathrm{ic})+L(-\mathrm{ic})\}E-\tfrac{1}{2}\mathrm{i}\{L(\mathrm{ic})-L(-\mathrm{ic})\}H. \tag{3.346}$$

The boundary condition at $r=\infty$ requires that the coefficient B of the exponentially increasing part should vanish. We therefore have to look for the zeros, with respect to W, of B. By means of the eigenvalue condition

$$B=0 \tag{3.347}$$

the coefficient A may be rewritten

$$A=E\{K(-\mathrm{ic})L(\mathrm{ic})-K(\mathrm{ic})L(-\mathrm{ic})\}/\{L(\mathrm{ic})-L(-\mathrm{ic})\}. \tag{3.348}$$

Using the definition of K and L we obtain in a similar way as in Section 3.3.4, by somewhat lengthy manipulations

$$A=\{c/(\alpha Z)\}(W/\lambda)Ei/\{L(\mathrm{ic})-L(-\mathrm{ic})\}, \tag{3.349}$$

an expression which shows that A becomes real whenever the eigenvalue condition is satisfied. This was not evident from eqn (3.348), for while $L(\mathrm{ic})$ and $L(-\mathrm{ic})$ are conjugate complex numbers, $K(\mathrm{ic})$ and $K(-\mathrm{ic})$ are not.

We are going to look more closely at the eigenvalue condition (3.346) and (3.347). Introducing the real number χ by

$$E=(E^2+H^2)^{1/2}\sin\chi, \tag{3.350a}$$

$$H=(E^2+H^2)^{1/2}\cos\chi \tag{3.350b}$$

and the real phase Θ by

$$L(\mathrm{ic})=|L(\mathrm{ic})|\exp(\mathrm{i}\Theta) \tag{3.351}$$

we may obtain, by omitting the non-vanishing factors $(E^2+H^2)^{1/2}$ and $|L(\mathrm{ic})|$, the eigenvalue condition in the form

$$\sin(\chi+\Theta)=0, \tag{3.352}$$

where

$$\Theta=\arg\Gamma(1+2\mathrm{ic})-\arg(-\kappa\lambda-\alpha ZT+\mathrm{ic}\lambda)-\arg\Gamma\{-\alpha Z(W/\lambda)+\mathrm{ic}\}-c\ln(2\lambda). \tag{3.353}$$

Finally, the eigenvalue condition may be written

$$\arctan(E/H)+\mathrm{Im}\ln\Gamma(1+2\mathrm{ic})+\arctan\{c\lambda/(\kappa\lambda+\alpha ZT)\}+\mathrm{Im}\ln\Gamma\{-\alpha Z(W/\lambda)-\mathrm{ic}\}-c\ln(2\lambda)=0 \bmod \pi \tag{3.354}$$

or alternatively, by means of the reflection formula for the gamma function,

$$\arctan(E/H)+\operatorname{Im}\ln\Gamma(1+2ic)+\arctan\{c\lambda/(\kappa\lambda+\alpha ZT)\}$$
$$-\arctan\{\operatorname{tgh}(\pi c)/\operatorname{tg}(\pi\alpha ZW/\lambda)\}-\arctan\{c\lambda/(\alpha ZW)\}$$
$$-\operatorname{Im}\ln\Gamma\{\alpha Z(W/\lambda)+ic\}-c\ln(2\lambda)=0 \bmod \pi. \quad (3.355)$$

Equation (3.354) or (3.355) is to be preferred according to whether we are looking for an eigenvalue of W which is less or greater than zero, respectively. Then the gamma functions are never needed in the left half plane.

Although finally the eigenvalues have to be computed numerically from eqns (3.354) or (3.355) by any suitable iterative method, it is instructive to try a qualitative discussion, in particular for the case when λ is small so that $\alpha Z|W|/\lambda$ is large. We first wish to show that then the only term which rapidly varies with λ is the term $\arctan\{\operatorname{tgh}(\pi c)/\operatorname{tg}(\pi\alpha ZW/\lambda)\}$. If $\alpha Z|W|/\lambda$ is sufficiently large we may use Stirling's asymptotic formula

$$\ln\Gamma(z)\simeq\{z-\tfrac{1}{2}\}\ln(z)-z+\tfrac{1}{2}\ln(2\pi) \quad (3.356)$$

which yields

$$\operatorname{Im}\ln\Gamma(z)\simeq\operatorname{Im}(z)\ln|z|+\operatorname{Re}(z)\arg(z)-\tfrac{1}{2}\arg(z)-\operatorname{Im}(z). \quad (3.357)$$

If, in addition, $\operatorname{Re}(z)\gg|\operatorname{Im}(z)|$ we have

$$\operatorname{Re}(z)\arg(z)-\operatorname{Im}(z)=\operatorname{Re}(z)\arctan\{\operatorname{Im}(z)/\operatorname{Re}(z)\}-\operatorname{Im}(z)\simeq 0$$

so that then eqn (3.357) may be simplified to give

$$\operatorname{Im}\ln\Gamma(z)\simeq\operatorname{Im}(z)\ln|z|-\tfrac{1}{2}\arg(z). \quad (3.358)$$

It follows that the strong non-linear λ-dependence of the two terms $-c\ln(2\lambda)$ and $\pm\operatorname{Im}\ln\Gamma\{\mp\alpha Z(W/\lambda)\mp ic\}$ essentially cancels. For we have, from eqn (3.358),

$$\pm\operatorname{Im}\ln\Gamma\{\mp\alpha Z(W/\lambda)\mp ic\}-c\ln(2\lambda)\simeq-\tfrac{1}{2}c\ln\{(\mp\alpha ZW/\lambda)^2+c^2\}$$
$$+\tfrac{1}{2}\arctan\{c\lambda/(\mp\alpha ZW)\}-c\ln(2\lambda).$$

This becomes

$$\simeq-c\ln(\mp\alpha ZW/\lambda)+\tfrac{1}{2}c\lambda/(\mp\alpha ZW)-c\ln(2\lambda)$$
$$=\tfrac{1}{2}c\lambda/(\mp\alpha ZW)-c\ln(\mp 2\alpha ZW)$$

if $c\ll\mp\alpha ZW/\lambda$. Because of our assumption $\lambda\ll W$, we have $-T\approx\lambda^2/(2m)$ and therefore

$$\arctan\{c\lambda/(\kappa\lambda+\alpha ZT)\}\simeq\arctan(c/\kappa).$$

Equation (3.354) then becomes approximately

$$\arctan(E/H)+\operatorname{Im}\ln\Gamma(1+2ic)+\arctan(c/\kappa)$$
$$+\tfrac{1}{2}c\lambda/(-\alpha ZW)-c\ln(-2\alpha ZW)\simeq 0 \bmod \pi. \quad (3.359)$$

Since E/H varies slowly† with λ and $W\simeq -m+(2m)^{-1}\lambda^2$ varies slowly too, eqn (3.359) may be looked on as an approximately linear equation for λ. Therefore only one eigenvalue, if any, corresponding to $W<0$ is expected. Although, because of our various assumptions and approximations, it might be that a few more eigenvalues are possible, it remains true that their number is finite, in accordance with the physical expectation. The available numerical results (Pieper and Greiner 1969) give some indication that, for every κ, the number of eigenvalues in the range‡ $-m<W\leqslant 0$ is, in fact, one or zero.

Turning our attention now to eqn (3.355) we find that it becomes approximately

$$\arctan(E/H)+\operatorname{Im}\ln\Gamma(1+2ic)+\arctan(c/\kappa)$$
$$-\arctan\{\operatorname{tgh}(\pi c)/\operatorname{tg}(\pi\alpha ZW/\lambda)\}-\tfrac{1}{2}c\lambda/(\alpha ZW)-c\ln(2\alpha ZW)$$
$$\simeq 0 \bmod \pi. \quad (3.360)$$

Besides some other minor differences, there is now a strong λ-dependence due to the term containing $\operatorname{tg}(\pi\alpha ZW/\lambda)$. Because of the periodicity of the tangens function, an infinite number of eigenvalues occurs near $\lambda=0$, $W=m$, as expected. It seems reasonable to label them by an integer $n'\geqslant 0$ according to

$$\alpha ZW/\lambda = n'+\varepsilon \quad (3.361)$$

where $\varepsilon=\varepsilon(n')$ is real and depends on n'. The corresponding eigenvalues

† In order to achieve this behaviour, it was necessary to chose $\mu=\frac{1}{2}$ (or 1) rather than $\mu=\lambda$ in eqn (3.321*a*).

‡ It should be mentioned that even for W equal to $-m$ a bound state solution may occur (Rafelski, Fulcher, and Klein 1978). To see this we have to consider, for r large, *WKB*-type solutions which contain the exponential factor

$$\exp\left\{\pm\int_{r_0}^{r}[m^2-\{W-V(x)\}^2]^{1/2}\,dx\right\}$$

with an r_0 such that the radicand does not change sign. For $W=-m$ the exponential factor becomes

$$\exp\left\{\pm\int_{r_0}^{r}[-V(x)\{2m+V(x)\}]^{1/2}\,dx\right\}.$$

Since $V(r)$ is negative in the attractive case under consideration, the integral is real so that an exponentially decaying solution becomes possible. In another context and for continuum rather than bound states, an approximate *WKB*-type solution is considered in Section 4.5.2.

then are

$$W_{n'} = (n'+\varepsilon)m\{(\alpha Z)^2+(n'+\varepsilon)^2\}^{-1/2}, \tag{3.362}$$

$$\lambda_{n'} = \alpha Z m\{(\alpha Z)^2+(n'+\varepsilon)^2\}^{-1/2}. \tag{3.363}$$

For every n', all the quantities may be looked at as functions of ε, and we are looking for an ε such that eqn (3.354) or eqn (3.355) is satisfied. Unfortunately, here we do not know the sign of ε nor do we have any reason to expect that ε might be confined to an interval of unit length. In particular, if n' is small or zero, a large negative value of ε may be needed if the eigenvalue $W_{n'}$ happens to be negative. To find numerically the appropriate root, with respect to ε, of eqn (3.354) or eqn (3.355) is therefore here less straightforward than in Section 3.3.4 above for the corresponding eigenvalue condition (3.282). In particular, the correct assignment of the quantum number n' requires more effort.

3.4.2.2. Continuum state radial wave functions for the uniform model

Again we may proceed essentially in the same way as in Section 3.2.3.2.2, but using the new standard Coulomb solutions $\{f_{0j}(r), g_{0j}(r)\}$ with $j = 3, 4$ rather than those with $j = 1, 2$. Denoting the desired normalized solution by $\{f(r), g(r)\}$ we have

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \alpha_\kappa \begin{Bmatrix} f^{\mathrm{I}}(r) \\ g^{\mathrm{I}}(r) \end{Bmatrix} \quad \text{for} \quad 0 \leqslant r \leqslant R \tag{3.364a}$$

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \alpha_\kappa E \begin{Bmatrix} f_{03}(r) \\ g_{03}(r) \end{Bmatrix} + \alpha_\kappa H \begin{Bmatrix} f_{04}(r) \\ g_{04}(r) \end{Bmatrix} \quad \text{for} \quad R \leqslant r. \tag{3.364b}$$

Here $\{f^{\mathrm{I}}(r), g^{\mathrm{I}}(r)\}$ is the local solution (3.129) for the uniform model inside the nucleus, and the amplitude α_κ is the normalization factor we are interested in. Because of the required continuity at $r = R$ we have for E and H the expressions (3.338) to (3.343) just as in the bound state case above. In order to exhibit the asymptotic behaviour, when $r \to \infty$, of the solution we write in place of eqn (3.364*b*)

$$\begin{Bmatrix} f(r) \\ g(r) \end{Bmatrix} = \alpha_\kappa A \begin{Bmatrix} f_{\infty 3}(r) \\ g_{\infty 3}(r) \end{Bmatrix} + \alpha_\kappa B \begin{Bmatrix} f_{\infty 4}(r) \\ g_{\infty 4}(r) \end{Bmatrix}, \tag{3.365}$$

where the coefficients A, B, which may be found by means of eqn (3.333), are

$$A = EQ\cos[\zeta + \tfrac{1}{2}\pi\{l(\kappa)+1\}] + HS\cos[\xi + \tfrac{1}{2}\pi\{l(\kappa)+1\}], \tag{3.366a}$$

$$B = EQ\sin[\zeta + \tfrac{1}{2}\pi\{l(\kappa)+1\}] + HS\sin[\xi + \tfrac{1}{2}\pi\{l(\kappa)+1\}]. \tag{3.366b}$$

The desired amplitude α_κ and the scattering phase shift Δ then are given by

$$\alpha_\kappa = (A^2 + B^2)^{-1/2}, \tag{3.367}$$

$$\cos(\Delta) = \alpha_\kappa A, \tag{3.368a}$$

$$\sin(\Delta) = \alpha_\kappa B. \tag{3.368b}$$

If desired, the amplitude might be written more explicitly as

$$\alpha_\kappa = \{(EQ)^2 + 2EHQS\cos(\zeta - \xi) + (HS)^2\}^{-1/2}. \tag{3.369}$$

4

ELECTRON RADIAL WAVE FUNCTIONS IN THE CONTEXT OF NUCLEAR BETA-DECAY

4.1. Quantities needed for nuclear beta-decay

OF THE modifications of the potential, as compared with the pure Coulomb potential, the finite extension of the nuclear charge is most important since it modifies the characteristic exponents of the electron radial wave functions at $r=0$ and restores their familiar angular momentum dependence, which is independent of αZ. Screening by the atomic electrons, on the other hand, although it would remove at $r=\infty$ the exponent y responsible for the pathologic logarithmic r-dependence y ln$(2pr)$ of the phase shift, is less important since its presence or absence does not qualitatively change the quantities of physical interest. Also, since the atom after beta-decay is usually ionized rather than neutral, such a Coulomb pathology, corresponding to the charge of the ion, still occurs at $r=\infty$ even if screening is allowed for. In the context of nuclear beta-decay we therefore have to consider radial wave functions corresponding to the potential of an extended nuclear charge distribution which is screened by the atomic electrons, but the case without screening is of considerable interest too. The particular solution needed is the solution which at $r=0$ is bounded and is suitably normalized at $r=\infty$. This solution may be characterized by its behaviour in a neighbourhood of $r=0$,

$$\begin{Bmatrix} f_\kappa(r) \\ g_\kappa(r) \end{Bmatrix} = \alpha_\kappa \{(2k-1)!!\}^{-1} (pr)^{k-1} \sum_{n=0}^{\infty} \begin{Bmatrix} a_{\kappa n} \\ b_{\kappa n} \end{Bmatrix} (r/R)^n \qquad (4.1a)$$

with

$$a_{\kappa 0}=1, \quad b_{\kappa 0}=0 \quad \text{if} \quad \kappa = k \qquad (4.1b)$$

or

$$a_{\kappa 0}=0, \quad b_{\kappa 0}=1 \quad \text{if} \quad k=-k, \qquad (4.1c)$$

and by its asymptotic behaviour when $r \to \infty$,

$$\begin{Bmatrix} f_\kappa(r) \\ g_\kappa(r) \end{Bmatrix} \sim r^{-1} \begin{Bmatrix} -(T/W)^{1/2} \sin[pr+\{\alpha(Z-\tilde{Z})W/p\}\ln(2pr)+\delta_\kappa] \\ \{(T+2m)/W\}^{1/2} \cos[pr+\{\alpha(Z-\tilde{Z})W/p\}\ln(2pr)+\delta_\kappa] \end{Bmatrix}. \qquad (4.2)$$

Here we have $\tilde{Z}=0$ if screening is not allowed for, or $\tilde{Z}=Z$ if the daughter atom is neutral, or, more realistically, $Z-\tilde{Z}=1$ in the case of a singly charged daughter atom. The phases δ_κ are related to the scattering phase shifts Δ_κ by

$$\delta_\kappa = \Delta_\kappa - (\pi/2)\{l(\kappa)+1\}. \tag{4.3}$$

It is, on the one hand, the amplitudes α_κ and the phases δ_κ or Δ_κ which are needed for nuclear beta-decay. These quantities are complicated and require electronic computation on the basis of the methods explained in Chapter 3. They are, however, not directly the quantities which are suitable for tabulation. What actually enters the beta-decay observables are certain combinations of the amplitudes and phases, bilinear in the amplitudes. These will be considered in further detail in Section 4.2.

It is, on the other hand, the r-dependence of the radial wave functions which has to be considered in detail, since the electron radial wave functions enter the integrands of integrals over r from zero to infinity. Since also the nuclear wave functions enter, the main contribution to the integrals comes from the region between $r=0$ and r equal to a few times the nuclear radius R. It is then clear from eqn (4.1) that the main radial dependence is given by r^{k-1} in the case of $f_{+k}(r)$ or $g_{-k}(r)$ or by r^k in the case of $f_{-k}(r)$ or $g_{+k}(r)$. Also we may notice by inspection of the differential eqn (3.1) that there is some similarity between $f_{+k}(r)$ and $g_{-k}(r)$ or between $f_{-k}(r)$ and $g_{+k}(r)$, the corresponding functions becoming essentially equal in the limit of $m \to 0$. It is therefore advantageous to write, in place of eqn (4.1),

$$f_{+k}(r) = \alpha_{+k}\{(2k-1)!!\}^{-1}(pr)^{k-1}\{H_k(r)+h_k(r)\}, \tag{4.4a}$$

$$g_{-k}(r) = \alpha_{-k}\{(2k-1)!!\}^{-1}(pr)^{k-1}\{H_k(r)-h_k(r)\}, \tag{4.4b}$$

$$f_{-k}(r) = -\alpha_{-k}\{(2k-1)!!\}^{-1}(pr)^{k-1}(r/R)\{D_k(r)-d_k(r)\}, \tag{4.4c}$$

$$g_{+k}(r) = \alpha_{+k}\{(2k-1)!!\}^{-1}(pr)^{k-1}(r/R)\{D_k(r)+d_k(r)\}, \tag{4.4d}$$

with the initial conditions

$$H_k(0)=1, \qquad h_k(0)=0, \tag{4.5a}$$

$$|D_k(0)|<\infty, \qquad |d_k(0)|<\infty. \tag{4.5b}$$

Here the main radial dependence has been extracted and the remaining weak r-dependence has been split into functions $H_k(r)$ or $D_k(r)$ consisting of those terms which agree and functions $h_k(r)$ or $d_k(r)$ containing those terms which have different sign for the corresponding similar functions. Furthermore, the notation is to suggest that the functions $H_k(r)$ or $D_k(r)$ will give more prominent contributions than $h_k(r)$ or $d_k(r)$, respectively.

In the case of the uniform model, the first few terms of the expansion in powers of r of these functions may be found, from eqns (3.126) to (3.127), to be

$$H_k(r) = 1 - \{2(2k+1)\}^{-1}(\bar{p}R)^2(r/R)^2 + \ldots, \tag{4.6a}$$

$$h_k(r) = \{4(2k+1)(2k+3)\}^{-1}\alpha ZmR(r/R)^4 + \ldots, \tag{4.6b}$$

$$D_k(r) = (2k+1)^{-1}\bar{W}R - [\{2(2k+1)(2k+3)\}^{-1}\bar{W}R(\bar{p}R)^2 + \{2(2k+3)\}^{-1}\alpha Z](r/R)^2 + \ldots, \tag{4.6c}$$

$$d_k(r) = (2k+1)^{-1}mR - [\{2(2k+1)(2k+3)\}^{-1}mR(\bar{p}R)^2](r/R)^2 + \ldots, \tag{4.6d}$$

where for brevity an 'effective' energy $\bar{W}$ and momentum $\bar{p}$ have been introduced by

$$\bar{W} = W + \tfrac{3}{2}\alpha Z/R, \tag{4.6e}$$

$$\bar{p} = \{\bar{W}^2 - m^2\}^{1/2}. \tag{4.6f}$$

4.2. Beta-decay Coulomb functions suitable for tabulation

The decomposition of the electron radial wave functions according to eqn (4.4) suggests a corresponding decomposition of the terms of the beta transition amplitude, which will be performed in detail later in Section 6.5. It then becomes evident that it is certain combinations of amplitudes α_κ and phases Δ_κ which enter the transition probability or the observables. It would not always be satisfactory, however, to tabulate just these combinations, which still might have a pronounced dependence on the energy T or the nuclear charge Z. Assuming this dependence to be known in advance and explicitly extracting appropriate factors we are led to quantities which have a comparatively weak dependence on the relevant parameters and, in fact, have been tabulated (Behrens and Jänecke 1969). For brevity of presentation we will first define these beta-decay Coulomb functions, as they will be called, and show afterwards that they have the desired properties.

4.2.1. Definition of beta-decay Coulomb functions

The beta-decay Coulomb functions L_0, λ_k, Λ_k, μ_k, $\eta_{kk'}$, $\hat{\eta}_{kk'}$, $\nu_{kk'}$, $\hat{\nu}_{kk'}$ mainly are considered as functions of two independent variables, the nuclear charge Z and the energy T or W or the momentum p, but do also depend to some extend on the details of the charge distribution or potential, in particular on the nuclear radius R. They are defined by the following equations, which are displayed so that the left-hand side of each equation is one of the combinations of amplitudes and phases which occur when the transition probability is rewritten using the decomposition eqn

(4.4). We have

$$(\alpha_{-1}^2+\alpha_{+1}^2)/(2p^2)=F_0L_0 \tag{4.7}$$

where

$$F_0=4(2pR)^{-2(1-\gamma_1)}\exp(\pi y)\,|\Gamma(\gamma_1+iy)|^2/\{\Gamma(2\gamma_1+1)\}^2, \tag{4.8a}$$

with†

$$\gamma_1=\{1-(\alpha Z)^2\}^{1/2}, \tag{4.8b}$$

is the famous Fermi function. It should be noted, however, that the term 'Fermi function' has no unique meaning,‡ but often

$$F=F_0L_0 \tag{4.9}$$

is called 'Fermi function' too. What actually occurs is $F=F_0L_0$ rather than F_0 alone. Nevertheless, the factorization of F is advantageous, for while F and L_0 are complicated functions and do depend on the model for the charge distribution, the function F_0, which already exhibits the main dependence on the energy and the nuclear charge number, is an explicitly well-defined analytic function, independent of the model except for the simple dependence on the nuclear radius R.

If the Fermi function F is extracted as a common factor of the transition probability, the following combinations of amplitudes and phase shifts occur:

$$(\alpha_{-k}^2+\alpha_{+k}^2)/(\alpha_{-1}^2+\alpha_{+1}^2)=\lambda_k, \tag{4.10}$$

$$(\alpha_{-k}^2-\alpha_{+k}^2)/(\alpha_{-k}^2+\alpha_{+k}^2)=(\gamma_k/k)(m/W)\mu_k, \tag{4.11}$$

$$2\alpha_{-k}\alpha_{+k}\cos(\Delta_{-k}-\Delta_{+k})/(\alpha_{-k}^2+\alpha_{+k}^2)=(p/W)\Lambda_k, \tag{4.12}$$

$$\{\alpha_{+k}\alpha_{-k'}\cos(\Delta_{+k}-\Delta_{-k'})+\alpha_{-k}\alpha_{+k'}\cos(\Delta_{-k}-\Delta_{+k'})\}/(\alpha_{-1}^2+\alpha_{+1}^2)=(p/W)\nu_{kk'}, \tag{4.13}$$

$$\{\alpha_{+k}\alpha_{-k'}\cos(\Delta_{+k}-\Delta_{-k'})-\alpha_{-k}\alpha_{+k'}\cos(\Delta_{-k}-\Delta_{+k'})\}/(\alpha_{-1}^2+\alpha_{+1}^2)=(\alpha Z)^2(m/p)\hat{\nu}_{kk'}, \tag{4.14}$$

$$\{\alpha_{-k}\alpha_{-k'}\cos(\Delta_{-k}-\Delta_{-k'})+\alpha_{+k}\alpha_{+k'}\cos(\Delta_{+k}-\Delta_{+k'})\}/(\alpha_{-1}^2+\alpha_{+1}^2)=\eta_{kk'}, \tag{4.15}$$

$$\{\alpha_{-k}\alpha_{-k'}\cos(\Delta_{-k}-\Delta_{-k'})-\alpha_{+k}\alpha_{+k'}\cos(\Delta_{+k}-\Delta_{+k'})\}/(\alpha_{-1}^2+\alpha_{+1}^2)=(m/W)\hat{\eta}_{kk'}. \tag{4.16}$$

† More generally we sometimes have to indicate the k-dependence of various quantities explicitly in this section, writing γ_k in place of γ where $\gamma=\{k^2-(\alpha Z)^2\}^{1/2}$ by eqn (3.19).

‡ It should also be noted that a slightly different definition of L_0 and of F is being used by some other authors, corresponding to radial wave functions evaluated at the nuclear radius R rather than at the origin if our definition (4.7) is rewritten in the form

$$[\{g_{-1}(0)\}^2+\{f_{+1}(0)\}^2]/(2p^2)=F_0L_0.$$

The functions depending on k and k' are needed for $k<k'$ only.† In addition other combinations do occur which, however, are related to the already introduced beta-decay Coulomb functions:

$$\{\alpha_{-k}\alpha_{+k}\cos(\Delta_{-k}-\Delta_{+k})\}/\{\alpha_{-1}\alpha_{+1}\cos(\Delta_{-1}-\Delta_{+1})\}=\lambda_k\Lambda_k/\Lambda_1, \tag{4.17}$$

$$\{2\alpha_{-k}\alpha_{+k}\cos(\Delta_{-k}-\Delta_{+k})\}/\{\alpha_{-1}^2+\alpha_{+1}^2\}=(p/W)\Lambda_k\lambda_k. \tag{4.18}$$

4.2.2. Approximate behaviour of the beta-decay Coulomb functions

We now want to show that, apart from the Fermi function, the beta-decay Coulomb functions just introduced are of the order of unity. For this purpose we use the amplitudes and phases obtained in Section 3.2.3 for the potential of the uniform nuclear charge distribution. Rather than to use the exact expressions, however, we make simplifications, omitting all the terms containing $U(H/E)$ or $U^2(H/E)^2$, as is suggested by the form of presentation of eqns (3.163), (3.165) and (3.166). This is a reasonable approximation because the nuclear radius R is small so that usually $WR\ll 1$ in nuclear beta-decay. It can be shown‡ by a somewhat detailed discussion of eqns (3.162) and (3.167) that roughly

$$|UH/E|\simeq\{(k-1)!/(2k)!\}^2\{(2k+1)(2k+3)\}^{-1}3k^2(W/p)^2(2pR)^{2\nu} \tag{4.19}$$

which is small as compared with unity.

With this simplification the phases Δ become essentially equal to the Coulomb phases Δ^C according to

$$\cos(\Delta)\simeq\mathrm{Sign}(E)\cos(\Delta^C),$$
$$\sin(\Delta)\simeq\mathrm{Sign}(E)\sin(\Delta^C),$$

and the amplitude α_κ is

$$\alpha_\kappa\simeq(2W)^{-1/2}(2p)^\nu\exp\{(\pi/2)y\}\,|\,\Gamma(\nu+iy)\,|\,\{\Gamma(1+2\nu)\}^{-1}\,|E^{-1}| \tag{4.20}$$

where

$$E^{-1}=\{f_{01}(R)g_{02}(R)-g_{01}(R)f_{02}(R)\}/\{f^1(R)g_{02}(R)-g^1(R)f_{02}(R)\}, \tag{4.21}$$

according to eqns (3.141), (3.143) and (3.164). Taking the exact normalization factors of the electron radial wave functions but keeping only the

† The beta-decay Coulomb function $\eta_{kk'}$ should not be confused with the phase $\eta=\eta_\kappa$ of eqn (3.72).

‡ This will be done later in Section 4.5.1.

leading terms from the series† which give unity we obtain

$$E^{-1} \simeq (2k-1)!!(pR)^{1-k}R^{-1+\gamma} * \begin{cases} \mathrm{Sign}(\alpha Z)\{(k+\gamma)(kW-\gamma m)\}^{1/2} & \text{if} \quad \kappa = k \\ \{(k+\gamma)(kW+\gamma m)\}^{1/2} & \text{if} \quad \kappa = -k. \end{cases} \tag{4.22}$$

For brevity we may introduce generalized Fermi functions

$$F_{k-1} = \{k(2k-1)!!\}^2 4^k (2pR)^{2(\gamma-k)} \exp(\pi y)\{|\Gamma(\gamma+iy)|/\Gamma(1+2\gamma)\}^2, \tag{4.23}$$

such that

$$F_{k-1} = 1 + O(\alpha Z) \quad \text{when} \quad \alpha Z \to 0. \tag{4.24}$$

We then have, from eqns (4.20), (4.22) and (4.23),

$$\alpha_\kappa \simeq (F_{k-1})^{1/2} k^{-1} p (2W)^{-1/2} \begin{cases} \{(k+\gamma)(kW-\gamma m)\}^{1/2} & \text{if} \quad \kappa = k \\ \{(k+\gamma)(kW+\gamma m)\}^{1/2} & \text{if} \quad \kappa = -k. \end{cases} \tag{4.25}$$

We may now find

$$\alpha_{-k}^2 + \alpha_{+k}^2 \simeq F_{k-1} p^2 (k+\gamma)/k, \tag{4.26}$$

$$\alpha_{-k}^2 - \alpha_{+k}^2 \simeq F_{k-1} p^2 \{(k+\gamma)\gamma/k^2\}(m/W), \tag{4.27}$$

so that, using again the more detailed notation γ_k rather than γ, we obtain

$$L_0 \simeq (1+\gamma_1)/2, \tag{4.28}$$

$$\mu_k \simeq 1, \tag{4.29}$$

$$\lambda_k \simeq \{F_{k-1}/F_0\}(k+\gamma_k)/\{k(1+\gamma_1)\}. \tag{4.30}$$

It follows immediately that‡

$$L_0 \simeq 1 + O\{(\alpha Z)^2\}, \tag{4.31}$$

$$\lambda_k \simeq 1 + O\{(\alpha Z)^2\}, \tag{4.32}$$

when $\alpha Z \to 0$.

† The numerator, which is a Wronskian and could have been evaluated by means of eqn (3.144), is not exactly accounted for in this way, but this is reasonable since the denominator is also evaluated approximately.

‡ The simplifications made so far in this section mean that some terms of the order of αZWR, αZmR, $(\alpha Z)^2$, $(WR)^2$, $WRmR$, $(mR)^2$, or of even higher order have been neglected as compared to 1. The order symbols like $O\{(\alpha Z)^2\}$ appearing in some of the following equations, eqns (4.31) to (4.59), concern the remaining dependence on αZ only. For this reason we use, for example in eqn (4.31), the sign $\simeq$ rather than $=$.

The other beta-decay Coulomb functions involve differences of phases, which we replace essentially by the Coulomb phases as motivated above.† It then follows that

$$\cos(\Delta_{+k}-\Delta_{-k}) \simeq \mathrm{Sign}(\alpha Z)\cos(\Delta^{C}_{+k}-\Delta^{C}_{-k}). \tag{4.33a}$$

We have, from eqns (3.79) and (3.82),

$$\begin{aligned}\cos(\Delta^{C}_{+k}-\Delta^{C}_{-k}) &= \cos[\eta_{+k}-\eta_{-k}+(\pi/2)\{l(k)-l(-k)\}] \\ &= \cos\{\eta_{+k}-\eta_{-k}+(\pi/2)\} = -\sin(\eta_{+k}-\eta_{-k}),\end{aligned} \tag{4.33b}$$

and, from eqn (3.156),

$$\begin{aligned}\exp\{\mathrm{i}(\eta_{+k}-\eta_{-k})\} &= [\{(k+\gamma)(kW+\gamma m)\}/\{(k-\gamma)(kW-\gamma m)\}]^{1/2} \\ &\quad *\{-(k-\gamma)p-\mathrm{i}\alpha ZT\}/\{(k+\gamma)p-\mathrm{i}\alpha ZT\}\end{aligned}$$

or

$$\begin{aligned}\eta_{+k}-\eta_{-k} &= \arg[\{-(k-\gamma)p-\mathrm{i}\alpha ZT\}\{(k+\gamma)p+\mathrm{i}\alpha ZT\}] \\ &= \arg[-\alpha Z\{\alpha Zm+\mathrm{i}kp\}],\end{aligned}$$

where positive real factors, which are irrelevant for the phase, have been

† We may recall that

$$\cos(\Delta) \simeq \mathrm{Sign}(E)\cos(\Delta^{C}),$$
$$\sin(\Delta) \simeq \mathrm{Sign}(E)\sin(\Delta^{C}).$$

Since

$$\mathrm{Sign}(E) = \begin{cases}\mathrm{Sign}(\alpha Z) & \text{if} \quad \kappa = k \\ 1 & \text{if} \quad \kappa = -k\end{cases}$$

according to eqn (4.22), the phases are approximately equal to the Coulomb phases Δ^{C} except when $\kappa > 0$ while $\alpha Z < 0$. In this exceptional case they differ by a shift of π. This behaviour is related to the fact that with respect to their dependence on the parameter αZ the Coulomb wave functions are not analytic at $\alpha Z = 0$. In fact, for $\alpha Z \to 0$ we have

$$\Delta^{C} \to \eta + (\pi/2)\{l(\kappa)+1-k\},$$

$$\eta \to \begin{cases}-(\pi/2)\mathrm{Sign}(\alpha Z) & \text{if} \quad \kappa = k \\ 0 & \text{if} \quad \kappa = -k,\end{cases}$$

$$l(\kappa)+1-k = \begin{cases}1 & \text{if} \quad \kappa = k \\ 0 & \text{if} \quad \kappa = -k\end{cases}$$

and therefore

$$\Delta^{C} \to \begin{cases}\pi & \text{if} \quad \kappa = k \quad \text{and} \quad \alpha Z \to -0 \\ 0 & \text{if} \quad \kappa = k \quad \text{and} \quad \alpha Z \to +0 \\ 0 & \text{if} \quad \kappa = -k\end{cases}$$

This means that if $\kappa > 0$ while $\alpha Z < 0$ the 'regular' Coulomb wave function in the limit of $\alpha Z \to -0$ tends to the corresponding regular free-particle solution, eqn (3.7), multiplied by the factor (-1). Our wave functions for an extended nuclear charge like eqns (3.130), (3.131) or (4.1) and (4.2) do not show such an anomaly, but behave as expected.

dropped. Restoring the normalization we obtain

$$\exp\{i(\eta_{+k}-\eta_{-k})\} = -\text{Sign}(\alpha Z)\{\alpha Zm + ikp\}/\{(\alpha Z)^2m^2+(kp)^2\}^{1/2}$$

or

$$\cos(\eta_{+k}-\eta_{-k}) = -|\alpha Z|m/\{(\alpha Z)^2m^2+(kp)^2\}^{1/2}, \tag{4.34}$$

$$\sin(\eta_{+k}-\eta_{-k}) = -\text{Sign}(\alpha Z)kp/\{(\alpha Z)^2m^2+(kp)^2\}^{1/2}. \tag{4.35}$$

From eqns (4.35), (4.33), (4.26), (4.25), and (4.12) it now follows that

$$\Lambda_k \simeq 1. \tag{4.36}$$

The investigation of the other beta-decay Coulomb functions is more laborious. We start by rewriting eqn (4.25) in a slightly modified form as

$$\alpha_\kappa \simeq (F_{k-1})^{1/2}k^{-1}p(2W)^{-1/2}\{(\kappa-\gamma)(\kappa W-\gamma m)\}^{1/2}\,|N(\kappa)| \tag{4.37}$$

with

$$N(\kappa) = \begin{cases} (\kappa+\gamma)/(\alpha Z) & \text{if} \quad \kappa = k \\ 1 & \text{if} \quad \kappa = -k \end{cases} \tag{4.38}$$

where

$$\text{Sign}\{N(\kappa)\} = \text{Sign}(E)$$

according to eqn (4.22). Using

$$\exp(i\Delta_\kappa) \simeq \text{Sign}(E)\exp(i\Delta_\kappa^C)$$

we have from eqns (3.154) to (3.156), (4.37) and (4.38)

$$\alpha_\kappa \exp(i\Delta_\kappa) \simeq (F_{k-1})^{1/2}(2k)^{-1}(WT)^{-1/2}p\exp\{-i\,\text{Im}\ln\Gamma(\gamma+iy)\}$$
$$*\exp[i(\pi/2)\{l(\kappa)+1-\gamma\}]N(\kappa)\{-(\kappa-\gamma)p-i\alpha ZT\}. \tag{4.39}$$

Introducing for brevity†

$$V = \{F_0p^2(1+\gamma_1)\}^{-1}(F_{k-1}F_{k'-1})^{1/2}(4kk')^{-1}(WT)^{-1}p^2$$
$$*\exp[i\{-\text{Im}\ln\Gamma(\gamma+iy)+\text{Im}\ln\Gamma(\gamma'+iy)\}], \tag{4.40}$$

$$\Phi(\kappa,\kappa') = \exp[i(\pi/2)\{l(\kappa)-l(\kappa')-\gamma+\gamma'\}], \tag{4.41}$$

$$X(\kappa,\kappa') = \{-(\kappa-\gamma)p-i\alpha ZT\}\{-(\kappa'-\gamma')p+i\alpha ZT\}$$
$$= \{(\kappa-\gamma)(\kappa'-\gamma')p^2+(\alpha Z)^2T^2\}-i\alpha ZTp(\kappa-\gamma-\kappa'+\gamma'), \tag{4.42}$$

we have

$$\{\alpha_{-1}^2+\alpha_{+1}^2\}^{-1}\alpha_\kappa\alpha_{\kappa'}\exp\{i(\Delta_\kappa-\Delta_{\kappa'})\} \simeq V\Phi(\kappa,\kappa')N(\kappa)N(\kappa')X(\kappa,\kappa') \tag{4.43}$$

† We simply write γ' rather than $\gamma_{k'}$.

and therefore

$$(p/W)\nu_{\kappa\kappa'} \simeq \mathrm{Re}[V\{\Phi(k,-k')N(k)N(-k')X(k,-k') + \Phi(-k,k')N(-k)N(k')X(-k,k')\}],$$

$$(\alpha Z)^2(m/p)\hat{\nu}_{kk'} \simeq \mathrm{Re}[V\{\Phi(k,-k')N(k)N(-k')X(k,-k') - \Phi(-k,k')N(-k)N(k')X(-k,k')\}],$$

$$\eta_{kk'} \simeq \mathrm{Re}[V\{\Phi(-k,-k')N(-k)N(-k')X(-k,-k') + \Phi(k,k')N(k)N(k')X(k,k')\}],$$

$$(m/W)\hat{\eta}_{kk'} \simeq \mathrm{Re}[V\{\Phi(-k,-k')N(-k)N(-k')X(-k,-k') - \Phi(k,k')N(k)N(k')X(k,k')\}].$$

Using

$$\Phi(-k,k') = -\Phi(k,-k'),$$

$$\Phi(k,k') = \Phi(-k,-k'),$$

and performing the sums or differences of the terms we obtain

$$(p/W)\nu_{kk'} \simeq \mathrm{Re}\{V\Phi(k,-k')[2\alpha ZTW(k+\gamma-k'-\gamma') - i2Tp\{(\alpha Z)^2+(k+\gamma)(k'+\gamma')\}]\}, \quad (4.44)$$

$$(\alpha Z)^2(m/p)\hat{\nu}_{kk'} \simeq \mathrm{Re}[V\Phi(k,-k')\{-2\alpha ZmT(k+\gamma+k'+\gamma')\}], \quad (4.45)$$

$$\eta_{kk'} \simeq \mathrm{Re}\{V\Phi(-k,-k')[2TW\{(k+\gamma)(k'+\gamma')+(\alpha Z)^2\} + i2\alpha ZTp(k+\gamma-k'-\gamma')]\}, \quad (4.46)$$

$$(m/W)\hat{\eta}_{kk'} \simeq \mathrm{Re}[V\Phi(-k,-k')2mT\{(k+\gamma)(k'+\gamma')-(\alpha Z)^2\}]. \quad (4.47)$$

We may now examine the behaviour, when $\alpha Z \to 0$, of these expressions, eqns (4.44) to (4.47) including the terms up to $O(\alpha Z)$. We have†

$$\Phi(k,-k') = \exp\{i(\pi/2)(k-k'+1-\gamma+\gamma')\} = i[1+iO\{(\alpha Z)^2\}],$$

$$\Phi(-k,-k') = \exp\{i(\pi/2)(k-k'-\gamma+\gamma')\} = 1+iO\{(\alpha Z)^2\},$$

$$\exp[i\{-\mathrm{Im}\ln\Gamma(\gamma+iy)+\mathrm{Im}\ln\Gamma(\gamma'+iy)\}] = 1+iy\{-\psi(\gamma)+\psi(\gamma')\}+O\{(\alpha Z)^2\} = 1+iy\{-\psi(k)+\psi(k')\}+O\{(\alpha Z)^2\} = 1+iy\sum_{n=k}^{k'-1}(1/n)+O\{(\alpha Z)^2\},$$

$$V = (8kk'WT)^{-1}\left\{1+iy\sum_{n=k}^{k'-1}(1/n)\right\}+O\{(\alpha Z)^2\}.$$

† We use $\psi(z) = \{d\ln\Gamma(z)\}/dz$.

Keeping the low order terms we obtain from eqns (4.44) to (4.47)

$$\nu_{kk'} \simeq 1 + O\{(\alpha Z)^2\},$$

$$\hat{\nu}_{kk'} \simeq \{(k+k')/(2kk')\}\left\{\sum_{n=k}^{k'-1}(1/n)\right\}\{1+O(\alpha Z)\},$$

$$\eta_{kk'} \simeq 1 + O\{(\alpha Z)^2\},$$

$$\hat{\eta}_{kk'} \simeq 1 + O\{(\alpha Z)^2\}.$$

Regarding $\hat{\nu}_{kk'}$, we do not have $\hat{\nu}_{kk'} \simeq 1 + O(\alpha Z)$, not even in the case of $\hat{\nu}_{12} \simeq \frac{3}{4} + O(\alpha Z)$. A modified definition of $\hat{\nu}_{kk'}$, therefore, would be more reasonable. The deviation from the desired behaviour, in particular in case of $\hat{\nu}_{12}$, which alone seems to be important in applications, is not, however, relevant enough to justify a change of definition as compared with the tables of Behrens and Jänecke (1969).

4.2.3. Some less important beta-decay Coulomb functions

Besides the beta-decay Coulomb functions introduced so far, there are other ones occurring in the formulas for certain observables which are more difficult to measure and therefore have played a minor role in the past. These beta-decay Coulomb functions contain phase shift differences and correspond to those considered above in so far as the sine of the phase shift difference occurs everywhere in place of the cosine. Since some of them show a more complicated behaviour, we deviate from our previous policy of introducing beta-decay Coulomb functions which are of the order of unity, but define symbols for the combinations of phase shifts and amplitudes just as they occur without extracting any energy or Z-dependent factors. The beta-decay Coulomb functions in question are

$$sl_k = 2\alpha_{-k}\alpha_{+k}\sin(\Delta_{-k} - \Delta_{+k})/(\alpha_{-k}^2 + \alpha_{+k}^2), \tag{4.48}$$

$$sn_{kk'} = \{\alpha_{+k}\alpha_{-k'}\sin(\Delta_{+k} - \Delta_{-k'}) + \alpha_{-k}\alpha_{+k'}\sin(\Delta_{-k} - \Delta_{+k'})\}/(\alpha_{-1}^2 + \alpha_{+1}^2), \tag{4.49}$$

$$s\hat{n}_{kk'} = \{\alpha_{+k}\alpha_{-k'}\sin(\Delta_{+k} - \Delta_{-k'}) - \alpha_{-k}\alpha_{+k'}\sin(\Delta_{-k} - \Delta_{+k'})\}/(\alpha_{-1}^2 + \alpha_{+1}^2), \tag{4.50}$$

$$se_{kk'} = \{\alpha_{-k}\alpha_{-k'}\sin(\Delta_{-k} - \Delta_{-k'}) + \alpha_{+k}\alpha_{+k'}\sin(\Delta_{+k} - \Delta_{+k'})\}/(\alpha_{-1}^2 + \alpha_{+1}^2), \tag{4.51}$$

$$s\hat{e}_{kk'} = \{\alpha_{-k}\alpha_{-k'}\sin(\Delta_{-k} - \Delta_{-k'}) - \alpha_{+k}\alpha_{+k'}\sin(\Delta_{+k} - \Delta_{+k'})\}/(\alpha_{-1}^2 + \alpha_{+1}^2). \tag{4.52}$$

Their approximate behaviour may be found in a similar way as above. Because of

$$sl_k = (p/W)\Lambda_k \operatorname{tg}(\Delta_{-k} - \Delta_{+k}) \tag{4.53}$$

we need from eqns (4.33) to (4.35) the relation

$$\operatorname{tg}(\Delta^C_{-k} - \Delta^C_{+k}) = \cot(\eta_{+k} - \eta_{-k}) = (1/k)\alpha Z(m/p) \tag{4.54}$$

in order to show that

$$sl_k \simeq (1/k)\alpha Z(m/W)[1 + O\{(\alpha Z)^2\}]. \tag{4.55}$$

In order to evaluate the quantities, eqns (4.49) to (4.52), we may look at the right-hand sides of the corresponding eqns (4.44) to (4.47), but taking the imaginary part in place of the real part. It then follows that

$$sn_{kk'} \simeq -\left[\{(k'-k)/(2kk')\} + \sum_{n=k}^{k'-1} (1/n)\right]\alpha Z\{1 + O(\alpha Z)\}, \tag{4.56}$$

$$s\hat{n}_{kk'} \simeq -\{(k+k')/(2kk')\}\alpha Z(m/W)[1 + O\{(\alpha Z)^2\}], \tag{4.57}$$

$$se_{kk'} \simeq \alpha Z\left[\left\{\sum_{n=k}^{k'-1} (1/n)\right\}(W/p) - \{(k'-k)/(2kk')\}(p/W)\right]\{1 + O(\alpha Z)\}, \tag{4.58}$$

$$s\hat{e}_{kk'} \simeq \left\{\sum_{n=k}^{k'-1} (1/n)\right\}\alpha Z(m/p)\{1 + O(\alpha Z)\}. \tag{4.59}$$

4.3. Expansion of the radial dependence of the wave functions

4.3.1. The differential equations and their solution

We are now going to investigate the functions $H_k(r)$, $h_k(r)$, $D_k(r)$, $d_k(r)$ which contain the non-trivial radial dependence of the wave functions. For this purpose we first want to obtain a system of linear differential equations of which these functions are a solution. Substituting eqn (4.4) into the Dirac differential eqn (3.1), for $\kappa = k$ and $\kappa = -k$ separately, and dropping common factors we have

$$\begin{aligned}
r\{H_k' + h_k'\} + \{Tr - rV(r)\}(r/R)\{D_k + d_k\} &= 0, \\
r[(r/R)\{D_k + d_k\}]' + 2k(r/R)\{D_k + d_k\} - \{(T+2m)r - rV(r)\}\{H_k + h_k\} &= 0, \\
-r[(r/R)\{D_k - d_k\}]' - 2k(r/R)\{D_k - d_k\} + \{Tr - rV(r)\}\{H_k - h_k\} &= 0, \\
r\{H_k' - h_k'\} + \{(T+2m)r - rV(r)\}(r/R)\{D_k - d_k\} &= 0.
\end{aligned}$$

Forming suitable linear combinations of these differential equations, such that each of the resulting equations contains the derivative of one function only, we obtain

$$rH_k' + \lambda[\{Wr - rV(r)\}(r/R)D_k - mr(r/R)d_k] = 0, \tag{4.60a}$$

$$rh_k' + \lambda[-mr(r/R)D_k + \{Wr - rV(r)\}(r/R)d_k] = 0, \tag{4.60b}$$

$$r\{(r/R)D_k\}' + 2k(r/R)D_k - \lambda[\{Wr - rV(r)\}H_k + mrh_k] = 0, \tag{4.60c}$$

$$r\{(r/R)d_k\}' + 2k(r/R)d_k - \lambda[mrH_k + \{Wr - rV(r)\}h_k] = 0. \tag{4.60d}$$

Here the factor λ is equal to 1. It has been introduced for later use in order to avoid rewriting the equations.

What is needed now is a suitable expansion of the functions $H_k(r)$, $D_k(r)$, $h_k(r)$, $d_k(r)$ such that each of them, when multiplied by the nuclear wave functions (and powers of r from the neutrino radial wave function) may be integrated term by term from 0 to ∞. Simple expansions in powers of r like eqn (4.6), which are based on the assumption that the potential has been expanded in powers of r too, are not satisfactory. The reason is that then the potential of a uniformly charged nucleus of radius R, which is very useful otherwise, could not be used since the power series expansion of the associated electron radial wave functions is valid on the finite interval $0 \leqslant r \leqslant R$ only. With the potential of a smooth nuclear charge distribution like the Fermi distribution (eqn (3.207) with $R_1 = \infty$) the trouble is essentially the same.† Because of the poles of this potential in the finite part of the complex plane, the power series expansion of the electron radial wave functions converges for $0 \leqslant r < \{c^2 + (\pi/\beta)^2\}^{1/2}$ only. Even with the potential of the Gaussian distribution (eqn (3.197) with $R_1 = \infty$) the difficulties still persist although the power series expansion of the electron radial wave functions converges everywhere, but the convergence of the integrand is not uniform near $r = \infty$, so that integration of the series term by term is a dubious procedure. In fact, it can be shown that the resulting series would be divergent (Behrens and Bühring 1970). For these reasons we are looking for another type of expansion of the electron radial wave functions, such that the expansion of the potential is avoided.

We now may make use of the factor λ introduced in the differential eqn (4.60) and expand in powers of λ the solutions

$$H_k(r) = \sum_{n=0}^{\infty} \lambda^n H_{kn}(r), \tag{4.61a}$$

† This fact looks quite reasonable since the uniform distribution may be thought of as being the limiting case of a Fermi distribution when $\beta \to \infty$.

$$h_k(r) = \sum_{n=0}^{\infty} \lambda^n h_{kn}(r), \tag{4.61b}$$

$$D_k(r) = \sum_{n=0}^{\infty} \lambda^n D_{kn}(r), \tag{4.61c}$$

$$d_k(r) = \sum_{n=0}^{\infty} \lambda^n d_{kn}(r). \tag{4.61d}$$

Substituting these expansions into the differential eqn (4.60), rearranging the terms according to their powers of λ, and equating the factor of each power of λ equal to zero, we obtain for $n = 0$ the homogeneous differential equations

$$rH'_{k0} = 0,$$
$$rh'_{k0} = 0,$$
$$r\{(r/R)D_{k0}\}' + 2k(r/R)D_{k0} = 0,$$
$$r\{(r/R)d_{k0}\}' + 2k(r/R)d_{k0} = 0,$$

and for $n = 1, 2, 3 \ldots$ the recursive system of inhomogeneous differential equations

$$rH'_{kn} = -\{Wr - rV(r)\}(r/R)D_{kn-1} + mr(r/R)d_{kn-1}, \tag{4.62a}$$

$$rh'_{kn} = mr(r/R)D_{kn-1} - \{Wr - rV(r)\}(r/R)d_{kn-1}, \tag{4.62b}$$

$$r\{(r/R)D_{kn}\}' + 2k(r/R)D_{kn} = \{Wr - rV(r)\}H_{kn-1} + mrh_{kn-1}, \tag{4.62c}$$

$$r\{(r/R)d_{kn}\}' + 2k(r/R)d_{kn} = mrH_{kn-1} + \{Wr - rV(r)\}h_{kn-1}. \tag{4.62d}$$

The general solution of the equations for $n = 0$ is

$$H_{k0} \equiv C_1, \qquad h_{k0} \equiv C_2,$$
$$(r/R)D_{k0} = C_3 r^{-2k},$$
$$(r/R)d_{k0} = C_4 r^{-2k},$$

where with $j = 1, 2, 3, 4$ all the C_j are constants of integration. Because of our initial condition, eqn (4.5), this reduces to

$$H_{k0}(r) \equiv 1, \qquad h_{k0}(r) = D_{k0}(r) = d_{k0}(r) \equiv 0. \tag{4.63}$$

Integration of the equations for $n \geqslant 1$ yields

$$H_{kn}(r) = \int_0^r [-\{W - V(x)\}(x/R)D_{kn-1}(x) + m(x/R)d_{kn-1}(x)]\,\mathrm{d}x, \tag{4.64a}$$

$$h_{kn}(r) = \int_0^r [m(x/R)D_{kn-1}(x) - \{W - V(x)\}(x/R)d_{kn-1}(x)]\,dx, \tag{4.64b}$$

$$(r/R)D_{kn}(r) = r^{-2k} \int_0^r [\{W - V(x)\}H_{kn-1}(x) + mh_{kn-1}(x)]x^{2k}\,dx, \tag{4.64c}$$

$$(r/R)d_{kn}(r) = r^{-2k} \int_0^r [mH_{kn-1}(x) + \{W - V(x)\}h_{kn-1}(x)]x^{2k}\,dx. \tag{4.64d}$$

Here the lower limit of the integrals is due to the initial conditions

$$H_{kn}(0) = h_{kn}(0) = 0, \tag{4.65a}$$

$$|D_{kn}(0)| < \infty, \qquad |d_{kn}(0)| < \infty, \tag{4.65b}$$

a consequence of eqn (4.5).

Using in place of the potential $V(x)$ the function $U(x)$ defined by

$$V(x) = -(\alpha Z/R)U(x) \tag{4.66}$$

and introducing the matrix function

$$K(r, x) = R^{-1} \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ (x/r)^{2k}\{WR + \alpha ZU(x)\} & (x/r)^{2k}mR \\ (x/r)^{2k}mR & (x/r)^{2k}\{WR + \alpha ZU(x)\} \\ -\{WR + \alpha ZU(x)\} & mR \\ mR & -\{WR + \alpha ZU(x)\} \\ 0 & 0 \\ 0 & 0 \end{Bmatrix} \tag{4.67}$$

we may rewrite eqn (4.64), for $n = 1, 2, 3, \ldots$, in the form

$$\begin{Bmatrix} H_{kn}(r) \\ h_{kn}(r) \\ (r/R)D_{kn}(r) \\ (r/R)d_{kn}(r) \end{Bmatrix} = \int_0^r K(r, x) \begin{Bmatrix} H_{kn-1}(x) \\ h_{kn-1}(x) \\ (x/R)D_{kn-1}(x) \\ (x/R)d_{kn-1}(x) \end{Bmatrix} dx. \tag{4.68}$$

By repeated use of eqn (4.68) with the initial functions (4.63) the complete expansion of the solution of eqn (4.60) in powers of $\lambda = 1$ may be constructed:

$$\begin{Bmatrix} H_k(r) \\ h_k(r) \\ (r/R)D_k(r) \\ (r/R)d_k(r) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \sum_{n=1}^{\infty} \int_0^r K^n(r, x)\,dx \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.69}$$

where

$$K^n(r, x) = K(r, x) \int_0^x K^{n-1}(x, y)\, dy \tag{4.70}$$

for $n = 2, 3, 4, \ldots$ and

$$K^1(r, x) = K(r, x). \tag{4.71}$$

4.3.2. Convergence of the expansion of the solution

In order to prove the convergence of the expansion we may look at eqn (4.69) as a vector equation

$$Y(r) = Y_0(r) + \sum_{n=1}^{\infty} \int_0^r K^n(r, x)\, dx\, Y_0(r) \tag{4.72}$$

for the four dimensional vector function†

$$Y(r) = \{Y_1(r), Y_2(r), Y_3(r), Y_4(r)\}^t = \{H_k(r), h_k(r), (r/R)D_k(r), (r/R)d_k(r)\}^t$$

with

$$Y_0(r) \equiv \{1, 0, 0, 0\}^t.$$

Using a suitable vector norm $\|Y(r)\|$ and an associated matrix norm $\|K^n(r, x)\|$ we may obtain from eqns (4.72) and (4.70) the estimates

$$\|Y(r)\| \leq \|Y_0(r)\| + \sum_{n=1}^{\infty} \int_0^r \|K^n(r, x)\|\, dx\, \|Y_0(r)\|, \tag{4.73}$$

$$\|K^n(r, x)\| \leq \|K(r, x)\| \int_0^x \|K^{n-1}(x, y)\|\, dy. \tag{4.74}$$

As a vector norm we may choose the length

$$\|Y(r)\|_2 = \left[\sum_{j=1}^{4} \{Y_j(r)\}^2\right]^{1/2}$$

of the vector.‡ The associated matrix norm is the spectral norm. We then

† The symbol t in $\{.,.,.,.\}^t$ means transposed and is to indicate that $Y(r)$ is a column vector rather than a row vector.

‡ Alternatively we might choose any of the vector norms

$$\|Y(r)\|_1 = \sum_{j=1}^{4} |Y_j(r)| \quad \text{or} \quad \|Y(r)\|_\infty = \max_j |Y_j(r)|.$$

Owing to the special structure of the matrix $K(r, x)$, its norm has the same value in all the three cases.

find

$$\|K(r,x)\| = \begin{cases} m + |W + (\alpha Z/R)U(x)| & \text{if } 0 \leq x \leq r \\ (x/r)^{2k}\{m + |W + (\alpha Z/R)U(x)|\} & \text{if } r \leq x \end{cases} \tag{4.75}$$

Since $K(r,x)$ and all the $K^n(r,x)$ are needed for $0 \leq x \leq r$ only, we may use the estimate

$$\|K(r,x)\| \leq \tilde{K}(x)$$

where

$$\tilde{K}(x) = m + |W + (\alpha Z/R)U(x)| \tag{4.76}$$

is independent of r. From eqn (4.74) with $n=2$ we then obtain

$$\|K^2(r,x)\| \leq \tilde{K}(x) \int_0^x \tilde{K}(y)\,dy$$

and by induction

$$\|K^n(r,x)\| \leq \tilde{K}(x)\{(n-1)!\}^{-1}\left\{\int_0^x \tilde{K}(y)\,dy\right\}^{n-1}$$

for $n \geq 2$. Finally, because $\|Y_0(r)\| \equiv 1$, eqn (4.73) yields

$$\|Y(r)\| \leq 1 + \sum_{n=1}^{\infty} (n!)^{-1}\left\{\int_0^r \tilde{K}(x)\,dx\right\}^n = \exp\left\{\int_0^r \tilde{K}(x)\,dx\right\}, \tag{4.77a}$$

where

$$\int_0^r \tilde{K}(x)\,dx \leq (W+m)r + (|\alpha Z|/R)\int_0^r |U(x)|\,dx. \tag{4.77b}$$

Defining positive constants

$$\bar{U} = \max_{0 \leq x \leq \infty} |U(x)| \tag{4.78a}$$

and

$$\mu = W + m + (|\alpha Z|/R)\bar{U} \tag{4.78b}$$

we have

$$\int_0^r \tilde{K}(x)\,dx \leq \mu r \tag{4.78c}$$

and consequently

$$\|Y(r)\| \leq \exp(\mu r). \tag{4.78d}$$

We therefore have proved that the expansion (4.69) converges, however large W and the strength $|\alpha Z|$ of the potential might be, as long as the potential remains bounded on the positive real axis. We also have shown that on the positive real r-axis† the solution $Y(r)$ is bounded by

$$\exp\left\{(W+m)r+(|\alpha Z|/R)\int_0^r |U(x)|\,dx\right\}$$

or by $\exp(\mu r)$ with μ from eqns (4.78a–b).

4.3.3. Convergence of the expansion of the transition amplitude

It is the product of the electron and neutrino radial wave functions multiplied by the product of the nuclear radial wave functions which has to be integrated from zero to infinity. With $m=0$ and $\alpha Z=0$ it follows from the estimate for the electron, eqn (4.78), that the non-trivial radial dependence of the neutrino wave function is bounded by $\exp(W_\nu r)$ if W_ν denotes the energy or momentum of the neutrino. Since $W+W_\nu=W_0$, the product of the lepton wave functions essentially is bounded by

$$\exp(\mu_0 r)=\exp[\{W_0+m+(|\alpha Z|/R)\bar{U}\}r], \tag{4.79}$$

where now the maximal electron energy or transition energy W_0 enters rather than W. The relevant behaviour of the product of the nuclear wave functions may be represented by $\exp(-\lambda r)$ with a suitable positive constant λ. Replacing the lepton wave functions by the bound, eqn (4.79), and omitting irrelevant factors (even positive powers of r depending on the angular momentum quantum numbers) we have to consider the integral

$$J=\int_0^\infty \exp(\mu_0 r)\exp(-\lambda r)\,dr, \tag{4.80}$$

which exists provided that

$$\mu_0<\lambda \tag{4.81}$$

and then has the value

$$J=(\lambda-\mu_0)^{-1}. \tag{4.82}$$

Expansion of the lepton part and integration of the series term by term, on the other hand, yields

$$J=\sum_{n=0}^{\infty}(n!)^{-1}\mu_0^n\int_0^\infty r^n\exp(-\lambda r)\,dr=\sum_{n=0}^{\infty}\mu_0^n\lambda^{-n-1},$$

† This bound, in fact, holds in a considerably larger domain of the complex plane.

a series which converges just under the condition, eqn (4.81), necessary for the existence of the integral, eqn (4.80), and then has as its sum the value, eqn (4.82), of the integral.

We therefore have shown that with the expansion, eqn (4.69), of the electron radial wave functions the usual procedure of forming nuclear matrix elements by integrating the series term by term is certainly justified if

$$W_0 + m + (|\alpha Z|/R)\bar{U} < \lambda$$

or

$$(W_0 + m)R + |\alpha Z|\bar{U} < \lambda R. \tag{4.83a}$$

Since we have $\bar{U} = U(0) = \frac{3}{2}$ for the uniform model, $W_0 R \ll 1$ usually holds in nuclear beta-decay, and λ is related to the nuclear radius according to $\lambda R = 1$, we may conclude that there are no convergence problems in the case of sufficiently light nuclei. However, we expect that this is true even for the heavy nuclei, since eqn (4.83*a*) is a sufficient condition for convergence, but not a necessary one, for we have used in our proof strict bounds on $U(r)$ rather than appropriate mean values.

In the case of the uniform model this expectation can be supported in the following way: outside the nucleus the electron radial wave function is a linear combination of any two linearly independent Coulomb wave functions, the behaviour of which in a neighbourhood of infinity is known from Section 3.2.2.4. Apart from irrelevant factors, the electron radial wave function is therefore essentially bounded by $\exp(p\,|r|)$, and this bound holds uniformly in the whole complex plane irrespective of the phase of r. With the neutrino wave function bounded accordingly by $\exp(W_\nu\,|r|)$, we may conclude, because $p + W_\nu \leqslant W_0$, that finally the transition energy W_0 rather than $\mu_0 = W_0 + m + (|\alpha Z|/R)\bar{U}$ enters the bound for the product of the lepton wave functions, so that the condition justifying term by term integration in place of eqn (4.83*a*) becomes

$$W_0 R < \lambda R \tag{4.83b}$$

and therefore is always satisfied in nuclear beta-decay.

In the case of potentials originating from more refined analytic charge distributions, the behaviour of the potential on the positive real axis is not significantly changed nor are the integrals of the potential, so that the condition for term by term integration is expected to remain $W_0 R < \lambda R$. This statement applies to the expansion of the electron wave function, eqn (4.69), which contains the potential function unexpanded. Here the potential function enters 'parametrically' and its behaviour on the positive real axis only is relevant for the convergence question, while it does not matter that the bound $\exp(p\,|r|)$ for the wave function does not hold in

the whole complex plane but only in a certain sector corresponding to sufficiently small values of arg(r).

The power series expansion of the potential, on the other hand, would reflect the behaviour of the potential in the whole complex plane, where it generally does not remain bounded. Consequently the corresponding total power series expansion of the wave function too would reflect this singular behaviour of the potential at infinity, with the consequence that term by term integration is no longer justified, and, in fact, would lead to a divergent series.

We may discuss this matter in detail by using the more complicated bound

$$\exp\left\{(W_0+m)r+(|\alpha Z|/R)\int_0^r |U(x)|\,\mathrm{d}x\right\}$$

from eqn (4.77) in place of eqn (4.79) for the product of the lepton radial wave functions. The integral corresponding to eqn (4.80) is then

$$\tilde{J}=\int_0^\infty \exp\left\{(W_0+m)r+(|\alpha Z|/R)\int_0^r |U(x)|\,\mathrm{d}x\right\}\exp(-\lambda r)\,\mathrm{d}r.$$

Expanding the lepton part in powers of the potential we have

$$\tilde{J}=\int_0^\infty \sum_{n=0}^{\infty}(n!)^{-1}\left\{(W_0+m)r+(|\alpha Z|/R)\int_0^r |U(x)|\,\mathrm{d}x\right\}^n \exp(-\lambda r)\,\mathrm{d}r.$$

Since $U(x)$, when $x\to\infty$, vanishes like (R/x) or even faster, for every $\varepsilon>0$ an $R_0>0$ exists such that

$$\int_0^r |U(x)|\,\mathrm{d}x<\varepsilon r \quad \text{if} \quad r\geqslant R_0.$$

Term by term integration of the series therefore is certainly justified if

$$W_0+m+(|\alpha Z|/R)\varepsilon<\lambda.$$

However, since, with R_0 chosen sufficiently large, ε can be made arbitrarily small, the condition finally becomes

$$W_0+m<\lambda$$

in accordance† with our earlier conclusion, eqn (4.83b).

We are now going to look at the situation where the potential has been expanded in powers of r, so that the total power series expansion of the lepton radial wave functions is under consideration. In this case we have to refer to a specific model of the potential in order to discuss the

† Apart from the fact that here W_0+m appears in place of W_0.

convergence question. While the potential of a Gaussian charge distribution according to eqn (3.197) with $A=0$ and $R_1=\infty$ could be used without difficulty, it may nevertheless be advantageous to use the following potential

$$V(r)=-(\alpha Z/r)[1-\{1+\tfrac{1}{2}\nu r\}\exp(-\nu r)]$$

corresponding to the simple nuclear charge distribution

$$\rho(r)=-\tfrac{1}{2}\alpha Z\nu^3\exp(-\nu r)$$

with the parameter ν related to the nuclear radius R (defined as the equivalent uniform radius) by

$$\nu R=2\sqrt{5}.$$

The reason for preferring this model in the present context is that its exponential decay $\exp(-\nu r)$, as opposed to $\exp\{-(r/a)^2\}$ in the case of the Gaussian distribution, fits better with our model for the decay of the nuclear wave functions $\exp(-\lambda r)$.

It is now not necessary to investigate the complete sum in the integrand of $\tilde{J}$, but it suffices to consider the terms linear in αZ, which may be written

$$(|\alpha Z|/R)\tilde{J}_1$$

with

$$\tilde{J}_1=\int_0^\infty \sum_{n=1}^{\infty}\{(n-1)!\}^{-1}\{(W_0+m)r\}^{n-1}\int_0^r |U(x)|\,\mathrm{d}x\exp(-\lambda r)\,\mathrm{d}r$$

Inserting the power series expansion of

$$|U(x)|=U(x)=(R/x)[1-\{1+\tfrac{1}{2}\nu x\}\exp(-\nu x)],$$

which yields

$$\int_0^r U(x)\,\mathrm{d}x=\tfrac{1}{2}\nu R\sum_{m=1}^{\infty}(-1)^m\{(m-2)/m\}(m!)^{-1}(\nu r)^m,$$

we obtain

$$\tilde{J}_1=\tfrac{1}{2}\nu R\int_0^\infty\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}(\lambda r)^{m+n-1}\exp(-\lambda r)\,\mathrm{d}r$$

with

$$A_{nm}=(-1)^m\{(m-2)/m\}\{(n-1)!\,m!\}^{-1}\{(W_0+m)/\lambda\}^{n-1}(\nu/\lambda)^m.$$

Since

$$\max_{0\leqslant r\leqslant\infty}\{(\lambda r)^{m+n-1}\exp(-\lambda r)\}=(m+n-1)^{m+n-1}\exp\{-(m+n-1)\}$$

increases faster than A_{nm} decreases for m or n or both large, the series does not uniformly converge, and in fact, because

$$\lambda \int_0^r (\lambda r)^{m+n-1} \exp(-\lambda r)\, dr = (m+n-1)!$$

term by term integration would yield a divergent series.

4.3.4. Explicit form of the expansion of the radial dependence

We are now going to look at the expansion (4.69) in more detail. Up to and including terms of order $n = 1$, the solution is given explicitly by

$$H_k(r) = 1 + \ldots, \tag{4.84}$$

$$h_k(r) = 0 + \ldots, \tag{4.85}$$

$$D_k(r) = (2k+1)^{-1} WR + \alpha Z r^{-1} \int_0^r (x/r)^{2k} U(x)\, dx + \ldots, \tag{4.86}$$

$$d_k(r) = (2k+1)^{-1} mR + \ldots. \tag{4.87}$$

The expansion with respect to different orders n can be regarded as a simultaneous expansion in powers of the three parameters (αZ), (WR), and (mR), where all the terms $(mR)^a (WR)^b (\alpha Z)^c$ are grouped together with the total power $a+b+c$ equal to n. Owing to the special structure of the matrix $K(r, x)$, which has zero submatrices in its diagonal, the functions $H_k(r)$ and $h_k(r)$ contain terms of even order n and the functions $D_k(r)$ and $d_k(r)$ terms of odd order n only. Furthermore, the functions $H_k(r)$ and $D_k(r)$ contain even powers of (mR) and the functions $h_k(r)$ and $d_k(r)$ odd powers of (mR) only. Finally, $h_k(r)$ would vanish identically if $U(r)$ did, and therefore does not have terms independent of αZ. Thus we may write

$$H_k(r) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \sum_{\rho=0}^{2\nu} \frac{(2k-1)!!}{(2\mu)!!(2\mu+2k-1)!!} (-1)^{\nu} \binom{\mu}{\nu} \binom{2\nu}{\rho} (r/R)^{2\mu} * I(k, 2\mu, 2\nu, \rho; r)(mR)^{2\mu-2\nu} (WR)^{2\nu-\rho} (\alpha Z)^{\rho}, \tag{4.88}$$

$$h_k(r) = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\mu} \sum_{\rho=1}^{2\nu-1} \frac{(2k-1)!!}{(2\mu)!!(2\mu+2k-1)!!} (-1)^{\nu} \binom{\mu}{\nu} \binom{2\nu-1}{\rho} * (r/R)^{2\mu} I(k, 2\mu, 2\nu-1, \rho; r)(mR)^{2\mu-2\nu+1} * (WR)^{2\nu-1-\rho} (\alpha Z)^{\rho}, \tag{4.89}$$

$$D_k(r) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \sum_{\rho=0}^{2\nu+1} \frac{(2k-1)!!}{(2\mu)!!(2\mu+2k+1)!!} (-1)^{\nu} \binom{\mu}{\nu} \binom{2\nu+1}{\rho} * (r/R)^{2\mu} I(k, 2\mu+1, 2\nu+1, \rho; r)(mR)^{2\mu-2\nu} * (WR)^{2\nu+1-\rho} (\alpha Z)^{\rho}, \tag{4.90}$$

$$d_k(r) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \sum_{\rho=0}^{2\nu} \frac{(2k-1)!!}{(2\mu)!!(2\mu+2k+1)!!} (-1)^{\nu} \binom{\mu}{\nu} \binom{2\nu}{\rho} * (r/R)^{2\mu} I(k, 2\mu+1, 2\nu, \rho; r)(mR)^{2\mu+1-2\nu} * (WR)^{2\nu-\rho}(\alpha Z)^{\rho}. \quad (4.91)$$

The coefficients of these expansions are functions of r which, after some factors have been extracted for later convenience, are denoted by $I(k, n, \sigma, \rho; r)$. They depend on the parameters k, n, σ, ρ, which, apart from k, specify the various terms in the expansions to which these functions are associated. The order n gives the total power of (mR), (WR), and (αZ), the integer σ the total power of (WR) and (αZ), and the integer ρ the power of (αZ). The functions with $\rho = 0$ are trivial:

$$I(k, n, \sigma, 0; r) \equiv 1. \quad (4.92)$$

The functions with $\rho > 0$ up to order $n = 3$ are shown in Tables 4.1 and 4.2. Table 4.1 gives the original definition as obtained by comparison of eqns (4.67) to (4.71) with the expansions (4.88) to (4.91). Here some of the integrations can be performed without knowledge of $U(x)$. Table 4.2 contains the accordingly simplified expressions. By extracting appropriate factors in eqns (4.88) to (4.91), we have defined the functions $I(k, n, \sigma, \rho; r)$ in such a way that they become equal to unity identically in r when $\rho = 0$. Also, if $U(r)$ were equal to unity identically, then all the functions which $\rho > 0$ would also be equal to unity except for $I(k, 2\mu, 2\nu-1, \rho; r)$ which would vanish. If, as usual, $U(r)$ decreases as $1/r$ for $r \to \infty$, then all the functions with $\rho > 0$ decrease as $1/r$ or faster for $r \to \infty$. At $r = 0$, on the other hand, they are equal to $\{U(0)\}^{\rho}$ except for $I(k, 2\mu, 2\nu-1, \rho; 0)$ which is equal to zero. The relevant functions $I(k, n, \sigma, \rho; r)$ for the simple uniform model without screening, eqn (3.122), a case where all the integrals of Table 4.2 become elementary, are shown in Table 4.3.

4.4. Dependence of nuclear beta-decay on the model for the charge distribution

4.4.1. Model dependence via the functions $I(k, n, \sigma, \rho; r)$

The quantity $U(0)$, which determines the behaviour near $r = 0$ for most of the functions $I(k, n, \sigma, \rho; r)$, is expected to be sensitive to the detailed shape of the nuclear charge distribution. It is well-known from electron scattering (Überall 1971) that the most relevant shape parameter of any

TABLE 4.1 *Original definition of the functions $I(k, n, \sigma, \rho; r)$ up to order $n = 3$*

$$I(k,1,1,1;r) = (2k+1)r^{-2k-1}\int_0^r x^{2k}U(x)\,dx$$

$$I(k,2,2,2;r) = 2(2k+1)r^{-2}\int_0^r U(y)y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy$$

$$I(k,2,2,1;r) = (2k+1)r^{-2}\left\{\int_0^r U(y)y^{-2k}\int_0^y x^{2k}\,dx\,dy + \int_0^r y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\right\}$$

$$I(k,2,1,1;r) = 2(2k+1)r^{-2}\left\{\int_0^r U(y)y^{-2k}\int_0^y x^{2k}\,dx\,dy - \int_0^r y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\right\}$$

$$I(k,3,3,3;r) = 2(2k+1)(2k+3)r^{-2k-3}\int_0^r z^{2k}U(z)\int_0^z U(y)y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\,dz$$

$$\begin{aligned} I(k,3,3,2;r) = \tfrac{2}{3}(2k+1)(2k+3)r^{-2k-3}\Bigg\{&\int_0^r z^{2k}U(z)\int_0^z U(y)y^{-2k}\int_0^y x^{2k}\,dx\,dy\,dz \\ &+ \int_0^r z^{2k}U(z)\int_0^z y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\,dz \\ &+ \int_0^r z^{2k}\int_0^z U(y)y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\,dz\Bigg\} \end{aligned}$$

$$\begin{aligned} I(k,3,2,2;r) = 2(2k+1)(2k+3)r^{-2k-3}\Bigg\{&\int_0^r z^{2k}\int_0^z U(y)y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\,dz \\ &+ \int_0^r z^{2k}U(z)\int_0^z U(y)y^{-2k}\int_0^y x^{2k}\,dx\,dy\,dz \\ &- \int_0^r z^{2k}U(z)\int_0^z y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\,dz\Bigg\} \end{aligned}$$

$$\begin{aligned} I(k,3,3,1;r) = \tfrac{2}{3}(2k+1)(2k+3)r^{-2k-3}\Bigg\{&\int_0^r z^{2k}\int_0^z y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\,dz \\ &+ \int_0^r z^{2k}\int_0^z U(y)y^{-2k}\int_0^y x^{2k}\,dx\,dy\,dz + \int_0^r z^{2k}U(z)\int_0^z y^{-2k}\int_0^y x^{2k}\,dx\,dy\,dz\Bigg\} \end{aligned}$$

$$I(k,3,2,1;r) = 2(2k+1)(2k+3)r^{-2k-3}\int_0^r z^{2k}\int_0^z U(y)y^{-2k}\int_0^y x^{2k}\,dx\,dy\,dz$$

$$\begin{aligned} I(k,3,1,1;r) = 2(2k+1)(2k+3)r^{-2k-3}\Bigg\{&\int_0^r z^{2k}U(z)\int_0^z y^{-2k}\int_0^y x^{2k}\,dx\,dy\,dz \\ &- \int_0^r z^{2k}\int_0^z U(y)y^{-2k}\int_0^y x^{2k}\,dx\,dy\,dz + \int_0^r z^{2k}\int_0^z y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\,dz\Bigg\} \end{aligned}$$

TABLE 4.2 *Simplified definition of the functions* $I(k, n, \sigma, \rho; r)$ *up to order* $n = 3$

$$I(k,1,1,1;r) = (2k+1)r^{-2k-1}\int_0^r x^{2k}U(x)\,dx$$

$$I(k,2,2,2;r) = 2(2k+1)r^{-2}\int_0^r U(y)y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy$$

$$I(k,2,2,1;r) = -\frac{2k+1}{2k-1}r^{-2k-1}\int_0^r x^{2k}U(x)\,dx + \frac{4k}{2k-1}r^{-2}\int_0^r xU(x)\,dx$$

$$I(k,2,1,1;r) = +\frac{2(2k+1)}{2k-1}r^{-2k-1}\int_0^r x^{2k}U(x)\,dx - \frac{4}{2k-1}r^{-2}\int_0^r xU(x)\,dx$$

$$I(k,3,3,3;r) = 2(2k+1)(2k+3)r^{-2k-3}\int_0^r z^{2k}U(z)\int_0^z U(y)y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy\,dz$$

$$I(k,3,3,2;r) = \tfrac{2}{3}(2k+3)r^{-2}\int_0^r U(y)y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy + \frac{8k(2k+3)}{3(2k-1)}r^{-2k-3} * \int_0^r y^{2k}U(y)\int_0^y xU(x)\,dx\,dy - \frac{8k(2k+3)}{3(2k-1)}r^{-2k-3}\int_0^r yU(y)\int_0^y x^{2k}U(x)\,dx\,dy$$

$$I(k,3,2,2;r) = 2(2k+3)r^{-2}\int_0^r U(y)y^{-2k}\int_0^y x^{2k}U(x)\,dx\,dy - \frac{4(2k+3)}{2k-1}r^{-2k-3} * \int_0^r y^{2k}U(y)\int_0^y xU(x)\,dx\,dy + \frac{4(2k+3)}{2k-1}r^{-2k-3}\int_0^r yU(y)\int_0^y x^{2k}U(x)\,dx\,dy$$

$$I(k,3,3,1;r) = \frac{4k(2k+3)}{3(2k+1)}r^{-2k-3}\int_0^r x^{2k+2}U(x)\,dx - \frac{(2k+1)(2k+3)}{3(2k-1)}r^{-2k-1} * \int_0^r x^{2k}U(x)\,dx + \frac{8k(2k+3)}{3(2k+1)(2k-1)}r^{-2}\int_0^r xU(x)\,dx$$

$$I(k,3,2,1;r) = \frac{2(2k+3)}{2k+1}r^{-2}\int_0^r xU(x)\,dx - \frac{2(2k+3)}{2k+1}r^{-2k-3}\int_0^r x^{2k+2}U(x)\,dx$$

$$I(k,3,1,1;r) = \frac{4(k+1)(2k+3)}{2k+1}r^{-2k-3}\int_0^r x^{2k+2}U(x)\,dx - \frac{(2k+1)(2k+3)}{2k-1}r^{-2k-1} * \int_0^r x^{2k}U(x)\,dx + \frac{4(2k+3)}{(2k+1)(2k-1)}r^{-2}\int_0^r xU(x)\,dx$$

charge distribution is its root mean square radius (rms-radius) defined by

$$R_{\text{rms}} = \left\{\int_0^\infty r^2\rho(r)r^2\,dr \Big/ \int_0^\infty \rho(r)r^2\,dr\right\}^{1/2}. \tag{4.93}$$

Alternatively, one often works with the equivalent uniform radius, equal to $(\frac{5}{3})^{1/2}R_{\text{rms}}$, which is the radius of a uniform distribution having the same root mean square radius as the distribution under consideration. In the

TABLE 4.3 *The most important functions $I(k, n, \sigma, \rho; r)$ for the uniform distribution (3.122)*

$$I(k,1,1,1;r)=\begin{cases}\dfrac{3}{2}-\dfrac{2k+1}{2(2k+3)}\left(\dfrac{r}{R}\right)^2 & \text{if } 0\leqslant r\leqslant R\\ \dfrac{2k+1}{2k}\dfrac{R}{r}-\dfrac{3}{2k(2k+3)}\left(\dfrac{R}{r}\right)^{2k+1} & \text{if } R\leqslant r\end{cases}$$

$$I(k,2,2,2;r)=\begin{cases}\dfrac{9}{4}-\dfrac{3(k+1)}{2(2k+3)}\left(\dfrac{r}{R}\right)^2+\dfrac{2k+1}{12(2k+3)}\left(\dfrac{r}{R}\right)^4 & \text{if } 0\leqslant r\leqslant R\\ \dfrac{2k+1}{k}\left(\dfrac{R}{r}\right)^2\ln\left(\dfrac{r}{R}\right)+\left[\dfrac{19}{12}+\dfrac{7}{12(2k+3)}-\dfrac{3}{2k^2(2k+3)}\right]\left(\dfrac{R}{r}\right)^2 & \\ \qquad+\dfrac{3}{2k^2(2k+3)}\left(\dfrac{R}{r}\right)^{2k+2} & \text{if } R\leqslant r\end{cases}$$

$$I(k,2,2,1;r)=\begin{cases}\dfrac{3}{2}-\dfrac{k+1}{2(2k+3)}\left(\dfrac{r}{R}\right)^2 & \text{if } 0\leqslant r\leqslant R\\ \dfrac{4k+1}{2k}\left(\dfrac{R}{r}\right)-\dfrac{3k}{2(2k-1)}\left(\dfrac{R}{r}\right)^2+\dfrac{3}{2k(2k+3)(2k-1)}\left(\dfrac{R}{r}\right)^{2k+1} & \text{if } R\leqslant r\end{cases}$$

$$I(k,2,1,1;r)=\begin{cases}\dfrac{-1}{2(2k+3)}\left(\dfrac{r}{R}\right)^2 & \text{if } 0\leqslant r\leqslant R\\ -\dfrac{1}{k}\dfrac{R}{r}+\dfrac{3}{2(2k-1)}\left(\dfrac{R}{r}\right)^2-\dfrac{3}{k(2k+3)(2k-1)}\left(\dfrac{R}{r}\right)^{2k+1} & \text{if } R\leqslant r\end{cases}$$

$$I(k,3,3,3;r)=\begin{cases}\dfrac{27}{8}-\dfrac{9(4k+5)}{8(2k+5)}\left(\dfrac{r}{R}\right)^2+\dfrac{8k+7}{8(2k+7)}\left(\dfrac{r}{R}\right)^4-\dfrac{2k+1}{24(2k+9)}\left(\dfrac{r}{R}\right)^6 & \text{if } 0\leqslant r\leqslant R\\ \dfrac{(2k+1)(2k+3)}{2k^2}\left(\dfrac{R}{r}\right)^3\ln\left(\dfrac{r}{R}\right)+\left[\dfrac{19}{12}+\dfrac{5}{3k}-\dfrac{2}{k^2}-\dfrac{3}{2k^3}\right]\left(\dfrac{R}{r}\right)^3 & \\ \qquad+\dfrac{3}{2k^2}\left(\dfrac{R}{r}\right)^{2k+3}\ln\left(\dfrac{r}{R}\right)+\left[\dfrac{45}{8(2k+5)}-\dfrac{21}{8(2k+7)}\right. & \\ \qquad\left.+\dfrac{1}{3(2k+9)}-\dfrac{5}{3k}+\dfrac{2}{k^2}+\dfrac{3}{2k^3}\right]\left(\dfrac{R}{r}\right)^{2k+3} & \text{if } R\leqslant r\end{cases}$$

$$I(k,3,3,2;r)=\begin{cases}\dfrac{9}{4}-\dfrac{4k+5}{2(2k+5)}\left(\dfrac{r}{R}\right)^2+\dfrac{8k+7}{36(2k+7)}\left(\dfrac{r}{R}\right)^4 & \text{if } 0\leqslant r\leqslant R\\ \dfrac{2k+3}{3k}\left(\dfrac{R}{r}\right)^2\ln\left(\dfrac{r}{R}\right)+\left[\dfrac{67}{36}+\dfrac{23}{12(2k+1)}+\dfrac{1}{k}-\dfrac{1}{2k^2}\right]\left(\dfrac{R}{r}\right)^2 & \\ \qquad-\dfrac{2k+3}{2(2k-1)}\left(\dfrac{R}{r}\right)^3+\dfrac{4k-1}{2k^2(2k-1)}\left(\dfrac{R}{r}\right)^{2k+2} & \\ \qquad+\left[-\dfrac{23}{12(2k+1)}+\dfrac{5}{2(2k+5)}-\dfrac{7}{12(2k+7)}\right]\left(\dfrac{R}{r}\right)^{2k+3} & \text{if } R\leqslant r\end{cases}$$

$$I(k,3,2,2;r)=\begin{cases}\dfrac{9}{4}-\dfrac{3(2k+3)}{4(2k+5)}\left(\dfrac{r}{R}\right)^2+\dfrac{k+2}{6(2k+7)}\left(\dfrac{r}{R}\right)^4 & \text{if } 0\leqslant r\leqslant R\\ \dfrac{2k+3}{k}\left(\dfrac{R}{r}\right)^2\ln\left(\dfrac{r}{R}\right)+\left[\dfrac{19}{12}+\dfrac{69}{12(2k+1)}-\dfrac{3}{k}-\dfrac{3}{2k^2}\right]\left(\dfrac{R}{r}\right)^2 & \\ \qquad+\dfrac{3(2k+3)}{4k(2k-1)}\left(\dfrac{R}{r}\right)^3-\dfrac{3}{2k^2(2k-1)}\left(\dfrac{R}{r}\right)^{2k+2} & \end{cases}$$

$$I(k,3,3,1;r)=\begin{cases}+\left[\dfrac{3}{2(2k+5)}-\dfrac{1}{4(2k+7)}-\dfrac{23}{4(2k+1)}+\dfrac{9}{4k}\right]\left(\dfrac{R}{r}\right)^{2k+3} & \text{if } R\leqslant r\\ \dfrac{3}{2}-\dfrac{4k+5}{6(2k+5)}\left(\dfrac{r}{R}\right)^2 & \text{if } 0\leqslant r\leqslant R\\ \dfrac{(5k+1)(2k+3)}{6k(k+1)}\dfrac{R}{r}-\dfrac{k(2k+3)}{(2k-1)(2k+1)}\left(\dfrac{R}{r}\right)^2 & \\ +\dfrac{1}{2k(2k-1)}\left(\dfrac{R}{r}\right)^{2k+1}+\left[\dfrac{7}{6}+\dfrac{5}{6(2k+5)}-\dfrac{(5k+1)(2k+3)}{6k(k+1)}\right. & \\ \left.+\dfrac{k(2k+3)}{(2k-1)(2k+1)}-\dfrac{1}{2k(2k-1)}\right]\left(\dfrac{R}{r}\right)^{2k+3} & \text{if } R\leqslant r\end{cases}$$

case of the refined nuclear charge distributions our 'nuclear radius' R therefore is to be identified with the equivalent uniform radius. It is reasonable then to look at the nuclear radius R as a quantity which is known for each nucleus. It therefore remains to discuss the shape dependence of $U(0)$ and other quantities in the sense that only different distributions with the same nuclear radius are admitted for comparison. In the case of the modified Gaussian distribution, eqn (3.197), with cut-off radius $R_1=\infty$, we find

$$R=a\{\tfrac{5}{2}(2+5A)/(2+3A)\}^{1/2}, \tag{4.94}$$

so that, for a given R, of the two parameters a and A, only one, preferentially A, is a free parameter for generating different distributions. In the case of the Fermi distribution, eqn (3.207), again with cut-off radius $R_1=\infty$, we find by means of the method of Schucan (1965)

$$R=c[\{1+\tfrac{10}{3}\pi^2(\beta c)^{-2}+\tfrac{7}{3}\pi^4(\beta c)^{-4}-120(\beta c)^{-5}B_5\}/\{1+\pi^2(\beta c)^{-2}-6(\beta c)^{-3}B_3\}]^{1/2} \tag{4.95}$$

where

$$B_n=\sum_{m=1}^{\infty} m^{-n}\{-\exp(-\beta c)\}^m, \tag{4.96}$$

and the normalization factor of the distribution is

$$N(\infty)=(-3\alpha Z/c^3)\{1+\pi^2(\beta c)^{-2}-6(\beta c)^{-3}B_3\}^{-1}. \tag{4.97}$$

Again, for any given R, of the two parameters c and β only one, preferentially β, remains a free parameter for generating different distributions, while c is a function of R and β given implicitly by eqn (4.95). In place of β, often the 'skin thickness' $t=\{4\ln(3)\}/\beta$ is used.

To consider here just the modified Gaussian and the Fermi distribution

is quite reasonable since these distributions have extensively been used in the context of electron scattering (Collard *et al.* 1965; Überall 1971). The modified Gaussian distribution, appropriate for the lighter nuclei, allows quite drastic changes of its shape by means of its parameter A, while the Fermi distribution, appropriate for the heavier nuclei, remains, if its parameter β is varied within reasonable limits, to a larger extent similar to the uniform distribution, to which it reduces in the limit of $\beta \to \infty$. It might be interesting to note that the 'symmetrized Fermi distribution'

$$\rho(r) = -(3\alpha Z/c^3)\{1+\pi^2(\beta c)^{-2}\}^{-1}\{[1+\exp\{\beta(r-c)\}]^{-1} + [1+\exp\{\beta(-r-c)\}]^{-1} - 1\}, \quad (4.98)$$

which (for $r>0$) nearly coincides with the original Fermi distribution if $\beta c \gg 1$ as it is the case for the heavier nuclei, is technically more convenient in so far as the normalization factor, already included explicitly in eqn (4.98), and the expression for the associated nuclear radius

$$R = c[\{1+\tfrac{10}{3}\pi^2(\beta c)^{-2}+\tfrac{7}{3}\pi^4(\beta c)^{-4}\}/\{1+\pi^2(\beta c)^{-2}\}]^{1/2} \quad (4.99)$$

are considerably simpler.

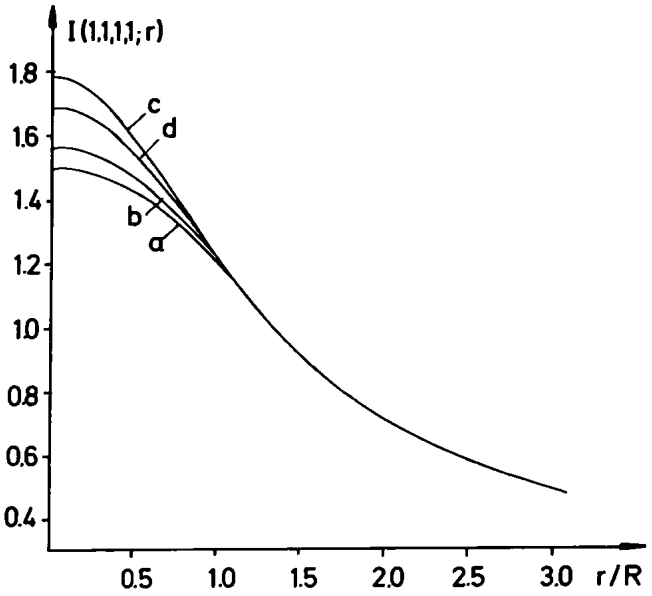


FIG. 4.1. Function $I(1, 1, 1, 1; r)$ for different charge distributions (Behrens and Bühring 1971). (a) uniform; (b) Fermi with $t = 0{\cdot}4$ R; (c) Gauss ($A = 0$); (d) modified Gauss with $A = 1$.

TABLE 4.4 *Analytic expressions of $U(0)$ for the various distributions*

Distribution	$U(0)$
Uniform	$U(0)=\frac{3}{2}$
Modified Gauss	$U(0)=8\pi^{-1/2}\{5(2+5A)\}^{1/2}\{2(2+3A)\}^{-3/2}(1+A)$
Fermi	$U(0)=\frac{3}{2}(R/c)\{1+\frac{1}{3}\pi^2(\beta c)^{-2}+2(\beta c)^{-2}B_2\}/\{1+\pi^2(\beta c)^{-2}-6(\beta c)^{-3}B_3\}$ with c from eqn (4.95)
Symmetrized Fermi	$U(0)=\frac{3}{2}(R/c)\{1+\frac{1}{3}\pi^2(\beta c)^{-2}\}/\{1+\pi^2(\beta c)^{-2}\}$ with c from eqn (4.99)

We are now prepared to discuss further the functions $I(k, n, \sigma, \rho; r)$. The most important one, $I(1, 1, 1, 1; r)$, which gives the large Coulomb terms in non-unique first forbidden transitions, is shown in Fig. 4.1. While its tail is independent of the model, its value at $r=0$, which is equal to $U(0)$, shows a significant model dependence. The analytic expressions of $U(0)$ are therefore given in Table 4.4, and numerical results for some specific values of the parameters are shown in Table 4.5.

On the basis of Fig. 4.1 and Table 4.5 the reader might get the impression that the differences between the various models would significantly influence nuclear beta-decay. This is not true, however, since the functions $I(k, n, \sigma, \rho; r)$ enter the nuclear matrix elements, so that an integration

$$\int_0^\infty (r/R)I(k, n, \sigma, \rho; r)\ldots r^2\,\mathrm{d}r$$

TABLE 4.5 *Numerical values of $U(0)$ for some specific values of the parameters*

Model		$U(0)$
Uniform		1·5000
Gauss	$(A=0)$	1·7841
Modified Gauss	$A=1$	1·6888
	$A=2$	1·6388
	$A=\infty$	1·5355
Fermi	$t/R=0{\cdot}2$	1·5156
	$t/R=0{\cdot}4$	1·5655
	$t/R=0{\cdot}6$	1·6622
	$t/R=0{\cdot}8$	1·8449
Symmetrized Fermi	$t/R=0{\cdot}2$	1·5156
	$t/R=0{\cdot}4$	1·5655
	$t/R=0{\cdot}6$	1·6629
	$t/R=0{\cdot}8$	1·8905

is involved which because of the factor r^3 suppresses the influence of the region near the origin where the differences are large. An impression of the remaining model dependence of these radial integrals may be obtained from Table 4.6, which gives some numerical examples. It should be noted that screening is quite unimportant in this context, for near the origin up to a few nuclear radii its influence on $U(r)$ is very small, as may be seen from the model, eqn (3.188) or eqn (3.189). The long-range tail of $U(r)$, on the other hand, which is modified by screening, does not

TABLE 4.6 *Radial integrals* $\int_0^\infty g_f(r)(r/R)I(1,1,1,1;r)\, g_i(r)r^2\,\mathrm{d}r$ *taken between oscillator radial wave functions* $g_i(r) = g_i(\nu_i, l_i; r)$ *and* $g_f(r) = g_f(\nu_f, l_f; r)$, *where* l *is the orbital angular momentum and* ν *the radial quantum number for the initial* (i) *or final* (f) *nuclear state, respectively. The normalization of the nuclear radial wave functions is* $\int_0^\infty \{g(r)\}^2 r^2\,\mathrm{d}r = 1$. *The nuclear radius* R *has been chosen equal to* 7·12 fm, *and the oscillator constant* $\hbar\omega$ *of the nuclear model equal to* 6·91 MeV. *The spectroscopic notation is used for the angular momenta.*

Transition $\nu_i, l_i \rightarrow \nu_f, l_f$	Charge distribution		Value of the radial integral
3p → 3s	Modified Gauss	$A = 0$	0·797
		$A = 1$	0·791
		$A = 2$	0·788
	uniform		0·744
3p → 2d	modified Gauss	$A = 0$	0·559
		$A = 1$	0·559
		$A = 2$	0·560
	uniform		0·559
2f → 2d	modified Gauss	$A = 0$	0·938
		$A = 1$	0·928
		$A = 2$	0·924
	uniform		0·902
2g → 2f	modified Gauss	$A = 0$	1·004
		$A = 1$	0·995
		$A = 2$	0·991
	uniform		0·971
2g → 1h	modified Gauss	$A = 0$	0·346
		$A = 1$	0·349
		$A = 2$	0·351
	uniform		0·360
1i → 1h	modified Gauss	$A = 0$	1·115
		$A = 1$	1·107
		$A = 2$	1·104
	uniform		1·084

significantly contribute to the radial integrals containing the nuclear radial wave functions which become very small outside of a few nuclear radii. Our conclusion is that for the functions $I(k, n, \sigma, \rho; r)$ the simple uniform model of the nuclear charge distribution without screening remains a satisfactory model, so that the explicit expressions of Table 4.3 may be used.

4.4.2. Model dependence via the beta-decay Coulomb functions

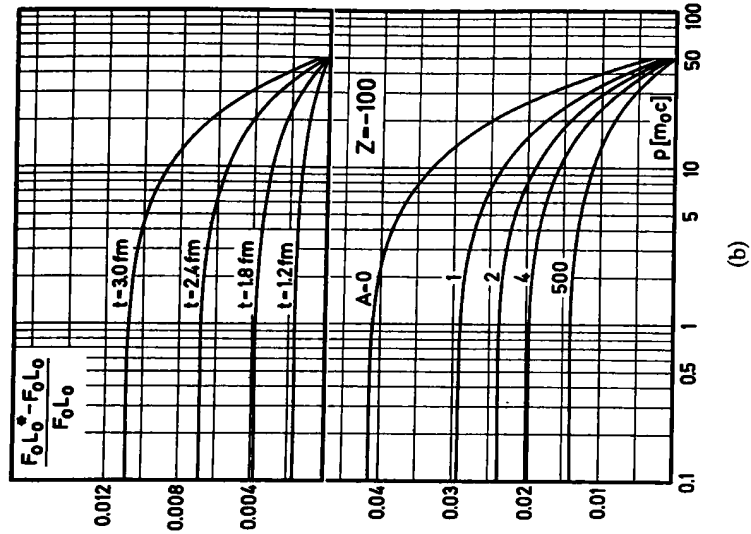
It remains to discuss the model dependence of the beta-decay Coulomb functions, again in the sense that the nuclear radius R is assumed to be known† and only different nuclear charge distributions having the same nuclear radius (defined as the equivalent uniform radius) are admitted for comparison.

Examples of the beta-decay Coulomb functions for several different charge distributions are displayed in Figs. 4.2–4.5 relative to the corresponding functions for the uniform distribution (without screening) tabulated by Behrens and Jänecke (1969). On the basis of these figures and some additional material not displayed here, the following statements can be made (Behrens and Bühring 1970):

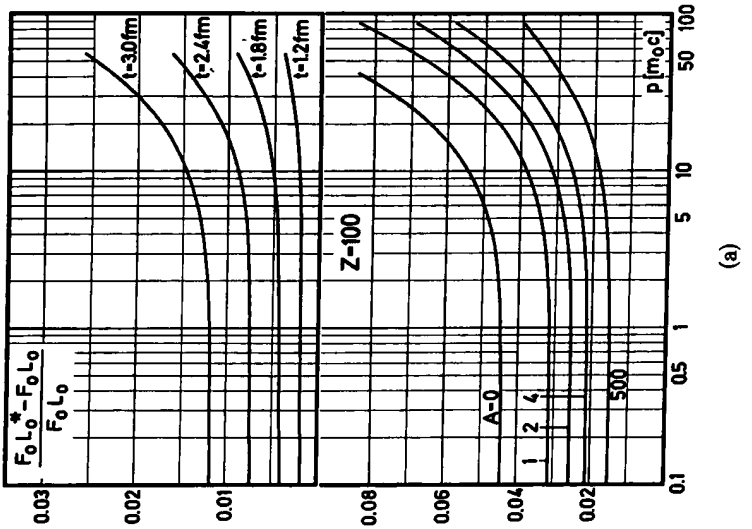
The energy dependence of the Fermi function is insensitive to the shape of the distribution for energies smaller than 3 MeV, as may be seen from Fig. 4.2. Beta transitions with larger endpoint energies are known for light nuclei, but then the sensitivity to the model occurring at energies larger than 3 MeV is small because of the smaller nuclear charge. We therefore can conclude that the energy dependence of the Fermi function and the shape of allowed beta transitions are not sensitive to the model of the charge distribution. The absolute value of the Fermi function does depend on the model, but this small dependence is practically unimportant, even when extreme accuracy is required, as it is the case for the superallowed Fermi transitions (Behrens and Bühring 1972). The Z-dependence of the Fermi function is displayed in Fig. 4.3 for several distributions: its model dependence rapidly becomes smaller as Z decreases.

The other beta-decay Coulomb functions have a similar behaviour as the Fermi function in so far as their energy dependence is insensitive to the model for energies below 3 MeV. Also the sensitivity of the Z-dependence is qualitatively similar for all the functions.

† The dependence on the nuclear radius is shown in Table 8 on pp. 23–24 of Behrens and Jänecke (1969). It should be noted, however, that in addition the nuclear radius enters the beta-decay observables explicitly. Their dependence on the nuclear radius has been investigated in case of the ft-values of superallowed Fermi transitions, where high accuracy is required, by Behrens and Bühring (1972), Damgaard (1969), Wilkinson (1973*a*), and others.


$\frac{F_0L_0^* - F_0L_0}{F_0L_0}$
t=3.0fm
t=2.4fm
t=1.8fm
t=1.2fm
0.012
0.008
0.004
Z=100
A=0
1
2
4
500
0.04
0.03
0.02
0.01
p[m₀c]
0.1
0.5
1
5
10
50
100

(b)


$\frac{F_0L_0^* - F_0L_0}{F_0L_0}$
t=3.0fm
t=2.4fm
t=1.8fm
t=1.2fm
0.03
0.02
0.01
Z=100
A=0
1
2
4
500
0.08
0.06
0.04
0.02
p[m₀c]
0.1
0.5
1
5
10
50
100

(a)

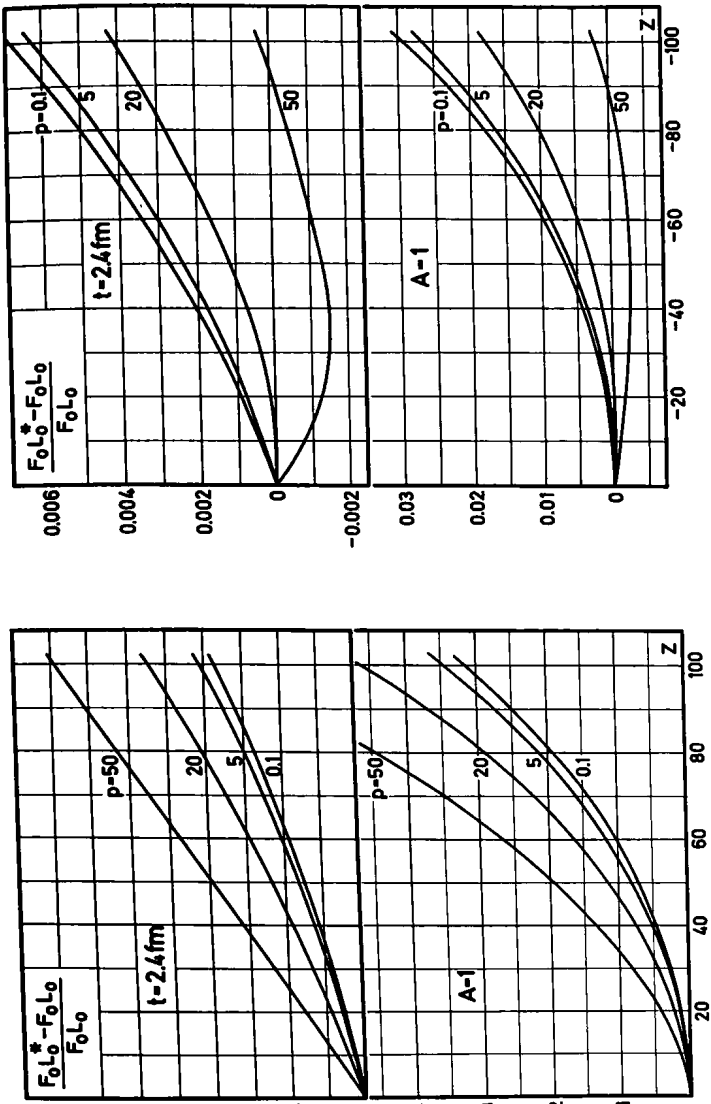
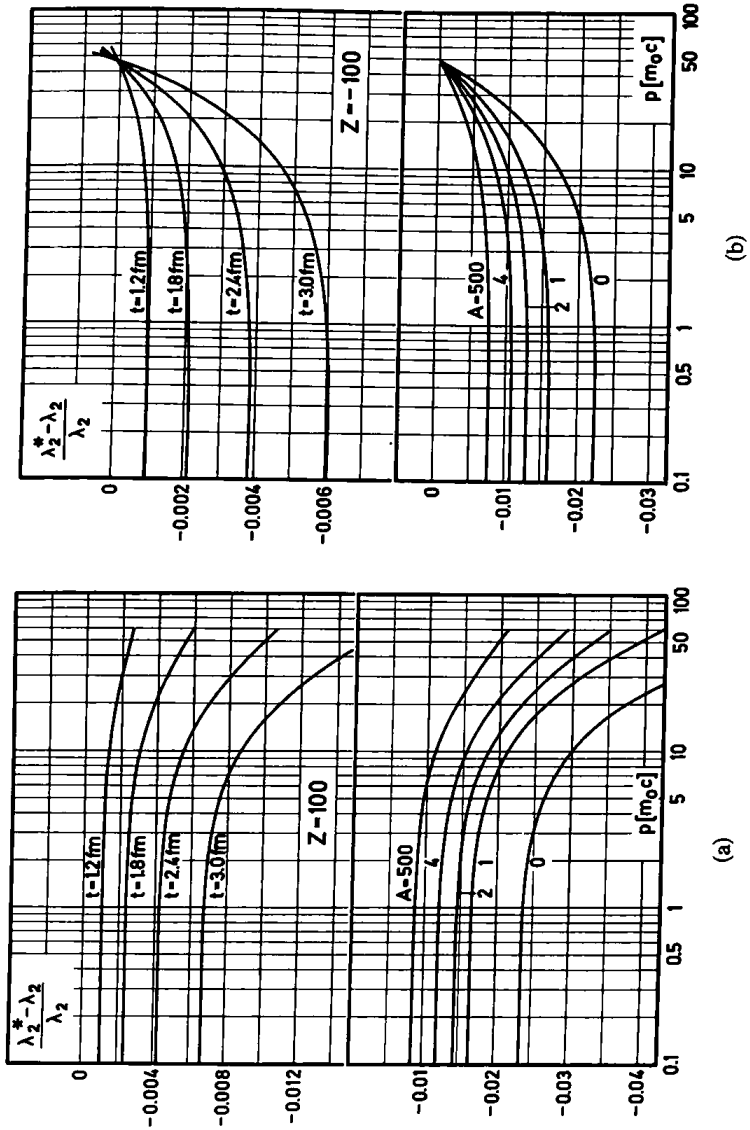
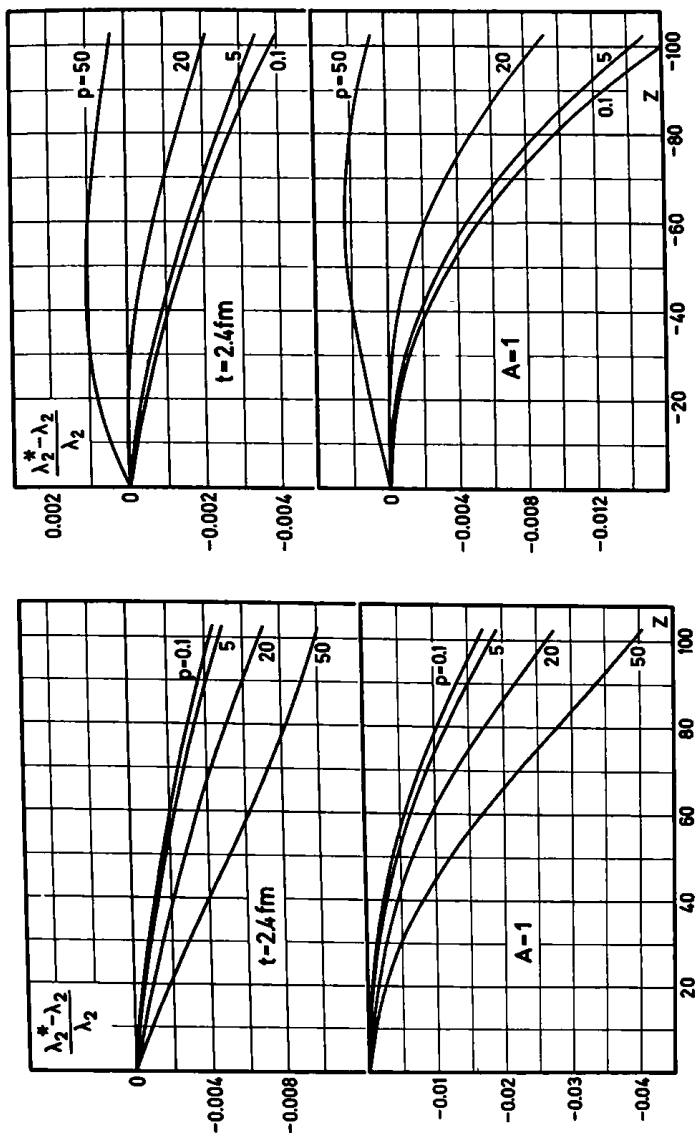


FIG. 4.3. Z-dependence of Fermi functions $F_0L_0^*$ for different charge distributions relative to the Fermi function F_0L_0 of the uniform distribution. The upper and lower part of each figure are for the Fermi-type and modified Gaussian distributions, respectively, (taken from Behrens and Bühring 1970). (a) electrons; (b) positrons.


(a)
(b)
Z=100
A=500
t=1.2fm
t=1.8fm
t=2.4fm
t=3.0fm
p[m₀c]
0
0.1
0.5
1
5
10
50
100
-0.004
-0.008
-0.012
-0.01
-0.02
-0.03
-0.04
-0.002
-0.006


(a)
(b)
$\frac{\lambda_2^*-\lambda_2}{\lambda_2}$
t=2.4fm
A=1
p=0.1
5
20
50
p=50
0.1
Z
0
-0.004
-0.008
-0.01
-0.02
-0.03
-0.04
20
40
60
80
100
-20
-40
-60
-80
-100
0.002
-0.002
-0.012

The function λ_2 is particularly important for first unique forbidden transitions, and it should be recalled that not only its energy dependence but also its absolute value is relevant to the spectrum shape. Nevertheless, the relatively small model dependence of λ_2, as shown in Figs. 4.4 and 4.5, seems to be not yet important in view of the experimentally attainable accuracy.

The model dependence of the functions μ_1, μ_2, Λ_1, and Λ_2 is completely negligible, for even if $|Z|$ is as large as 100, numerical computations have shown that

$$|(\mu_1^* - \mu_1)/\mu_1| < 4 \times 10^{-3} \quad \text{if} \quad p < 5{\cdot}0,$$
$$|(\mu_2^* - \mu_2)/\mu_2| < 3 \times 10^{-4} \quad \text{if} \quad p < 5{\cdot}0,$$
$$|(\Lambda_1^* - \Lambda_1)/\Lambda_1| < 1 \times 10^{-3} \quad \text{if} \quad p > 0{\cdot}1,$$
$$|(\Lambda_2^* - \Lambda_2)/\Lambda_2| < 1 \times 10^{-4} \quad \text{if} \quad p > 0{\cdot}1.$$

Here the functions with or without an asterisk are for the refined or uniform distributions, respectively.

The model dependence of the functions $\nu_{12}, \hat{\nu}_{12}, \eta_{12}$, and $\hat{\eta}_{12}$, which enter the angular correlation functions, seems not yet be important in view of the accuracy of angular correlation measurements. For computations with $|Z| = 100$ have shown that

$$|(\nu_{12}^* - \nu_{12})/\nu_{12}| < 0{\cdot}014 \quad \text{if} \quad p < 5{\cdot}0,$$
$$|(\hat{\nu}_{12}^* - \hat{\nu}_{12})/\hat{\nu}_{12}| < 0{\cdot}014 \quad \text{if} \quad p < 5{\cdot}0,$$
$$|(\eta_{12}^* - \eta_{12})/\eta_{12}| < 0{\cdot}014 \quad \text{if} \quad p < 5{\cdot}0,$$
$$|(\hat{\eta}_{12}^* - \hat{\eta}_{12})/\hat{\eta}_{12}| < 0{\cdot}012 \quad \text{if} \quad p < 5{\cdot}0.$$

The model dependence discussed so far in this section refers to the model for the nuclear charge distribution. As to the model for screening by the distribution of the atomic electrons, the most important shape parameter is the difference V_0 between the values of the potential at the nuclear radius R with and without screening (Rose 1936; Longmire and Brown 1949; Good 1954; Matese and Johnson 1966; Garret and Bhalla 1967). In the case of the models (3.188) or (3.189) presented in Section 3.2.4.5 we have

$$V_0 = \alpha \tilde{Z} \sum_{n=1}^{N} a_n \beta_n \exp(-\beta_n R) \tag{4.100}$$

or

$$V_0 = (\alpha \tilde{Z}/R) \sum_{n=1}^{N} a_n \{1 - \exp(-\beta_n R)\}, \tag{4.101}$$

respectively. Because $\beta_n R \ll 1$, these two values are not significantly

different, but in both cases

$$V_0 \simeq \tilde{V}_0, \tag{4.102}$$

where

$$\tilde{V}_0 = \alpha \tilde{Z} \sum_{n=1}^{N} a_n \beta_n \tag{4.103}$$

is a quantity which is independent of the nuclear radius R. The dependence of the beta-decay Coulomb functions with screening on the value of $\tilde{V}_0$ will be considered in Section 4.5.2. We expect that the remaining model dependence, if different models with the same value of $\tilde{V}_0$ are admitted for comparison, is very small and unimportant, although a detailed investigation of this point has not been made.

4.5. Approximate expressions for some of the beta-decay Coulomb functions

4.5.1. Approximate expression for L_0

Although we have approximately evaluated L_0 already in Section 4.2.2 in order to show that it is of the order of unity, we now come back to this point. It is sometimes useful (Huffacker and Laird 1967; Calaprice and Holstein 1976) to have a simple analytical expression which approximately represents L_0, but more accurately than eqn (4.31). In evaluating the amplitudes α_κ, two modifications are then necessary.

One modification, as compared with the treatment of Section 4.2.2, is that we must not replace the factor

$$FA1 = \{1 + 2U(H/E)\cos(\Delta^C - \bar{\Delta}^C) + U^2(H/E)^2\}^{-1/2}$$

by 1 in eqn (3.163), but have to evaluate it approximately for the case when $\alpha Z \to 0$. Since we know from Section 3.2.2.8 that then $\Delta^C - \bar{\Delta}^C \to 0$, it follows that

$$FA1 \simeq \{1 + U(H/E)\}^{-1}. \tag{4.105}$$

We are now going to derive in more detail the result for UH/E quoted above by eqn (4.19). Using the identity

$$\gamma^2 + y^2 = p^{-2}(kW - \gamma m)(kW + \gamma m) \tag{4.106}$$

and restoring the gamma functions, the exact expression for U from eqn (3.162) may be rewritten

$$U = \{|\Gamma(\gamma + iy)|/\Gamma(1 + 2\gamma)\}^2 2\gamma(2p)^{2\gamma} p^{-1}\{(kW - \gamma m)(kW + \gamma m)\}^{1/2} * \{\sin(2\pi\gamma)\}^{-1}[\{\sin(\pi\gamma)\}^2 + \{\sinh(\pi y)\}^2]^{1/2}. \tag{4.107}$$

When $\alpha Z \to 0$ we have

$$\gamma - k = -(2k)^{-1}(\alpha Z)^2 + O\{(\alpha Z)^4\}$$

and therefore

$$\sin(2\pi\gamma) \simeq -\pi k^{-1}(\alpha Z)^2,$$

$$\sin(\pi\gamma) \simeq -(-1)^k \pi (2k)^{-1}(\alpha Z)^2,$$

$$[\{\sin(\pi\gamma)\}^2 + \{\sinh(\pi y)\}^2]^{1/2} \simeq |\sinh(\pi y)| \simeq \pi |y| = \pi \operatorname{Sign}(\alpha Z)\alpha Z W/p.$$

We then obtain

$$U \simeq -\operatorname{Sign}(\alpha Z)\{|\Gamma(\gamma + iy)|/\Gamma(1+2\gamma)\}^2 2\gamma k (2p)^{2\gamma} p^{-2} W(\alpha Z)^{-1} * \{(kW - \gamma m)(kW + \gamma m)\}^{1/2}. \quad (4.108)$$

According to eqn (3.167) we have

$$-H/E = \{f^1(R)g_{01}(R) - g^1(R)f_{01}(R)\}/\{f^1(R)g_{02}(R) - g^1(R)f_{02}(R)\}. \quad (4.109)$$

Taking the exact normalization factors, which yield

$$\operatorname{Sign}(\alpha Z)\{(kW - \gamma m)/(kW + \gamma m)\}^{1/2} R^{2\gamma}$$

if $\kappa = k$, or

$$-\operatorname{Sign}(\alpha Z)\{(kW + \gamma m)/(kW - \gamma m)\}^{1/2} R^{2\gamma}$$

if $\kappa = -k$, but keeping the leading terms only of the series which, when γ is approximated by k, give

$$\pm[1 * \{(2k)^{-1}\alpha Z + (2k+1)^{-1}(W \pm m)R\} - \{(2k+1)^{-1}(W \pm m)R + (2k+1)^{-1}(2k+3)^{-1}2(2+k)\alpha Z\} * 1]/1 = \pm\{2k(2k+1)(2k+3)\}^{-1} 3\alpha Z,$$

we obtain

$$-H/E \simeq \operatorname{Sign}(\alpha Z) R^{2\gamma}\{2k(2k+1)(2k+3)\}^{-1} * 3\alpha Z \begin{cases} \{(kW - \gamma m)/(kW + \gamma m)\}^{1/2} & \text{if } \kappa = k \\ \{(kW + \gamma m)/(kW - \gamma m)\}^{1/2} & \text{if } \kappa = -\kappa \end{cases}. \quad (4.110)$$

We therefore have, from eqns (4.108) and (4.110),

$$UH/E \simeq \{|\Gamma(\gamma + iy)|/\Gamma(1+2\gamma)\}^2\{(2k+1)(2k+3)\}^{-1} * 3\gamma(2pR)^{2\gamma} p^{-2} W \begin{cases} kW - \gamma m & \text{if } \kappa = k \\ kW + \gamma m & \text{if } \kappa = -k \end{cases}. \quad (4.111)$$

Now replacing γ by k, except in the exponent, and y by zero, we obtain

$$UH/E \simeq \{(k-1)!/(2k)!\}^2\{(2k+1)(2k+3)\}^{-1} 3k^2 * (2pR)^{2\gamma} p^{-2} W \begin{cases} W - m & \text{if } \kappa = k \\ W + m & \text{if } \kappa = -k \end{cases}, \quad (4.112)$$

a result which, when the term $\mp m$ is omitted, reduces to the estimate, eqn (4.19), quoted above.

When eqn (4.112) with γ approximated by k is evaluated for $k=1$, eqn (4.105) yields

$$FA1 \simeq 1 - U(H/E) \simeq 1 - \tfrac{1}{5}WR(W \mp m)R, \tag{4.113}$$

where the upper or lower sign is for $\kappa = 1$ or $\kappa = -1$, respectively.

The other modification as compared with Section 4.2.2 is that we now have to evaluate the quantity

$$E^{-1} = \{f_{01}(R)g_{02}(R) - g_{01}(R)f_{02}(R)\}/\{f^{\mathrm{I}}(R)g_{02}(R) - g^{\mathrm{I}}(R)f_{02}(R)\} \tag{4.114}$$

of eqn (4.21) more carefully. Rather than replace the non-trivial radial dependence from the series expansion of the wave functions by 1 or 0, respectively, we have to use $H_k(R)+h_k(R)$ in the case of $f^{\mathrm{I}}(R)$, and $D_k(R)+d_k(R)$ in the case of $g^{\mathrm{I}}(R)$ if $\kappa = k$, or $H_k(R)-h_k(R)$ in the case of $g^{\mathrm{I}}(R)$, and $-\{D_k(R)-d_k(R)\}$ in the case of $f^{\mathrm{I}}(R)$ if $\kappa = -\kappa$ according to eqn (4.4). With $k=1$ we obtain by means of eqns (4.88)–(4.91) and Table 4.3, or alternatively† from eqns (3.126) and (3.127)

$$H_1(R) \pm h_1(R) = 1 - \tfrac{13}{30}\alpha ZWR \pm \tfrac{1}{60}\alpha ZmR - \tfrac{17}{60}(\alpha Z)^2 - \tfrac{1}{6}(pR)^2 + \ldots, \tag{4.115}$$

$$D_1(R) \pm d_1(R) = \tfrac{1}{3}(W \pm m)R + \tfrac{2}{5}\alpha Z + \ldots \tag{4.116}$$

Denoting the non-trivial radial dependence of the Coulomb wave functions $f_{02}(r)$ and $g_{02}(r)$ by

$$Sf_{02}(r) = \sum_{n=0}^{\infty} a_n r^n, \tag{4.117a}$$

$$Sg_{02} = \sum_{n=0}^{\infty} b_n r^n, \tag{4.117b}$$

respectively, we have from eqns (3.20) and (3.21), if $\kappa = k$,

$$a_0 = \alpha Z/(k+\gamma), \tag{4.118a}$$

$$b_0 = 1, \tag{4.118b}$$

$$a_1 = \{(1-2\gamma)(k+\gamma)\}^{-1}\{-2(\alpha Z)^2 W - (k+\gamma)(W-m)\}, \tag{4.118c}$$

$$b_1 = \{(1-2\gamma)(k+\gamma)\}^{-1}\alpha Z\{-2(k+\gamma)W + W + m\}, \tag{4.118d}$$

† It should be noted, however, that, while the coefficients up to $n=3$ are needed for $D_1(r)$, they have to be included up to $n=6$ for $H_1(r)$. The terms explicitly displayed in eqn (4.6) therefore do not suffice in the case of $H_1(R)$.

$$a_2 = \{4(1-\gamma)(1-2\gamma)(k+\gamma)\}^{-1}\alpha Z\{4(\alpha Z)^2W^2 + 4(k+\gamma)W(W-m) + 2(\gamma-1)p^2\}, \quad (4.118e)$$

$$b_2 = \{4(1-\gamma)(1-2\gamma)(k+\gamma)\}^{-1}\{-4(\alpha Z)^2W(W+m) + 4(\alpha Z)^2(k+\gamma)W^2 + 2(k+\gamma)(\gamma-1)p^2\}, \quad (4.118f)$$

$$b_3 = \{3(3-2\gamma)\}^{-1}\{(3-k-\gamma)(W+m)a_2 - \alpha Z(W-m)b_2\}. \quad (4.118g)$$

It should be noted that when $k=1$, which is just the case we are interested in, the coefficients a_2 and b_3 contain comparatively large terms due to the fact that the factor $1-\gamma$ appearing in the denominator then is of the order of $(\alpha Z)^2$. Evaluating eqns (4.117) and (4.118) for $k=1$, we have to use $\gamma = \gamma_1 = \{1-(\alpha Z)^2\}^{1/2} = 1-\frac{1}{2}(\alpha Z)^2 + O\{(\alpha Z)^4\}$.

Omitting some higher order terms and replacing $1-\gamma$ by $\frac{1}{2}(\alpha Z)^2$ or γ by 1 elsewhere, we obtain

$$Sf_{02}(R) = \tfrac{1}{2}(\alpha Z) + (W-m)R - 2(\alpha Z)^{-1}WR(W-m)R + \ldots, \quad (4.119a)$$

$$Sg_{02}(R) = 1 + 2\alpha ZWR - \tfrac{1}{2}\alpha Z(W+m)R + WR(W+m)R - 2(WR)^2 + \tfrac{1}{2}(pR)^2 - \tfrac{2}{3}(\alpha Z)^{-1}WR(pR)^2 + \ldots. \quad (4.119b)$$

In the case of $\kappa = -1$, the same result, but with m replaced by $-m$, is obtained for $-Sg_{02}$ in place of Sf_{02} and for Sf_{02} in place of Sg_{02}.

The denominator of eqn (4.114) therefore approximately yields a factor

$$\begin{aligned} FA2 \simeq {} & \{1 - \tfrac{13}{30}\alpha ZWR \pm \tfrac{1}{60}\alpha ZmR - \tfrac{17}{60}(\alpha Z)^2 - \tfrac{1}{6}(pR)^2\} \\ & * \{1 + 2\alpha ZWR - \tfrac{1}{2}\alpha Z(W \pm m)R + WR(W \pm m)R - 2(WR)^2 + \tfrac{1}{2}(pR)^2 \\ & - \tfrac{2}{3}(\alpha Z)^{-1}WR(pR)^2\} - \{\tfrac{1}{3}(W \pm m)R + \tfrac{2}{3}\alpha Z\}\{\tfrac{1}{2}\alpha Z \\ & + (W \mp m)R - 2(\alpha Z)^{-1}WR(W \mp m)R\} \end{aligned}$$

or after simplification

$$FA2 \simeq 1 + \tfrac{1}{2}\alpha ZWR \mp \tfrac{1}{4}\alpha ZmR - \tfrac{29}{60}(\alpha Z)^2 - \tfrac{1}{5}WR(W \mp m)R. \quad (4.120)$$

From the numerator, which is a Wronskian, we obtain an exact factor

$$FA3 = 1 - \{(\alpha Z)/(k+\gamma)\}^2 \quad (4.121)$$

or, for $k=1$, approximately

$$FA3 \simeq 1 - \tfrac{1}{4}(\alpha Z)^2. \quad (4.122)$$

The total correction factor to be applied to the value for E^{-1} as given by eqn (4.22) therefore, in the case of $k=1$, is

$$FA4 = FA3/FA2$$

or

$$FA4 \simeq 1 - \tfrac{1}{2}\alpha ZWR \pm \tfrac{1}{4}\alpha ZmR + \tfrac{7}{30}(\alpha Z)^2 + \tfrac{1}{5}WR(W \mp m)R. \quad (4.123)$$

The total correction factor to be applied to the amplitudes $\alpha_{\pm 1}$ of eqn (4.25) then is

$$FA5 = FA1 * FA4$$

or

$$FA5 \simeq 1 - \tfrac{1}{2}\alpha ZWR \pm \tfrac{1}{4}\alpha ZmR + \tfrac{7}{30}(\alpha Z)^2. \quad (4.124)$$

It follows that

$$\alpha_{\pm 1}^2 \simeq F_0 p^2 (2W)^{-1}(1+\gamma_1)(W \mp \gamma_1 m) * \{1 - \alpha ZWR \pm \tfrac{1}{2}\alpha ZmR + \tfrac{7}{15}(\alpha Z)^2\} \quad (4.125)$$

and therefore, from eqn (4.7),

$$L_0 \simeq \tfrac{1}{2}(1+\gamma_1)[\{1 - \alpha ZWR + \tfrac{7}{15}(\alpha Z)^2\} - \tfrac{1}{2}\gamma_1 (m/W)\alpha ZmR]$$

or, because

$$\tfrac{1}{2}(1+\gamma_1) \simeq 1 - \tfrac{1}{4}(\alpha Z)^2,$$

finally

$$L_0 \simeq 1 - \alpha ZWR + \tfrac{13}{60}(\alpha Z)^2 - \tfrac{1}{2}(m/W)\alpha ZmR. \quad (4.126)$$

It should be noted that this formula is applicable if $|Z|$ is sufficiently small. For $Z = 6$ or $Z = -6$ the values computed from this formula, in fact, nicely agree with the exact values tabulated by Behrens and Jänecke (1969). While the agreement is still rather good for $Z = 15$ or $Z = -15$, it gradually becomes poorer as $|Z|$ increases.

While eqn (4.126) is based on analytical work, expressions of a similar structure are available (Wilkinson and Mars 1972; Wilkinson 1978) the coefficients of which have been fitted to reproduce within a certain accuracy the exact numerical values of L_0 over a certain range of the relevant variables. The coefficients of these approximating expressions are slightly different from those of eqn (4.126). This is not unexpected, however, but related to the fact that the coefficients of an economized power series are different from the Taylor coefficients of the analytic function which is to be approximated uniformly on a certain interval.

4.5.2. Approximate screening corrections

The approximate expression obtained for L_0 in the preceding section is based on the model of the uniformly extended nuclear charge distribution without any allowance for the screening by the atomic electrons, but the

influence of screening is not always unimportant for nuclear beta-decay. It would therefore be useful to have approximate analytical results concerning the screening effect, in particular since the exact treatment requires expensive numerical computations. In fact, several authors (Rose 1936; Longmire and Brown 1949; Good 1954), of whom Rose (1936) was the first, have found a simple analytical prescription of how to obtain the Fermi function for a screened field from the Fermi function for the corresponding unscreened field. The additional information needed in the case of forbidden beta spectra was supplied later by Good (1954). In any case such results have been based on approximate WKB-type solutions of the radial wave equations.

Proceeding in a similar although not quite the same way, we first want to construct a WKB-type solution which approximately satisfies the Dirac radial equations when r is sufficiently large. For this purpose we may consider the simplified radial equations

$$(rf)' + \{W - m - V(r)\} rg(r) = 0, \tag{4.127a}$$

$$(rg)' - \{W + m - V(r)\} rf(r) = 0 \tag{4.127b}$$

obtained by omitting the angular momentum dependent terms which do not affect the leading terms of the radial dependence for $r \to \infty$. We try to solve eqn (4.127), at least approximately, by means of an ansatz†

$$rf(r) = A(r)\sin\{\chi(r)\}, \tag{4.128a}$$

$$rg(r) = B(r)\cos\{\chi(r)\}. \tag{4.128b}$$

The differential eqns (4.127) then are satisfied if

$$A' \sin(\chi) + A\chi' \cos(\chi) + B(W - m - V)\cos(\chi) = 0, \tag{4.129a}$$

$$B' \cos(\chi) - B\chi' \sin(\chi) - A(W + m - V)\sin(\chi) = 0. \tag{4.129b}$$

In order that the terms with $\cos(\chi)$ in the first and the terms with $\sin(\chi)$ in the second equation may cancel, the functions $A(r), B(r), \chi(r)$ have to satisfy the differential equations

$$A\chi' + B(W - m - V) = 0, \tag{4.130a}$$

$$A(W + m - V) + B\chi' = 0. \tag{4.130b}$$

If these differential equations are looked at as linear algebraic equations for A and B unknown, the condition for the existence of a non-trivial

† More generally, the appropriate ansatz is

$$\{rf(r), rg(r)\} = \{A(r), B(r)\}\exp\{i\chi(r)\},$$

but for our purpose we may immediately start with the explicitly real solution (4.128).

solution is

$$(\chi')^2-(W-m-V)(W+m-V)=0 \tag{4.131}$$

or

$$\chi'=\bar{p} \tag{4.132}$$

if we introduce for brevity an 'effective momentum' $\bar{p}(r)$ by

$$\bar{p}(r)=[\{W-m-V(r)\}\{W+m-V(r)\}]^{1/2}. \tag{4.133}$$

It then follows that eqns (4.130) are satisfied if

$$A(r)=-\{W-m-V(r)\}^{1/2}\psi(r), \tag{4.134a}$$

$$B(r)=\{W+m-V(r)\}^{1/2}\psi(r) \tag{4.134b}$$

with any function $\psi(r)$. We are going to look for a function $\psi(r)$ such that

$$A'\simeq 0, \tag{4.135a}$$

$$B'\simeq 0, \tag{4.135b}$$

so that the remaining terms in eqn (4.129) vanish too, at least approximately. The approximation, which consists of neglecting some terms $\pm m$ as compared with $W-V(r)$, will be necessary since eqns (4.135) are not exactly compatible. It follows from eqns (4.134) and (4.135) that $\psi(r)$ should satisfy the differential equations

$$-(W-m-V)^{1/2}\psi'-\tfrac{1}{2}(W-m-V)^{-1/2}(-V)'\psi=0 \tag{4.136a}$$

if we use eqn (4.135a) or

$$(W+m-V)^{1/2}\psi'+\tfrac{1}{2}(W+m-V)^{-1/2}(-V)'\psi=0 \tag{4.136b}$$

if we use eqn (4.135b). We may eliminate V' by means of eqn (4.133), which yields

$$\bar{p}\bar{p}'=(W-V)(-V'), \tag{4.137}$$

so that we have

$$(W-V)(W\mp m-V)\psi'+\tfrac{1}{2}\bar{p}\bar{p}'\psi=0. \tag{4.138}$$

Replacing $W-V$ approximately by $W+m-V$ or $W-m-V$, respectively, and applying eqn (4.133) we obtain in both cases the differential equation

$$\bar{p}\psi'+\tfrac{1}{2}\bar{p}'\psi=0, \tag{4.139}$$

the solution of which is easily found to be

$$\psi(r)=C\{\bar{p}(r)\}^{-1/2} \tag{4.140}$$

with an arbitrary constant of integration C. In order to conform with our

standard normalization of the wave functions for $r \to \infty$ according to eqn (4.2) we need

$$C = (p/W)^{1/2}. \tag{4.141}$$

Collecting the results we see that we have constructed radial wave functions

$$f(r) = -(p/W)^{1/2}[\{W - m - V(r)\}/\bar{p}(r)]^{1/2} r^{-1} \sin\left\{\int \bar{p}(r)\, \mathrm{d}r\right\}, \tag{4.142a}$$

$$g(r) = (p/W)^{1/2}[\{W + m - V(r)\}/\bar{p}(r)]^{1/2} r^{-1} \cos\left\{\int \bar{p}(r)\, \mathrm{d}r\right\}, \tag{4.142b}$$

approximately valid when r is sufficiently large, for any potential $V(r)$. What we want to learn from eqn (4.142) is that the amplitudes of the renormalized wave functions

$$\Phi(W, r) = (W/p)^{1/2} f(r), \tag{4.143a}$$

$$\Psi(W, r) = (W/p)^{1/2} g(r) \tag{4.143b}$$

approximately have the property of depending on W and $V(r)$ via the difference $W - V(r)$ only.

We now want to identify $V(r)$ with a screened potential and denote by $V_{\mathrm{us}}(r)$ the corresponding potential without screening. We then may write

$$V(r) = V_{\mathrm{us}}(r) + U(r) \tag{4.144a}$$

where

$$U(r) = V(r) - V_{\mathrm{us}}(r) \tag{4.144b}$$

is the difference between the screened and the unscreened potential. The renormalized wave functions $\Phi(W, r), \Psi(W, r)$ then have the property that their amplitudes depend also on W and $U(r)$ approximately via the difference $W - U(r)$ only. It is therefore not unreasonable to try for all r as an approximate solution, of the exact radial equations with a screened potential,

$$\begin{Bmatrix} f_{\mathrm{Rose}}(r) \\ g_{\mathrm{Rose}}(r) \end{Bmatrix} = \begin{Bmatrix} (p/W)^{1/2} \Phi_{\mathrm{us}}\{W - U(r), r\} \\ (p/W)^{1/2} \Psi_{\mathrm{us}}\{W - U(r), r\} \end{Bmatrix} \tag{4.145}$$

with functions $\Phi_{\mathrm{us}}(x, r), \Psi_{\mathrm{us}}(x, r)$ (of two independent variables x and r) such that

$$\{f_{\mathrm{us}}(r), g_{\mathrm{us}}(r)\} = (p/W)^{1/2}\{\Phi_{\mathrm{us}}(W, r), \Psi_{\mathrm{us}}(W, r)\} \tag{4.146}$$

is the exact, normalized solution for the unscreened potential. Because

$$f'_{\mathrm{Rose}}(r) = (p/W)^{1/2}\{-(\partial \Phi_{\mathrm{us}}/\partial W)(\mathrm{d}U/\mathrm{d}r) + (\partial \Phi_{\mathrm{us}}/\partial r)\} \tag{4.147}$$

the approximation consists in neglecting the term $(\partial \Phi_{\mathrm{us}}/\partial W)(\mathrm{d}U/\mathrm{d}r)$ as

compared with $\partial\Phi_{us}/\partial r$ and similarly for $g'_{Rose}(r)$ and $\Psi_{us}(x, r)$. Using relative and dimensionless quantities, we may write the condition in the form

$$|(R\Phi_{us})^{-1}(\partial\Phi_{us}/\partial W)(R^2\,\mathrm{d}U/\mathrm{d}r)| \ll |(R/\Phi_{us})(\partial\Phi_{us}/\partial r)|. \quad (4.148)$$

Let us now look more closely at the difference potential $U(r)$. If, for example,

$$V_{us}(r) = -\alpha Z r^{-1} \quad (4.149a)$$

is the pure Coulomb potential and we use a simple exponential screening function according to

$$V(r) = -\alpha(Z-\tilde{Z})r^{-1} - \alpha\tilde{Z}r^{-1}\exp(-\beta r) \quad (4.149b)$$

with some appropriate constant β, we have

$$U(r) = -\alpha\tilde{Z}r^{-1}\{\exp(-\beta r)-1\} = \alpha\tilde{Z}\beta\{1-\tfrac{1}{2}(\beta r)+\tfrac{1}{6}(\beta r)^2-\ldots\}. \quad (4.149c)$$

For sufficiently small values of r, such that $\beta r \ll 1$, the difference between the screened and the unscreened potential is essentially given by the constant term $\alpha\tilde{Z}\beta$. This means that near the origin screening simply effects a constant potential shift

$$\tilde{V}_0 = \alpha\tilde{Z}\beta. \quad (4.150a)$$

If a more refined screening function with N exponential terms is used, an appropriate mean value

$$\beta = \sum_{n=1}^{N} a_n\beta_n \quad (4.150b)$$

enters for β. The same result is obtained if a uniformly distributed rather than the point nuclear charge is considered according to eqn (3.189).

From our approximate wave functions, eqn (4.145), we therefore may conclude that the suitably renormalized amplitudes

$$(W/p)^{1/2}p^{k-1}\alpha_\kappa \quad (4.151)$$

for the screened potential may be obtained by evaluating the corresponding amplitudes for the unscreened potential at the shifted energy $W-\tilde{V}_0$.

For the Fermi function

$$F(Z, W) \equiv F_0 L_0$$

this result means that, because of eqn (4.7), the function

$$G(Z, W) = pWF(Z, W) \quad (4.152)$$

for the screened potential is approximately equal to the corresponding function $G^{us}(Z, W)$ for the unscreened potential but evaluated at the

shifted energy $W-\tilde{V}_0$ in place of W. What we have found in this way is the well-known Rose screening formula

$$G(Z, W) \simeq G^{\mathrm{us}}(Z, W-\tilde{V}_0) \tag{4.153}$$

where $\tilde{V}_0$ is the potential shift at the origin due to screening, which is

$$\tilde{V}_0 = \alpha \tilde{Z} \beta = \alpha \tilde{Z} \sum_{n=1}^{N} a_n \beta_n \tag{4.154a}$$

with†

$$\tilde{Z} = Z - 1 \tag{4.154b}$$

for our screening model, eqn (3.189). That we need $\tilde{V}_0$ rather than the potential shift V_0 at the nuclear radius as used by Rose and later workers, is related to the fact that we consider the amplitudes α_κ of the wave functions rather than their values at the nuclear radius. Nevertheless the convenient tables for V_0 by Matese and Johnson (1966) or Garrett and Bhalla (1967) may be used since the relative difference between $\tilde{V}_0$ and V_0, which is of the order of $\frac{1}{2}\beta R$ according to eqn (4.149c), is very small and not significant in view of the underlying approximation. Since β is of the order of $\alpha \tilde{Z}^{1/3} m$, it is reasonable to represent the potential shift $\tilde{V}_0 = \alpha \tilde{Z} \beta$ or V_0 in the form‡

$$V_0 = N(\tilde{Z}) \alpha^2 \tilde{Z}^{4/3} m \tag{4.154c}$$

where $N(\tilde{Z})$, which is slowly varying, is the function suitable for tabulation. Values of $N(\tilde{Z})$, taken from Matese and Johnson (1966), are shown in Table 4.7. These values are based on relativistic self-consistent field calculations similar to those of Liberman, Waber, and Cromer (1965). A more extended table by Garrett and Bhalla (1967), based on non-relativistic self-consistent field calculations similar to those of Herman and Skillman (1963), gives values which for the larger $\tilde{Z}$ are significantly smaller.

From examples by Matese and Johnson (1966), who tested the quality of the Rose formula by numerical comparison with exact results, it may be seen that the Rose formula yields rather good values in the case of electron decays at all energies, while the agreement is less satisfactory in the case of positron decays, in particular at low energies. This behaviour can easily be understood if we try to justify our approximation by

† It may be helpful in this context to recall that $\tilde{Z}$ and Z are formally negative integers in case of positron decays and that $|\tilde{Z}|$ is the nuclear charge number of the parent atom and $|Z|$ the nuclear charge number of the daughter atom in the beta-decay process under consideration.

‡ Following our general policy, we have kept the mass m of the electron, although it is equal to 1 in our units.

TABLE 4.7 *Function $N(\tilde{Z})$ for eqn (4.154c)*

$\tilde{Z}$	$N(\tilde{Z})$
1	1·000
8	1·420
13	1·484
16	1·497
17	1·502
23	1·520
27	1·544
29	1·561
49	1·637
84	1·838
92	1·907

checking the condition (4.148), the right-hand side of which is of the order of kpR. We have

$$\mathrm{d}U/\mathrm{d}r = -\tfrac{1}{2}\alpha\tilde{Z}\beta^2\{1-\tfrac{2}{3}\beta r+\ldots\} \tag{4.155}$$

where $|\tilde{Z}|=|Z|-1$ in the case of electrons or $|\tilde{Z}|=|Z|+1$ in the case of positrons. This difference in $|\tilde{Z}|$ is unimportant, $\mathrm{d}U/\mathrm{d}r$ being of the same order of magnitude at and near the origin in both cases so that

$$|R^2(\mathrm{d}U/\mathrm{d}r)| \simeq \tfrac{1}{2}\alpha\,|\tilde{Z}|\,(\beta R)^2. \tag{4.156}$$

In order to investigate the other factor, $\partial\Phi_{us}/\partial W$ or $\partial\Psi_{us}/\partial W$, we want to restrict our attention to the case $k=1$ and be content with a very rough treatment. Near the origin the main energy dependence of Φ_{us} or Ψ_{us} is given by the energy dependence of $(W/p)^{1/2}\alpha_\kappa$, and the main energy dependence of this term is given by $\{(p/W)F_0\}^{1/2}W$ according to eqn (4.25). The quantity $(p/W)F_0$, however, is known to vary slowly with W in the case of electrons but rapidly in the case of positrons if p becomes small.

We may also see this behaviour analytically if we evaluate F_0 for the case when $(\alpha Z)^2$ becomes small while πy remains large. In the limit $(\alpha Z)^2 \to 0$ we have

$$F_0 \to \exp(\pi y)\Gamma(1+\mathrm{i}y)\Gamma(1-\mathrm{i}y) = \exp(\pi y)\pi y/\sinh(\pi y). \tag{4.157}$$

Now while

$$F_0 \to 1+\pi y+O\{(\pi y)^2\} \quad \text{if} \quad \pi|y| \ll 1,$$

it follows also from eqn (4.157) that

$$F_0 \to 2\pi|y|\exp\{\pi(y-|y|)\} \quad \text{if} \quad \pi|y| \gg 1,$$

where $y>0$ for electrons but $y<0$ for positrons. We therefore have

$$(p/W)F_0 = 2\pi\alpha Z \tag{4.158}$$

in the case of electrons, but

$$(p/W)F_0 \simeq 2\pi\alpha\,|Z|\exp(-2\pi\alpha\,|Z|\,W/p) \tag{4.159}$$

in the case of positrons. This qualitatively different behaviour of F_0 for electrons or positrons is not unexpected since we know from eqn (4.8) or (4.23) that exactly

$$F_0(-Z, W) = F_0(Z, W)\exp\{-2\pi\alpha ZW/p\} \tag{4.160}$$

holds.

That the quality of the Rose screening approximation is different for electrons and positrons now follows from eqns (4.158) and (4.159). We have

$$(R\Phi_{us})^{-1}\,\partial\Phi_{us}/\partial W \simeq R^{-1}\{(p/W)F_0\}^{-1/2}\,\partial[\{(p/W)F_0\}^{1/2}]/\partial W + (WR)^{-1}. \tag{4.161}$$

While the first term nearly vanishes for electrons according to eqn (4.158), because of eqn (4.159) it gives for positrons the most important contribution

$$\simeq -\pi\alpha\,|Z|\,R^{-1}\partial(W/p)/\partial W = \pi\alpha\,|Z|\,(pR)^{-1}(m/p)^2, \tag{4.162}$$

an expression which may become very large for small p. Collecting the results, eqns (4.156) and (4.162), we may see that the left-hand side of the condition (4.148) in the case of positrons becomes

$$\simeq \tfrac{1}{2}\pi(\alpha Z)(\alpha\tilde{Z})(\beta m/p^2)^2(pR),$$

and, since the right-hand side is of the order of (pR), the whole condition becomes

$$\tfrac{1}{2}\pi(\alpha Z)(\alpha\tilde{Z})(\beta m/p^2)^2 \ll 1 \tag{4.163}$$

and ceases from being well satisfied for p sufficiently small depending on Z.

Finally we wish to show that the Rose screening prescription, eqn (4.153), for the Fermi function can be further simplified if we consider the screening correction factor†

$$Q(Z, W) = L_0(Z, W)/L_0^{us}(Z, W) = F(Z, W)/F^{us}(Z, W). \tag{4.164}$$

† Again, the functions with the superscript us belong to the unscreened potential, those without a superscript to the corresponding screened potential.

Introducing for brevity

$$\tilde{W} = W - \tilde{V}_0, \tag{4.165a}$$

$$\tilde{p} = (\tilde{W}^2 - m^2)^{1/2}, \tag{4.165b}$$

we have from eqn (4.153).

$$F(Z, W) \simeq (\tilde{p}/p)(\tilde{W}/W)F^{us}(Z, \tilde{W}). \tag{4.166}$$

For the screening correction factor it then follows that

$$\begin{aligned} Q(Z, W) &\simeq (\tilde{p}/p)(\tilde{W}/W)F^{us}(Z, \tilde{W})/F^{us}(Z, W) \\ &= (\tilde{p}/p)(\tilde{W}/W)\{F_0(Z, \tilde{W})/F_0(Z, W)\}\{L_0^{us}(Z, \tilde{W})/L_0^{us}(Z, W)\}. \end{aligned} \tag{4.167}$$

Since we know that the energy dependence of $L_0^{us}(Z, W)$ is rather weak, we may expect that the corresponding ratio can be omitted. From eqn (4.126) we may find

$$\begin{aligned} L_0^{us}(Z, \tilde{W})/L_0^{us}(Z, W) &\simeq 1 + \alpha Z\tilde{V}_0 R - \tfrac{1}{2}m\{(1/\tilde{W}) - (1/W)\}\alpha ZmR \\ &= 1 + \alpha Z\tilde{V}_0 R\{1 - \tfrac{1}{2}(m/W)(m/\tilde{W})\}, \end{aligned}$$

an expression of which the deviation from 1 is, in fact, very small and can be neglected. We therefore have

$$Q(Z, W) \simeq (\tilde{p}/p)(\tilde{W}/W)F_0(Z, \tilde{W})/F_0(Z, W) \tag{4.168}$$

or, more explicitly,

$$Q(Z, W) \simeq (\tilde{W}/W)(\tilde{p}/p)^{-1+2\gamma_1} \exp\{\pi(\tilde{y} - y)\}\,|\,\Gamma(\gamma_1 + i\tilde{y})|^2/|\,\Gamma(\gamma_1 + iy)|^2, \tag{4.169}$$

where

$$\tilde{y} = \alpha Z\tilde{W}/\tilde{p}. \tag{4.170}$$

The approximate screening correction factor, eqn (4.169), is seen to be independent of the nuclear radius R. To compute it is easy if a subroutine for the complex gamma function is available. It should be mentioned that additional theoretical support for the Rose screening formula, eqn (4.168), comes from the work of Durand (1964) on the basis of an entirely different approach.

It remains to comment on the screening corrections for the other beta-decay Coulomb functions. Concerning those which enter the beta spectra it follows in a similar way that the quantities $(m/W)\mu_k$ and $p^{2k-2}\lambda_k$ for the screened potential may be approximated by the corresponding quantities for the unscreened potential evaluated at the shifted energy $W - \tilde{V}_0$. It should be noted, however, that checking the condition (4.148) for $k > 1$ is more complicated, and therefore it is less evident that

the approximation is really justified for $k > 1$ too. The procedure therefore cannot be recommended before its accuracy has been demonstrated by numerical examples of comparison with exact results. Quite unclear is the situation for the other beta-decay Coulomb functions which contain phase shift differences in addition to the amplitudes.

While the existing tables (Behrens and Jänecke 1969) should be used for all of these beta-decay Coulomb functions, the approximate analytical treatment of the screening correction presented in this section is of practical importance in the case of the Fermi function for obtaining in an easy and inexpensive way integrated Fermi functions or *ft*-values. Moreover, this treatment gives support to our earlier statement that the most important parameter of screening models is the potential shift $\tilde{V}_0$ near the origin. The Rose screening formula, eqn (4.153), shows the dependence on this parameter for the Fermi function.

It seems appropriate to add here some further remarks concerning tables for electron and positron beta-decay. Although tables (with or without screening) have been prepared by many authors, it is only the work by Behrens and Jänecke (1969) which contains tables just for the beta-decay Coulomb functions as defined in this book. We therefore do not want to consider here the other tables, which have been discussed in some detail by Behrens and Jänecke (1969). The situation is different, however, for integrated Fermi functions, for which Behrens and Jänecke (1969) give some diagrams only. In applications where this material is not sufficient or too inconvenient to use, the tables by Gove and Martin (1971) or those by Dzhelepov, Zyrianova, and Suslov (1972)† may be consulted. It should be noted, however, that these authors use a different definition of the Fermi function F, corresponding to radial wave functions evaluated at the nuclear radius rather than at the origin. Consequently their values of the integrated Fermi function are slightly different too.

Values of the integrated Fermi function for allowed transitions may also be obtained from the work of Wilkinson and Macefield (1974), although not without a little computation. By parametrization of the 'exact' values, these authors have obtained a relatively simple formula which shows the main dependence on the transition energy explicitly. The coefficients appearing in this formula depend on the nuclear charge number and in addition weakly on the transition energy so that different sets of coefficients are needed in certain different energy regions. A table of all the required coefficients is contained in the paper by Wilkinson and Macefield (1974). It should be noted that the integrated Fermi function of these authors, while it conforms with our view in being based on wave functions evaluated at the origin, includes approximately the deviation of

† In this work Z denotes the nuclear charge number of the parent atom rather than that of the daughter atom, a fact which should not escape the attention of the prospective user.

the Gamow–Teller spectrum shape factor from unity and in addition some other small effects like a recoil correction or radiative corrections.

It is for such differences in definition that values of the integrated Fermi function obtained from the different sources generally do not agree exactly but may show deviations of a few percent.

4.6. Quantities needed for electron capture

4.6.1. Single-particle wave functions

In the case of electron capture the behaviour of the normalized bound state solutions in a neighbourhood of the origin may be characterized by

$$\begin{Bmatrix} f_\kappa(r) \\ g_\kappa(r) \end{Bmatrix} = \beta_\kappa \{(2k-1)!!\}^{-1} (p_\kappa r)^{k-1} \sum_{n=0}^{\infty} \begin{Bmatrix} a_{\kappa n} \\ b_{\kappa n} \end{Bmatrix} (r/R)^n \tag{4.171}$$

with

$$a_{\kappa 0} = 1, \qquad b_{\kappa 0} = 0 \quad \text{if} \quad \kappa = k \tag{4.172a}$$

or

$$a_{\kappa 0} = 0, \qquad b_{\kappa 0} = 1 \quad \text{if} \quad \kappa = -k. \tag{4.172b}$$

The amplitude β_κ, which is determined by the normalization condition, is complicated and requires electronic computation on the basis of the methods explained in Section 3.3.5. Tables of β_κ therefore are needed, but since for electron capture the positive quantity

$$p_\kappa = (-p^2)^{1/2} = \{(m+W)(m-W)\}^{1/2} \tag{4.173}$$

has a fixed value depending on the shell under consideration according to the values of κ and the radial quantum number, it is even more convenient to have tables for the product $p_\kappa^{k-1}\beta_\kappa$ or for $(p_\kappa^{k-1}\beta_\kappa)^2$. A modified, more informative notation is often used with the index κ replaced by the corresponding spectroscopic symbol K, L_1, etc. of the shell.

As to the radial dependence of the bound state solutions, we have the same situation as in case of the continuum state solutions treated in Section 4.3. In analogy to eqn (4.4) we may therefore write

$$f_{+k}(r) = \beta_{+k}\{(2k-1)!!\}^{-1}(p_{+k}r)^{k-1}\{H_k(r) + h_k(r)\}, \tag{4.174a}$$

$$g_{-k}(r) = \beta_{-k}\{(2k-1)!!\}^{-1}(p_{-k}r)^{k-1}\{H_k(r) - h_k(r)\}, \tag{4.174b}$$

$$f_{-k}(r) = -\beta_{-k}\{(2k-1)!!\}^{-1}(p_{-k}r)^{k-1}(r/R)\{D_k(r) - d_k(r)\}, \tag{4.174c}$$

$$g_{+k}(r) = \beta_{+k}\{(2k-1)!!\}^{-1}(p_{+k}r)^{k-1}(r/R)\{D_k(r) + d_k(r)\}, \tag{4.174d}$$

and use the appropriate expansions of Section 4.3 for the functions

$H_k(r)$, $h_k(r)$, $D_k(r)$, $d_k(r)$ with Z replaced by the nuclear charge number $\tilde{Z}$ of the parent atom.

Some further comments are necessary concerning the computation of the amplitudes β_κ. Only the general method of Section 3.3.5 is of interest in this context since screening is essential. As in the case of electron or positron decay, it seems reasonable to use an exponentially screened potential like†

$$V(r) = -(\alpha\tilde{Z}/r)\sum_{n=1}^{N} a_n \exp(-\beta_n r) \tag{4.175}$$

or a suitably modified version of this like eqn (3.188) or eqn (3.189). It is, however, necessary to add an exchange term to this potential. While for the continuum states an exchange term either does not exist, in the case of positrons, or its contribution is negligibly small, in the case of electrons as shown by Matese and Johnson (1966), it has to be included for bound states. The simplest way to do so approximately is to add a Slater exchange potential term (Slater 1960)

$$V_{\text{exch}}(r) = \alpha\{81/(8\pi)\}^{1/3}\{\tilde{\rho}(r)\}^{1/3} \tag{4.176}$$

where $\tilde{\rho}(r)$ is the number of electrons per unit volume (in natural units) and therefore is proportional to our charge distribution $\rho(r)$. In place of eqn (4.175), the total potential including the Slater exchange term then is

$$V(r) = -(\alpha\tilde{Z}/r)\sum_{n=1}^{N} a_n \exp(-\beta_n r) - (\alpha/r)\tilde{Z}^{1/3}\{81/(32\pi^2)\}^{1/3} * \left\{\sum_{n=1}^{N} a_n(\beta_n r)^2 \exp(-\beta_n r)\right\}^{1/3}. \tag{4.177}$$

The form of the exchange term is not so appropriate if the power series method for integrating the Dirac radial equations according to Section 3.3.5.2 is to be used. Behrens and Jänecke (1969) have therefore replaced it in the case of $N = 3$ by an analytically simpler expression, so that the total potential looks like

$$V(r) = -(\alpha\tilde{Z}/r)\left\{\sum_{n=1}^{3} a_n \exp(-\beta_n r) + r\sum_{n=1}^{3} \tilde{a}_n \exp(-\tilde{\beta}_n r)\right\} \tag{4.178}$$

with suitably chosen values of the new parameters $\tilde{a}_n$, $\tilde{\beta}_n$, as obtained by least squares fitting‡. Equation (4.178) need not necessarily be a poorer

† Here the potential parameters β_n should not be confused with the amplitude β_κ of the bound state wave function.

‡ It should be mentioned that more recently tables have become available which in numerical form contain atomic potentials based on relativistic Hartree-Fock computations (Lu, Carlson, Malik, Tucker and Nestor, Jr., 1971). Since exchange is usually included, such tables may be used for directly fitting the parameters of eqn (4.178). For positron or electron decay, on the other hand, the appropriate potential parameters are less simple to obtain, since here the electrostatic part of the potential without any exchange contribution is needed (Matese and Johnson 1966).

approximation, but might even be preferable since the pathological $r^{-1/3}$ branch point of the Slater exchange term is no longer present. Finally it should be mentioned that Behrens and Jänecke (1969) did not use eqn (4.178) as it stands but with the slight modifications corresponding to the transition from the potential, eqn (3.181) or (4.175), to the more refined potential, eqn (3.188). We do not want to discuss this matter in more detail since, besides the tables of bound state amplitudes which have been computed in this way (Behrens and Jänecke 1969), several other sources for bound state amplitudes or related quantities are available. This is so for the following reason: while we have assumed that the parameters of an appropriate atomic potential are already known and we only have to treat the bound state problem for a single electron in this potential, the computation of the potential itself is a much more complicated many-electron problem which, when solved by means of the self-consistent field method, furnishes as a by-product the single electron bound state energies and wave functions we are interested in for electron capture. There is a recent review article (Bambynek *et al.* 1977) which contains tables for electron capture based on such atomic structure calculations by Mann and Waber (1973) and includes a detailed comparison and discussion of the other published tables for electron capture. We do not want to reproduce this material here, but refer the interested reader to that article.

4.6.2. Many-particle effects

A complete description of the initial and final states for the electron capture process must include the electrons of the atomic cloud (as first shown by Benoist-Gueutal 1953 and by Odiot and Daudel 1956). An essential part of this atomic cloud effect was, on the other hand, taken into account by introducing an average screening function caused by the atomic electrons, as discussed in the previous section. Since the nuclear charge and the number of electrons are, however, different in the initial and final states, the atomic electron wave functions of these states are not orthogonal, and the overlap between them is not perfect. Besides some excitation and autoionization processes of the atomic cloud (shake-up and shake-off) this leads to a modification of the transition rate which can usually be expressed as an exchange and overlap correction factor of the form (Bahcall 1963)

$$B_x = \left. \frac{\sum_j |\langle \psi_x^{(j)} | O | \psi_g \rangle|^2}{g_x^2(r)} \right|_{r=0} \tag{4.179}$$

Here ψ_x and ψ_g are the many-electron wave functions for the initial state (ground state of an atom with atomic number $\tilde{Z}$) and the final state (of an atom with atomic number $\tilde{Z}-1$ and 1 hole in shell x) in second

quantization. The summation over j runs over different (for example, excited) final states. The operator which is responsible for the transition takes the form (see Chapter 6)

$$O = \sum_i a_i g_i(r) \tag{4.180}$$

where a_i is the annihilation operator for an electron in shell i and $g_i(r)$ the corresponding bound state electron radial wave function (see eqn (4.174*b*)). x stands for the shell under consideration i.e. for K, L_1, L_2, L_3, M_1 etc.

The above expression holds for capture from shells with $\kappa < 0$; for capture from shells with $\kappa > 0$, $g(r)$ has to be replaced by $f(r)$. The atomic electron wave functions can be written exactly as (see Sections 8.1.1.3 and 8.1.1.4 for many nucleon wave functions)

$$\psi(J) = \sum_k^{\text{Dim}} C_k \phi_k(J) \tag{4.181}$$

where J is the total angular momentum and the ϕ_k are the antisymmetric independent many-particle basis states (for example, Slater determinants of single-particle wave functions calculated for suitably screened nuclei). If, as usually, L–S coupling is applied, this equation can also be written as

$$\psi(LS) = \sum_k C'_k \phi_k(LS) \tag{4.182}$$

where L is the total orbital angular momentum, and S is the total spin angular momentum.

The atomic correlations, i.e. the coefficients C_k, have to be calculated by introducing the residual interaction between the different electrons i and j

$$V_{ij} = \alpha \frac{1}{r_{ij}} \tag{4.183}$$

and by making use of appropriate methods like perturbation theory, multi-configurational Hartree-Fock and others (for a detailed review see Froese Fischer 1977). The atomic matrix element then becomes

$$\langle \psi_x^{(j)} | O | \psi_g \rangle = \sum_{k'k} C_{k'}^{(j)} C_k \langle \phi_{k'}^{(j)} | O | \phi_k \rangle \tag{4.184}$$

When outer electrons are involved exact calculations of the exchange and overlap correction factors B_x according to the lines sketched above have to be carried out. As explicitly demonstrated in some recent publications (Chen and Crasemann 1978; Neumann 1979) in such cases electron correlations often substantially and sometimes even dramatically affect electron capture decay constants.

When only very inner electron shells are involved, on the other hand, use can be made of the independent-particle approximation. Effects due to the electron–electron residual interaction can be neglected in this case. Thus, the calculation of the exchange and overlap correction factor can explicitly be written down since the corresponding initial and final states are represented by a Slater determinant each. One then obtains, for example, for B_K

$$B_K = D\left\{\langle 2s' \mid 2s\rangle\langle 3s' \mid 3s\rangle - \langle 2s' \mid 1s\rangle\langle 3s' \mid 3s\rangle \frac{\beta_{2s}}{\beta_{1s}} - \langle 2s' \mid 2s\rangle\langle 3s' \mid 1s\rangle \frac{\beta_{3s}}{\beta_{1s}}\right\}^2 \tag{4.185}$$

where Bahcall has set (Bahcall 1963)

$$D = 1 \tag{4.186}$$

and Vatai (Vatai 1970)

$$D = \langle 1s' \mid 1s\rangle^2\langle 2s' \mid 2s\rangle^2\langle 2p' \mid 2p\rangle^{2n(2p)}\langle 3s' \mid 3s\rangle^{2[n(3s)-1]}\langle 3p' \mid 3p\rangle^{2n(3p)} \tag{4.187}$$

Here, $n(..)$ is the relative occupation number of the $(..)$-shell (for closed shells we have $n = 1$). β_x is the bound state radial wave function amplitude introduced before (see eqns (4.174*a–d*)). For other shells the quantities B_x have a similar form and can easily be obtained. The difference of the factor D between the two approaches has its origin in a different treatment of the overlap corrections.† In Bahcall's approach, the closure relation has been applied to perform the sum over the infinite number of final atomic states (sum over j in eqn (4.179)). This leads to an overestimate of the shake-up and shake-off contributions. On the contrary, these latter contributions are simply neglected in Vatai's treatment where the predominant configurations are considered only. Based on a recalculation a comparison of both methods was made some years ago by using the same Hartree-Fock wave functions and by including the hole in the daughter atom for both approaches (Bambynek *et al.* 1977). It then comes out that Vatai's approach underestimates L/K capture ratios at low $\bar{Z}$ while Bahcall's approach yields better agreement for L/K capture ratios. On the other hand, M/L capture ratios calculated by using Vatai's method are more reliable compared with those obtained by Bahcall's formulation which are overestimated.

Other earlier calculations of exchange and overlap correction factors B_x or of the corresponding correction factors $X^{x_2/x_1} = B_{x_2}/B_{x_1}$ for capture ratios, respectively, where either Bahcall's or Vatai's approach has been

† Originally, Vatai included the effect of the inner hole in the daughter atom (but in perturbation theory) whilst Bahcall neglected this contribution i.e. took neutral atomic states.

applied, also exist. Here, non-relativistic (Faessler *et al.* 1970) as well as relativistic wave functions (Suslov 1970; Martin and Blichert-Toft 1970) have been used. The latter two publications contain, by the way, comprehensive tables of the corresponding correction factors up to the highest atomic numbers $\bar{Z}$.

Beyond that, the whole topic has been critically reviewed recently (Bambynek *et al.* 1977). In that review article all methods based on the independent particle approximation are considered in detail and the various existing theoretical calculations are compared with one another (and with experiments). A number of values have also been recalculated there. Thus, for a more detailed survey and for comprehensive tables the reader is once again referred to that publication. An inspection of Table 4.8 shows us that electron capture from single shells is essentially in-

TABLE 4.8 *Exchange and overlap correction factors* B_K, B_{L_1}, B_{M_1} *(from Bambynek et al. 1977)*†

$\bar{Z}$	B_K		B_{L_1}		B_{M_1}	
	Bahcall	Vatai	Bahcall	Vatai	Bahcall	Vatai
5	0·924	0·866	2·432	1·875		
10	0·970	0·957	1·449	1·309		
15	0·975	0·968	1·219	1·152	1·733	1·411
20	0·980	0·975	1·141	1·099	1·414	1·239
30	0·986	0·983	1·085	1·067	1·258	1·186
40	0·989	0·987	1·060	1·045	1·162	1·112
50	0·991	0·990	1·045	1·035	1·121	1·086

† The values listed above have not been taken from Bahcall's and Vatai's original work, but they have been recalculated according to their methods.

fluenced by exchange and overlap effects, at least, for light nuclei. That is, however, not true for the total capture rate from all shells. Here, the net effect does not exceed a few percent (Bahcall 1963). As mentioned before, electron capture (and also beta-decay) is accompanied by atomic transitions. Namely, the atom reacts to the sudden change of its charge by an excitation of electrons to unoccupied bound states (shake-up) and by an ejection of bound electrons to continuum states (shake-off). Shaking processes, especially of outer atomic shells, are relatively probable events (20–30%). The contribution of the inner electron shaking is, on the other hand, very small in all but the lightest atoms. Although this special aspect does not fall as a whole within the scope of this text, for the convenience of the reader we will not renounce to quote some reviews where detailed information about atomic structure effects of this type can be found (see, for example, Freedman 1974; Walen and Briancon 1975; Bambynek *et al.* 1977).

5
BETA-DECAY KINEMATICS

5.1. General remarks

IN THE preceding chapters the solution of Dirac radial equations, the corresponding electron radial wave functions and the related combinations of them needed for beta-decay were treated in detail. Now, we begin with the beta-decay itself. First of all, however, we will illustrate briefly the consequences which follow simply from energy and momentum conservation. Then, decay rates, S- and T-matrix, phase space and kinematics of a decay into three particles are discussed but without giving a derivation of these quantities from first principles (for more details see, for example, Pilkuhn 1979 and Scadron 1979).

Generally, a three body decay, which is the situation in beta-decay, is complicated in its treatment. In our case we can, however, make use of the fact that the nuclei in initial and final state are very heavy compared with the Q-value of the beta-decay. Thus, in a very good approximation the participating nuclei can be assumed to be infinitely massive. Then, the whole treatment is remarkably simplified as we will see in the following. This is also the reason why neither a covariant S- and T-matrix nor covariantly normalized particle states have to be used throughout this text. Our choice of the T-matrix normalization which is the conventional one of the non-relativistic quantum mechanics has, on the other hand, the advantage that on the one side the T-matrix is approximately independent of the kinematical variables in certain cases and on the other side the non-relativistic limit is easy to make. All formulae are developed either in the laboratory system with the initial nucleus being at rest or in the Breit system whereby the difference between both reference systems can again be neglected because of the massive nuclei. The corresponding Lorentz transformation which transforms one reference system to the other one is briefly elucidated, too. In particular, results which simply follow from phase space kinematics alone are examined since the whole behaviour of some beta-decay observables is governed to a large extent by these kinematic effects.

5.2. Energetics

In nuclear beta-decay we have to consider the three basic processes

$$\begin{aligned} &(Z, A) \rightarrow (Z+1, A) + e^- + \bar{\nu}_e \quad (\beta^-\text{-decay}) \\ &(Z, A) \rightarrow (Z-1, A) + e^+ + \nu_e \quad (\beta^+\text{-decay}) \\ &(Z, A) + e^- \rightarrow (Z-1, A) + \nu_e \quad \text{(electron capture)} \end{aligned} \tag{5.1}$$

Here, (Z, A) signifies an atomic nucleus of mass number A and atomic number Z, $e^-(e^+)$ denotes an electron (positron) and $\nu_e(\bar{\nu}_e)$ is the neutrino (antineutrino).

First, let us discuss the momenta and energies of the different particles which follow from energy and momentum conservation. Assuming the initial nucleus is at rest before the decay we have the four momenta for initial and final nucleus

$$p_i = \{\mathbf{0}, \mathrm{i}M_i\} \tag{5.2}$$

$$p_f = \{\mathbf{p}_f, \mathrm{i}W_f\} \tag{5.3}$$

For electron and neutrino we have correspondingly (assuming neutrino rest mass to be zero).

$$p_e = \{\mathbf{p}_e, \mathrm{i}W_e\} \tag{5.4}$$

$$p_\nu = \{\mathbf{p}_\nu, \mathrm{i}W_\nu\} = \{\mathbf{p}_\nu, \mathrm{i}\,|\mathbf{p}_\nu|\} \tag{5.5}$$

In the case of β^-- and β^+-decay momentum and energy conservation requires

$$\mathbf{p}_f + \mathbf{p}_e + \mathbf{p}_\nu = 0 \tag{5.6}$$

$$W_f + W_e + W_\nu = M_i \tag{5.7}$$

For nearly all beta-decays the maximum kinetic energy of the emitted beta particles is smaller than 20 MeV. We have therefore $(W_e + W_\nu)_{\max} = W_{e_{\max}} \lesssim 20$ MeV. The nuclear mass is given by $M_i \simeq A \times 1000$ MeV. We therefore get

$$\frac{W_f - M_i}{M_i} \lesssim \frac{0{\cdot}02}{A} \ll 1 \tag{5.8}$$

Thus we can replace

$$W_f = \sqrt{(p_f^2 + M_f^2)} \tag{5.9}$$

by the non-relativistic formula

$$W_f = M_f + \underbrace{\frac{p_f^2}{2M_f}}_{E_R} \tag{5.10}$$

(since $M_f \simeq M_i$ and $(p_f^2/2M_f^2) \ll 1$).

The last term of eqn (5.10) represents the kinetic recoil energy E_R of the daughter nucleus.

Let us now introduce the Lorentz invariant quantities

$$P^2 = (p_i + p_f)^2 = \mathbf{p}_f^2 - (W_f + M_i)^2 \simeq -(M_f + M_i)^2 - 2M_i E_R \tag{5.11}$$

and the four momentum transfer to the nucleus

$$q^2 = (p_f - p_i)^2 = \mathbf{p}_f^2 - (W_f - M_i)^2$$
$$q^2 \simeq -\Delta^2 + 2M_i E_R \tag{5.12}$$

where

$$\Delta = -(M_f - M_i) \tag{5.13}$$

is the nuclear mass difference between the initial and final nuclear state.†

Since it is important for what follows later we will consider the three momentum transfer to the nucleus (which is, of course, not Lorentz invariant),

$$\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i = -(\mathbf{p}_e + \mathbf{p}_\nu). \tag{5.14}$$

In our system, where the initial nucleus is at rest, we have

$$\mathbf{q} = \mathbf{p}_f \tag{5.15}$$

and

$$\mathbf{q}^2 = \mathbf{p}_f^2 = 2M_f E_R. \tag{5.16}$$

Let us now turn to the special cases:

$$\begin{aligned} &\mathbf{p}_e = 0 \\ &\mathbf{p}_e = \mathbf{p}_\nu \\ &\mathbf{p}_\nu = 0 \end{aligned} \tag{5.17}$$

By combining eqns (5.6), (5.7) and (5.10) it can easily be shown that

$$\text{for } \mathbf{p}_e = 0 \qquad \mathbf{q}^2 = (\Delta - m_e)^2, \tag{5.18}$$

$$\text{for } \mathbf{p}_e = \mathbf{p}_\nu \qquad \mathbf{q}^2 = \frac{(\Delta^2 - m_e^2)^2}{\Delta^2}, \tag{5.19}$$

$$\text{for } \mathbf{p}_\nu = 0 \qquad \mathbf{q}^2 = \Delta^2 - m_e^2. \tag{5.20}$$

† In nuclear beta-decay the initial and final states do not consist of bare nuclei, but rather of atoms. Thus, for a complete description, the entire atom, including its electron cloud, has to be considered. The energy Δ which is at the disposal for the decay is therefore related to the difference of the atomic masses M_A by

$$\Delta - m_e = M_A(i) - M_A(f) \quad \beta^-\text{-decay}$$
$$\Delta + m_e = M_A(i) - M_A(f) \quad \beta^+\text{-decay}$$

The above relations can also be expressed in terms of the nuclear mass difference $(M_i - M_f)$ by

$$\Delta = M_i - M_f + |\Delta B| \quad \beta^-\text{-decay}$$
$$\Delta = M_i - M_f - |\Delta B| \quad \beta^+\text{-decay}$$

where ΔB is total change in the atomic electron binding energy. Contrary to a first belief, the atomic binding energy change ΔB is not negligible. It amounts, for example, to about $\simeq 20$ keV for $Z = 85$.

$\mathbf{q}^2$ is a maximum for the last case where $\mathbf{p}_\nu = 0$. Thus we obtain the following limits for three and four momentum transfer†

$$0 \leqslant \mathbf{q}^2 \leqslant \Delta^2 - m_e^2 \tag{5.21}$$

and

$$-\Delta^2 \leqslant q^2 \leqslant -m_e^2. \tag{5.22}$$

Note that in our metric the square of the four momentum transfer is always negative for the nuclear beta-decay.

As far as the maximal energy W_0 of the beta particles emitted is concerned we can deduce from eqn (5.7)

$$W_0 = -(W_f - M_i) = \Delta - |E_R|_{W_\nu = 0} \tag{5.23}$$

and by using eqn (5.20)

$$W_0 = \Delta\left\{1 - \frac{\Delta^2 - m_e^2}{2M_f\Delta}\right\} \tag{5.24}$$

The last term in eqn (5.24) takes into account the recoil energy correction. It is of the order $\leqslant 0{\cdot}01/A$ (see eqn (1.8)) and can usually be neglected. Then we have

$$W_0 \simeq \Delta \tag{5.25}$$

In the case of electron capture the final state is a two particle state (neutrino and recoil nucleus). Energy and momentum conservation requires here

$$M_i + m_e = W_f + W_\nu \tag{5.26}$$

$$\mathbf{p}_f + \mathbf{p}_\nu = 0 \tag{5.27}$$

Explicitly we then have

$$W_\nu = \Delta + m_e - E_R = \Delta + m_e\left\{1 - \frac{\Delta + m_e}{2M_f}\right\} \tag{5.28}$$

Further we obtain for the three and four momentum transfer

$$\mathbf{q}^2 = (\Delta + m_e)^2\left\{1 - \frac{\Delta + m_e}{2M_f}\right\}^2 \simeq (\Delta + m_e)^2 \tag{5.29}$$

$$q^2 = m_e^2 + 2m_e\left\{\Delta - \frac{(\Delta + m_e)^2}{2M_f}\right\}. \tag{5.30}$$

In contrast to the beta-decay the square of the four momentum transfer is

† Note that eqn (5.21) is not entirely exact. The exact value of $\mathbf{q}_{max}$ is

$$\mathbf{q}^2_{max} = 2M_f[M_f + \Delta - \sqrt{\{(M_f + \Delta)^2 - (\Delta^2 - m_e^2)\}}]$$

positive in this case. Beyond that the value of both momentum transfers is fixed.

Up to now we have only discussed the whole kinematical problem in the laboratory system, and here especially the case where the initial nucleus is at rest.

Since it is important for some calculations which follow in later chapters, we will give a short outline of the transformation in the Breit (or brick wall) system.

The Lorentz transformation of energy and three momentum from the laboratory system K to an other system K' is given by (Hagedorn 1963)

$$\mathbf{p}' = \mathbf{p} + \boldsymbol{\beta}\gamma\left(\frac{\gamma\boldsymbol{\beta}\mathbf{p}}{\gamma-1} - W\right) \tag{5.31}$$

$$W' = \gamma(W - \boldsymbol{\beta}\mathbf{p}). \tag{5.32}$$

$\mathbf{v} = \boldsymbol{\beta}$ is the velocity of the system K' as seen from K and γ is related to β by

$$\gamma = \frac{1}{\sqrt{(1-\beta^2)}}. \tag{5.33}$$

In the system K' the four momenta of the initial and final nuclei are

$$\begin{aligned} p_i' &= \{\mathbf{p}_i', \mathrm{i}W_i'\} \\ p_f' &= \{\mathbf{p}_f', \mathrm{i}W_f'\}. \end{aligned} \tag{5.34}$$

A special system called the Breit (or brick wall) system is defined as follows

$$\mathbf{p}_i' + \mathbf{p}_f' = 0. \tag{5.35}$$

Combining eqns (5.31) and (5.35) we obtain

$$\mathbf{p}_i' + \mathbf{p}_f' = 0 = \mathbf{p}_f + \boldsymbol{\beta}\gamma\left(\frac{\gamma}{1+\gamma}\boldsymbol{\beta}\mathbf{p}_f - W_f\right) - \boldsymbol{\beta}\gamma W_i \tag{5.36}$$

and

$$\boldsymbol{\beta} = \frac{\mathbf{p}_f}{W_f + M_i} = \frac{\mathbf{q}}{W_f + M_i} \simeq \frac{\mathbf{q}}{M_f + M_i}. \tag{5.37}$$

By inserting this result in eqn (5.31) we obtain

$$\mathbf{p}_f' = -\mathbf{p}_i' = \frac{M_i}{\sqrt{(M_f^2 + M_i^2 + 2W_f M_i)}}\mathbf{q} \simeq \frac{M_i}{M_f + M_i}\mathbf{q} \simeq \frac{\mathbf{q}}{2} \tag{5.38}$$

The energies W_i' and W_f' in the Breit system are obtained as

$$W_i' = \gamma M_i \tag{5.39}$$

$$W_f' = \gamma\left\{\frac{M_f^2 + W_f M_i}{W_f + M_i}\right\} \simeq \gamma M_f. \tag{5.40}$$

In the case of an elastic process where $M_i = M_f = M$ we would have $W_i' = W_f' = \gamma M$, $q = (\mathbf{q}, 0)$ and $P = \{\mathbf{0}, i\gamma 2M\}$. This is the usual situation in elastic scattering processes. In our problem of nuclear beta-decay, however, we have something like an inelastic process and therefore (if the recoil energy E_R is neglected)

$$q' \simeq \{\mathbf{q}, -i\Delta\} \tag{5.41}$$

$$P' = \{\mathbf{0}, i\gamma(M_f + M_i)\} \tag{5.42}$$

where P' has the direction of the four-axis.

5.3. *S*-Matrix, transition probability and phase space

It is our aim to calculate the transition probability between special initial and final states, i.e. all observables of the nuclear beta-decay. An useful formalism to do so is the scattering matrix or S-matrix. The symbol S designates the matrix whose elements connect initial and final states of a weak decay in the following way

$$S_{fi} = \langle f | S | i \rangle \tag{5.43}$$

(see, for example, Källen 1969; Perl 1974; Scadron 1979).

The transition probability for an initial state i to a final state f is then given by

$$P_{fi} = \langle f | S | i \rangle^* \langle f | S | i \rangle = S_{fi}^* S_{fi} = S_{if}^+ S_{fi} \tag{5.44}$$

The total decay probability of a state i can be obtained by summing over all final states f.

S is a unitary matrix, i.e.

$$S^+ S = 1 \tag{5.45}$$

It should, however, be noted that S includes the probability that there is an interaction plus the probability that there is no interaction.

In order to eliminate this last probability, another matrix called the transition matrix, T, is usually introduced. Both matrices are related by

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - p_i) N T_{fi}. \tag{5.46}$$

The four-dimensional δ-function is split off since the last term of eqn (5.46) is different from zero only if the total energy and momentum of initial and final states are equal. p_i and p_f are, therefore, the total four momenta of the initial and the final state, respectively.

N is a normalization factor which depends essentially on the normaliza-

tion of the continuum plane wave states in initial and final states

$$\begin{aligned} i\rangle &= |\mathbf{p}_i, s_i\rangle \\ f\rangle &= |\mathbf{p}_f, s_f; \mathbf{p}_e, s_e; \mathbf{p}_\nu, s_\nu\rangle. \end{aligned} \tag{5.47}$$

s is a discrete parameter representing the spin state of the particle. In our case we have chosen the normalization

$$\langle \mathbf{p}', s' \,|\, \mathbf{p}, s\rangle = (2\pi)^3 \delta^3(\mathbf{p}' - \mathbf{p}) \delta_{s's} \tag{5.48}$$

i.e. we have one particle per unit volume V, since

$$\langle \mathbf{p}, s \,|\, \mathbf{p}, s\rangle = (2\pi)^3 \delta^3(0) = V. \tag{5.49}$$

The normalization factor has then the value†

$$N = \frac{1}{V^{(n_i+n_f)/2}} \tag{5.50}$$

where n_i = number of particles in the initial state and n_f = number of particles in the final state. In our case of nuclear beta-decay we have $n_i = 1$ and $n_f = 3$ (for electron capture $n_i = 2$ and $n_f = 2$).

For the transition probability, we now obtain, by combining eqn (5.44) and eqn (5.46),

$$P_{fi} = \frac{[(2\pi)^4 \delta^4(p_f - p_i)]^2}{V^{n_i+n_f}} T_{if}^{+} T_{fi} \quad (f \neq i). \tag{5.51}$$

The next step is to simplify the $[\delta^4]^2$-function. We first write (for a proof see Eisele 1969)

$$\begin{aligned} [(2\pi)^4 \delta^4(p_f - p_i)]^2 &= (2\pi)^4 \delta^4(p_f - p_i)(2\pi)^4 \delta^4(0) \\ &= \lim_{V,T\to\infty} (2\pi)^4 \delta^4(p_f - p_i) VT \end{aligned} \tag{5.52}$$

where VT is the four dimensional volume element (space × time).

We are, however, not interested in the transition probability to a single state f. Instead we must sum over all final states.

For a particle in three dimensions of the volume and momentum space

† Note that the normalization N chosen in eqn (1.50) does not lead to Lorentz invariant T-matrix elements. A formulation of

$$N = \prod_{\substack{\text{initial}\\ \text{state}}} \frac{1}{\sqrt{(2W_i V)}} \prod_{\substack{\text{final}\\ \text{state}}} \frac{1}{\sqrt{(2W_f V)}}$$

would lead to such a Lorentz invariant form. Usually N is, therefore, chosen as given above. In our case of nuclear beta-decay the choice of eqn (5.50) has some advantages as we shall see later in this book.

the number of states in the six dimensional phase space $\int \mathrm{d}x^3\, \mathrm{d}^3p$ is

$$N_\Phi = \frac{\int \mathrm{d}^3x\, \mathrm{d}^3p}{(2\pi)^3} = \frac{V}{(2\pi)^3}\int \mathrm{d}^3p. \tag{5.53}$$

Therefore we must divide eqn (5.51) by the time T and multiply by the phase space factors of eqn (5.53) for the final particles. Then we obtain for the decay probability per unit time

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \sum_f \sum_s \int \frac{(2\pi)^4 \delta^4(p_f - p_i)V}{V^4} T_{if}^{+} T_{fi} \frac{\mathrm{d}^3p_f V}{(2\pi)^3} \frac{\mathrm{d}^3p_e V}{(2\pi)^3} \frac{\mathrm{d}^3p_\nu V}{(2\pi)^3}. \tag{5.54}$$

The summation over the final states consists of both the integration over p of every particle and the summation over all spin states together with all other final states.

By splitting off the δ^4 function in a momentum and energy dependent part we get finally

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{(2\pi)^5} \sum_f \sum_s \int & T_{if}^{+} T_{fi} \delta^3(\mathbf{p}_f + \mathbf{p}_e + \mathbf{p}_\nu - \mathbf{p}_i) \\ & \times \delta(W_f + W_e + W_\nu - W_i)\, \mathrm{d}^3p_f\, \mathrm{d}^3p_e\, \mathrm{d}^3p_\nu\end{aligned} \tag{5.55}$$

and in the rest frame of initial (parent) nucleus $\mathbf{p}_i = 0$

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{(2\pi)^5} \sum_f \sum_s \int & |T|^2 \delta^3(\mathbf{p}_f + \mathbf{p}_e + \mathbf{p}_\nu) \\ & \times \delta(W_f + W_e + W_\nu - M_i)\, \mathrm{d}^3p_f\, \mathrm{d}^3p_e\, \mathrm{d}^3p_\nu.\end{aligned} \tag{5.56}$$

The beta-decay process is caused by the interaction Hamiltonian H_β of the weak interaction. In first order perturbation theory the S-matrix element is given by (see, for example, Scadron 1979)

$$S_{fi} = \delta_{fi} - i\left\langle f \left| \int H_\beta(x)\, \mathrm{d}^4x \right| i \right\rangle + \ldots \tag{5.57}$$

The Hamiltonian H_β and the T-matrix are, therefore, connected by

$$(2\pi)^4 \delta^4(p_f - p_i) T_{fi} = -\left\langle f \left| \int H_\beta(x)\, \mathrm{d}^4x \right| i \right\rangle. \tag{5.58}$$

Coming back to eqn (5.55) we see that the dynamics of the decay process is contained in the T-matrix $|T|^2$ whereas all the rest, which is usually called 'phase space factor', is kinematics.

Let us now in a first approximation assume that

$$\sum_f \sum_s |T|^2 = \text{const},$$

i.e. that $|T|^2$ is independent of momenta, energies and angles of the three

particles in the final state.† The decay is then governed by the pure kinematical aspects and we have

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\text{const}}{(2\pi)^5} R \tag{5.59}$$

where R is the phase space integral

$$R = \int \delta^3(\mathbf{p}_f + \mathbf{p}_e + \mathbf{p}_\nu)\delta(W_f + W_e + W_\nu - M_i)\,\mathrm{d}^3p_f\,\mathrm{d}^3p_e\,\mathrm{d}^3p_\nu. \tag{5.60}$$

The solution of phase space integrals is, for example, extensively discussed by Hagedorn (1963) and Byckling and Kajantie (1974).

We turn now to the more concrete calculation of this integral. First we shall carry out the integration over the neutrino and recoil nucleus variables in order to obtain the electron energy distribution. Secondly we shall replace the integration over the recoil nucleus variables by the integration over the electron variables in order to get the recoil nucleus momentum distribution.

We begin with the integration over $\mathbf{p}_\nu$ with the result

$$\mathrm{d}R = \delta(W_f + W_e + W_\nu - M_i)\,\mathrm{d}^3p_e\,\mathrm{d}^3p_f \tag{5.61}$$

$$W_\nu^2 = (\mathbf{p}_e + \mathbf{p}_f)^2 = p_e^2 + p_f^2 + 2p_e p_f \cos\Theta_{fe} \tag{5.62}$$

Θ_{fe} is the angle between $\mathbf{p}_e$ and $\mathbf{p}_f$. Further we write‡

$$\mathrm{d}^3p_e\,\mathrm{d}^3p_f = p_e^2\,\mathrm{d}p_e\,\mathrm{d}\Omega_e p_f^2\,\mathrm{d}p_f\,\mathrm{d}\Omega_f = p_e W_e\,\mathrm{d}W_e\,\mathrm{d}(\cos\Theta_{fe})\,\mathrm{d}\phi_e\,\mathrm{d}\Omega_f \tag{5.63}$$

where $\mathrm{d}\Omega$ is the solid angle ($\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$).

Equation (5.61) could be integrated over $\mathrm{d}\phi_e$ to give 2π and over $\mathrm{d}\Omega_f$ to give 4π.

The next step is to integrate over $\mathrm{d}(\cos\Theta_{fe})$. For this purpose we use the relation

$$W_\nu\,\mathrm{d}W_\nu = p_e p_f\,\mathrm{d}(\cos\Theta_{fe}) \tag{5.64}$$

which follows from eqn (5.62).

Then we obtain

$$\mathrm{d}R = 8\pi^2 \int_{p_{f_1}}^{p_{f_2}} (W_f + W_e - M_i) p_f\,\mathrm{d}p_f W_e\,\mathrm{d}W_e. \tag{5.65}$$

p_{f_1} and p_{f_2} are given by the boundary of the physical region in the $p_e p_f$

† It should be noted that this approximation is partly a good one for certain types of transitions (allowed transitions), at least as far as the dependence from energy and momentum of the different particles is concerned. $|T|^2$ is, however, always dependent on the angles (for example, between electron and neutrino) to a certain degree.

‡ Here we use $p_e = |\mathbf{p}_e|$.

plane corresponding to $\cos\Theta_{fe} = \pm 1$ in eqn (5.62). Introducing the further approximation $W_f \simeq M_f$ already discussed in Section 5.2, we find

$$W_\nu = M_i - W_e - M_f = p_e \pm p_f$$

and

$$p_{f_{1,2}} = |p_e \pm (\Delta - W_e)|. \tag{5.66}$$

Integrating eqn (5.65) we get finally†

$$dR = 8\pi^2\{\Delta - W_e\}W_e\left\{\frac{p_{f_1}^2}{2} - \frac{p_{f_2}^2}{2}\right\}dW_e \tag{5.67}$$

$$= 16\pi^2(\Delta - W_e)^2 p_e W_e\, dW_e$$

(probability for an electron $P(W_e)$ to be emitted for an energy between W_e and $W_e + dW_e$).

The beta ray spectrum now becomes ($W_0 = \Delta$, see eqn (5.25))

$$P(W_e)\,dW_e = \frac{\sum_f \sum_s |T|^2}{2\pi^3}(W_0 - W_e)^2 p_e W_e\, dW_e \tag{5.68}$$

(see Fig. 5.1).

The factor $(W_0 - W_e)^2 p_e W_e\, dW_e$ is called the statistical shape. The last integration over the electron energy can also easily be carried out and we obtain

$$R = 16\pi^2 \int_1^{W_0} (W_0 - W_e)^2 p_e W_e\, dW_e$$

$$= \frac{4\pi^2}{15}[(2W_0^4 - 9W_0^2 - 8)\sqrt{(W_0^2 - m_e^2)} + 15 W_0 \ln\{W_0 + \sqrt{(W_0^2 - m_e^2)}\}]. \tag{5.69}$$

Besides the electron distribution we are also able to calculate the statistical recoil momentum distribution. For this purpose we have now to integrate eqn (5.65) over the electron energy W_e, i.e.

$$dR = 8\pi^2 \int_{W_{e_1}}^{W_{e_2}} (W_f + W_e - M_i) W_e\, dW_e p_f\, dp_f. \tag{5.70}$$

† For a finite neutrino rest mass m_ν this statistical shape factor will of course be modified. It then reads

$$P(W_e)\,dW_e = \frac{\sum_f \sum_s |T|^2}{2\pi^3}[\Delta - W_e]\{[\Delta - W_e]^2 - m_\nu^2\}^{1/2} p_e W_e\, dW_e.$$

We then also have $\Delta = W_0 + m_\nu$.

It is not certain that the neutrino rest mass is exactly zero (Bilenky and Pontecorvo 1978; Wolfenstein 1978), but, it is assumed that it is at least $m_\nu < 50$ eV, even if a finite value for m_ν of the order of a few eV exists (Lubimov *et al.* 1980; Reines *et al.* 1980; de Rujula *et al.* 1980).

Effects of such a finite neutrino mass can only be observed if beta-transitions of low energy are investigated like, for example, the decay of ^{3}H.

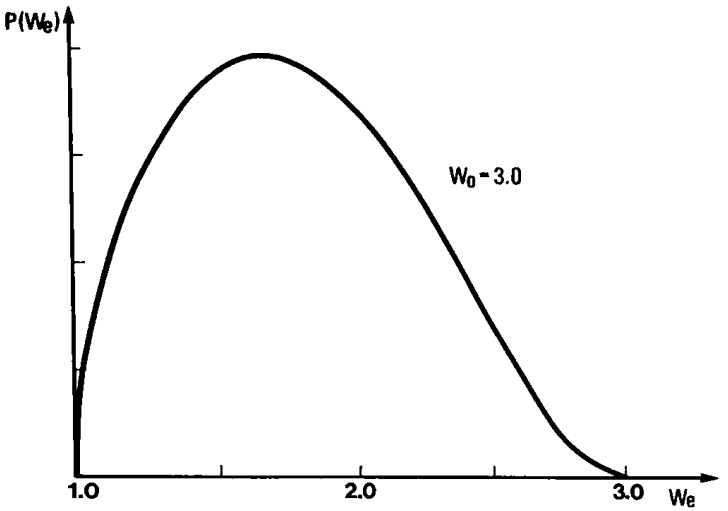


FIG. 5.1. The statistical energy distribution of beta-particles ($W_0 = \Delta$).

As before in the case of p_{f_1} and p_{f_2} the upper and lower limits W_{e_2} and W_{e_1} of the integration are determined by the boundary of the physical region in eqn (5.62) corresponding to $\cos\Theta_{fe} = \pm 1$ (as before we also approximate W_f by M_f, i.e. neglect the kinetic energy of the recoil nucleus)

$$W_\nu = \underbrace{M_i - W_e - M_f}_{\Delta - W_e} = \sqrt{(W_e^2 - m_e^2)} \pm p_f.$$

By squaring this equation and solving it for W_e we find

$$W_{e_{1,2}} = \frac{(\Delta \pm p_f)^2 + m_e^2}{2(\Delta \pm p_f)}. \tag{5.71}$$

Carrying out the integration in eqn (5.70) we obtain

$$\begin{aligned} dR &= 8\pi^2 \left\{ \frac{W_{e_1}^2 - W_{e_2}^2}{2} \Delta - \frac{W_{e_1}^3 - W_{e_2}^3}{3} \right\} p_f \, dp_f \\ &= 8\pi^2 p_f^2 \frac{(\Delta^2 - p_f^2 - m_e^2)^2}{12(\Delta^2 - p_f^2)^3} \cdot \{p_f^4 - p_f^2(4\Delta^2 - m_e^2) + 3\Delta^2(\Delta^2 + m_e^2)\} \, dp_f. \end{aligned} \tag{5.72}$$

This is a well-known result (see Kofoed-Hansen 1948).

As discussed before (eqn (5.15)) the three momentum transfer $\mathbf{q}$ is equal to the recoil momentum $\mathbf{p}_f$ in our reference frame. Equation (5.72)

applies therefore to the statistical three momentum distribution $P(|\mathbf{q}|)$ too. Introducing the dimensionless variable

$$x = \frac{|\mathbf{q}|}{|\mathbf{q}_{\max}|} \tag{5.73}$$

with $|\mathbf{q}_{\max}| = \sqrt{(\Delta^2 - m_e^2)}$ (see also eqn (5.21)), we obtain

$$P(x)\,\mathrm{d}x = \frac{\sum_s \sum_f |T|^2}{(2\pi)^3 6} |\mathbf{q}_{\max}|^5 x^2 \frac{(1-x^2)^2}{(A-x^2)^3} \{x^4 - Bx^2 + C\}\,\mathrm{d}x \tag{5.74}$$

where

$$A = \frac{\Delta^2}{\Delta^2 - m_e^2}$$

$$B = \frac{4\Delta^2 - m_e^2}{\Delta^2 - m_e^2}$$

$$C = \frac{3\Delta^2(\Delta^2 + m_e^2)}{(\Delta^2 - m_e^2)^2}.$$

We have, for example,

$$\left.\begin{aligned} A &= \tfrac{4}{3} \\ B &= 5 \\ C &= \tfrac{20}{3} \end{aligned}\right\} \text{ if } \Delta = 2m_e \quad \text{or} \quad \left.\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned}\right\} \text{ if } \Delta \gg m_e.$$

Of course, the final integration of eqn (5.72) or eqn (5.74), respectively, leads to the same result as given in eqn (5.69) (see Fig. 5.2).

We conclude this section with a short discussion of the electron capture. In this case the final state is a two particle state only (recoil nucleus and neutrino).

We have, therefore, to rewrite eqn (5.54) in the following way:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \sum_f \sum_s \int \frac{(2\pi)^4 \delta^4(p_f - p_i)}{V^3} T_{if}^+ T_{fi} \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3} V^3 \tag{5.75}$$

Splitting off the δ-function, as before, we get

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{(2\pi)^2} \sum_f \sum_s \int T_{if}^+ T_{fi} \delta(\mathbf{p}_f + \mathbf{p}_\nu - \mathbf{p}_i)\,\delta(W_f + W_\nu - W_i)\,\mathrm{d}^3 p_f\,\mathrm{d}^3 p_\nu. \tag{5.76}$$

With the rest frame chosen according to $\mathbf{p}_i = 0$ the integral over $\mathbf{p}_\nu$ gives

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{(2\pi)^2} \sum_{fs} \int T_{if}^+ T_{fi} \delta(W_f + W_\nu - W_i)\,\mathrm{d}^3 p_f \tag{5.77}$$

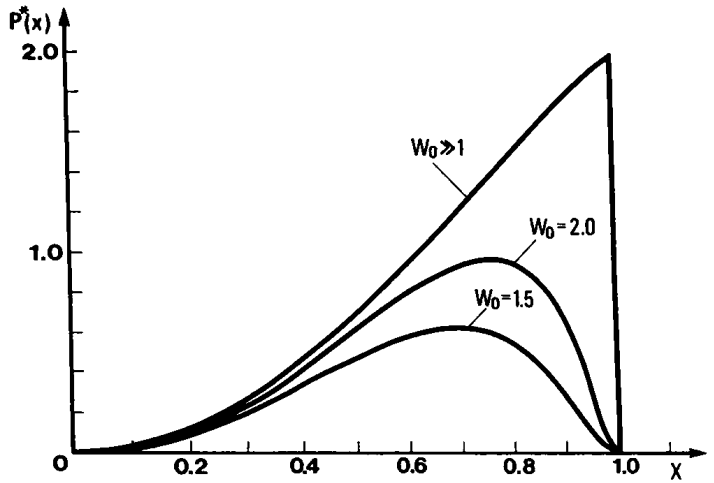


FIG. 5.2. Statistical recoil momentum distribution as a function of $x = |q|/|q_{max}|$. The distribution

$$P^*(x) = x^2 \frac{(1-x)^2)^2}{(A-x^2)^3}\{x^4 - Bx^2 + C\}$$

is not normalized (see eqn (5.74)).

with

$$W_\nu = |\mathbf{p}_\nu| = |\mathbf{p}_f|. \tag{5.78}$$

Neglecting the kinetic energy of the recoil nucleus we have

$$W_f = M_f$$

Also, W_i is now given by

$$W_i = M_i + m_e - E_B$$

where E_B is the binding energy of the electron.

Carrying out the next integration we then find

$$\frac{dW}{dt} = \frac{2}{2\pi} \sum_f \sum_s T_{if}^+ T_{fi} \mathbf{p}_f^2 \tag{5.79}$$

with

$$|\mathbf{p}_f| = \Delta + m_e - E_B.$$

After the detailed discussion of the kinematical aspects of beta-decay and electron capture we shall now consider the dynamical information of the weak interaction and nuclear structure which can be derived from beta-decay observables.

6

THE BETA-DECAY HAMILTONIAN

6.1. Weak interaction density

IT IS usually assumed that all weak interaction processes can be described by a universal fundamental Hamiltonian density. This interaction density has been very successfully expressed in terms of self-interacting currents (current–current interaction) (Schopper 1966; Marshak *et al.* 1969; Blin-Stoyle 1973; Ryder 1977).

For the special case of nuclear-beta-decay, this Hamiltonian density has the form†

$$H_\beta(x) = \frac{G_\beta}{\sqrt{2}}\{J_\mu^+(x)L_\mu(x) + h.c.\} \tag{6.1}$$

where $J_\mu(x)$ and $L_\mu(x)$ denote the Hadron (or here the nuclear) current and the lepton current, respectively.‡

The beta-decay coupling constant G_β is related to the universal weak coupling constant G by

$$G_\beta = G\cos\Theta \tag{6.2}$$

where Θ is the Cabibbo angle ($\sin\Theta = 0{\cdot}234 \pm 0{\cdot}003$, see Roos 1974).

In order to elucidate the general feature of this Hamiltonian density in more detail, we first consider the beta-decay (or electron capture) of a single neutron or proton.

For nuclear beta-decay, the electron part of the lepton current is the only important one. It is written as

$$L_\mu(x) = i\bar{\psi}_{\nu_e}(x)\gamma_\mu(1+\gamma_5)\psi_e(x) \tag{6.3}$$

where ψ_{ν_e} and ψ_e are the field operators for neutrino and electron, respectively, and γ_λ are the Dirac matrices.

In principle it is not possible to express the hadron current so simply in terms of field operators (see, for example, Blin-Stoyle 1973). This is a consequence of the fact that the nucleons, unlike the leptons, interact strongly as well. This leads to complications which will be discussed later in this book. For the moment, however, we treat the nucleons as point particles, neglecting the influence of the strong interaction, in order to

† *h.c.* = hermitian conjugate.

‡ Note that

$$J_\mu^+ L_\mu = \mathbf{J}^+\mathbf{L} - J_0^+ L_0 = J_1^+ L_1 + J_2^+ L_2 + J_3^+ L_3 - J_4^+ L_4.$$

make the reader more familiar with the concept of self-interaction currents. Then the hadron current is given analogously by

$$J_\mu^+(x) = i\bar{\psi}_p(x)\gamma_\mu(1+\lambda\gamma_5)\psi_n(x). \tag{6.4}$$

Here we have assumed the pure V–A interaction, where V stands for vector and A for axial vector. $\lambda = -C_A/C_V = 1{\cdot}251 \pm 0{\cdot}009$ (Kropf and Paul 1974) is the ratio of the axial vector† coupling constant C_A to the vector coupling C_V. ψ_p and ψ_n are the field operators for the nucleons (proton and neutron).

The Hamiltonian density can then be written as

$$H_\beta(x) = -\frac{G_\beta}{\sqrt{2}}\{\bar{\psi}_p(x)\gamma_\mu(1+\lambda_5)\psi_n(x)\bar{\psi}_e(x)\gamma_\mu(1+\gamma_5)\psi_{\nu_e}(x)+\text{h.c.}\} \tag{6.5}$$

The field operators which have to be inserted in eqns (6.3)–(6.5) are given by

$$\psi(x) = \frac{1}{\sqrt{V}}\sum_p\sum_r\{e^{ipx}a_r(p)u_r(p) + b_r^+(p)v_r(p)e^{-ipx}\} \tag{6.6}$$

$$\bar{\psi}(x) = \frac{1}{\sqrt{V}}\sum_p\sum_r\{e^{-ipx}a_r^+(p)\bar{u}_r(p) + b_r(p)\bar{v}_r(p)e^{ipx}\} \tag{6.7}$$

where $r = 1, 2$.

The $a_r(p)$ and $a_r^+(p)$ are annihilation and creation operators for a fermion of momentum p and spin label r, respectively, $b_r(p)$ and $b_r^+(p)$ are the corresponding operators for the antiparticles.

The $u_r(p)$ and $v_r(p)$ are both the free-particle Dirac spinors (see Chapter 2). We have $\bar{u}_r(p) = u_r^+(p)\gamma_4$ and $\bar{v}_r(p) = v_r^+\gamma_4$.

The different operators $a_r(p)$, $a_r^+(p)$, $b_r(p)$ and $b_r^+(p)$ satisfy the following anti-commutation relations

$$\{a_r(p), a_r^+(p')\} = \{b_r(p), b_r^+(p')\} = \delta_{rr'}\delta_{pp'}. \tag{6.8}$$

The corresponding S-matrix for the three basic processes (given in eqn (5.1)) in nuclear beta-decay is now‡

$$n \rightarrow p + e^- + \bar{\nu}_e \qquad S_{fi} = -i\langle pe^-\bar{\nu}_e|\int H_\beta(x)\,d^4x\,|n\rangle \tag{6.9a}$$

† Of course, in the case of point-like nucleons we would have $\lambda = 1$. For practical reasons, however, we are already introducing the true value for λ at this place.

‡ Note that this form of the interaction density is the consequence of a local interaction, i.e.

$$\int \bar{H}_\beta(x_n, x_p, x_e, x_{\nu_e})\,d^4x_n\,d^4x_p\,d^4x_e\,d^4x_{\nu_e}$$
$$= \int H_\beta\delta^4(x_n - x_p)\delta^4(x_n - x_e)\delta^4(x_n - x_{\nu_e})\,d^4x_n\,d^4x_p\,d^4x_e\,d^4x_{\nu_e}$$
$$= \int H_\beta(x_n)\,d^4x_n = \int H_\beta(x)\,d^4x.$$

$$p \to n + e^+ + \nu_e \qquad S_{fi} = -i \langle n e^+ \nu_e | \int H_\beta(x)\, d^4x\, |p\rangle \tag{6.9b}$$

$$p + e^- \to n + \nu_e \qquad S_{fi} = -i \langle n \nu_e | \int H_\beta(x)\, d^4x\, |p e^-\rangle. \tag{6.9c}$$

Taking into account the fact that

$$\begin{aligned} |p e^- \bar{\nu}_e\rangle &= a_p^+ a_e^+ b_{\nu e}^+ |0\rangle \\ |n\rangle &= a_n^+ |0\rangle \end{aligned} \tag{6.10a}$$

$$\begin{aligned} |n e^+ \nu_e\rangle &= a_n^+ b_e^+ a_{\nu_e}^+ |0\rangle \\ |p\rangle &= a_p^+ |0\rangle \end{aligned} \tag{6.10b}$$

$$\begin{aligned} |n \nu_e\rangle &= a_n^+ a_{\nu_e}^+ |0\rangle \\ |p e^-\rangle &= a_p^+ a_e^+ |0\rangle \end{aligned} \tag{6.10c}$$

we find (for simplicity the spin labels r have been omitted in the equations above):

β^--decay

$$S_{fi} = \frac{G_\beta}{\sqrt{2}} i\{\bar{u}_p \gamma_\mu (1 + \lambda\gamma_5) u_n\}\{\bar{u}_e \gamma_\mu (1 + \gamma_5) v_\nu\} \int e^{-i(p_p + p_{e^-} + p_{\bar{\nu}_e} - p_n)x}\, d^4x \tag{6.11a}$$

β^+-decay

$$S_{fi} = \frac{G_\beta}{\sqrt{2}} i\{\bar{u}_n \gamma_\mu (1 + \lambda\gamma_5) u_p\}\{\bar{u}_\nu \gamma_\mu (1 + \gamma_5) v_e\} \int e^{-i(p_n + p_{e^+} + p_{\nu_e} - p_p)x}\, d^4x \tag{6.11b}$$

electron capture

$$S_{fi} = \frac{G_\beta}{\sqrt{2}} i\{\bar{u}_n \gamma_\mu (1 + \lambda\gamma_5) u_p\}\{\bar{u}_\nu \gamma_\mu (1 + \gamma_5) u_e\} \int e^{-i(p_n + p_{\nu_e} - p_p - p_{e^-})x}\, d^4x. \tag{6.11c}$$

The integrals in the eqns. (6.11) can be replaced by the four dimensional delta function since

$$\delta^4(p) = \frac{1}{(2\pi)^4} \int e^{-ipx}\, d^4x. \tag{6.12}$$

Introducing the four dimensional delta functions we obtain from eqn (5.58) for the T-matrix:

β^--decay

$$T = \frac{G_\beta}{\sqrt{2}} [\bar{u}_p \gamma_\mu (1 + \lambda\gamma_5) u_n][\bar{u}_e \gamma_\mu (1 + \gamma_5) v_\nu]. \tag{6.13a}$$

β^+-decay

$$T = \frac{G_\beta}{\sqrt{2}} [\bar{u}_n \gamma_\mu (1 + \lambda \gamma_5) u_p][\bar{u}_\nu \gamma_\mu (1 + \gamma_5) v_e]. \tag{6.13b}$$

electron capture

$$T = \frac{G_\beta}{\sqrt{2}} [\bar{u}_n \gamma_\mu (1 + \lambda \gamma_5) u_p][\bar{u}_\nu \gamma_\mu (1 + \gamma_5) u_e]. \tag{6.13c}$$

It should be mentioned in this context that the Coulomb interaction between the charged hadrons and leptons in initial or final states has been neglected up to now. This topic will be discussed later in detail.

Although the beta-decay interaction density discussed in this chapter describes beta-decay (and other weak interaction processes) perfectly well it represents an incomplete theory because it is not renormalizable. This means, when a perturbation expansion is made in the usual way, the perturbation series diverges. Now in a first stage we have the same situation in quantum electrodynamics, but there divergencies can be overcome by a redefinition of masses and coupling constants. That is what we denote by renormalization, and quantum electrodynamics is therefore renormalizable.

In the weak interaction theory sketched above, each order in the perturbation expansion contains, however, new infinities. Renormalization is, therefore, not possible and the theory is totally divergent.

In the last few years, however, renormalizable models (first proposed by Weinberg (1967) and Salam (1968)) have been developed. These models are based on gauge theories unifying the weak and electromagnetic interactions (and maybe the strong also) (Lee 1973; Bernstein 1974; Weinberg 1974; Beg and Sirlin 1974; Taylor 1976; Ryder 1977, Harari 1978, Fadeev and Slavnov 1980). These gauge theories imply that the weak interaction density contains neutral currents in addition to the previously known (and above only discussed) charged currents. Phenomena induced by neutral currents occur mostly in high energy physics, but they can be found in nuclear and atomic physics as well.

Nevertheless, for the present consideration of nuclear beta-decay the so-called phenomenological form of the weak interaction theory discussed above is sufficient. An exception is the radiative corrections (discussed later) where a reliable solution can only be given by making use of the gauge form of the theory. Since radiative corrections lie, however, on the border line of the theme of this book, we shall deal only with the simple phenomenological form of the theory.

6.2. Multipole expansion of the nuclear current

Let us now consider the true situation in nuclear beta-decay, i.e. the fact that we have to deal with more or less complicated nuclei consisting of many bound nucleons in initial and final states.

The strong interactions of and between the nucleons can manifest themselves in three ways.

First, the nucleons inside the nucleus cannot be treated as free particles, i.e. the plane wave states have to be replaced by bound state wave functions.

Secondly, meson exchange-, off mass shell- and other many body effects are present. This results, for example, in the occurrence of two, three etc. body terms in the beta-decay Hamiltonian in addition to the single particle operators (see, for example, Rho and Wilkinson (1979)).

Thirdly, the nuclear current itself as given in eqn (6.4) is influenced by the presence of the strong interaction and has to be modified. The exact form of this current will be discussed later under the heading induced interactions.

To summarize the statements made above, all uncertainties of nuclear structure theory render it more, and maybe impossibly, difficult to give an explicit exact form of the nuclear current.

In the most general case, we must, therefore, make the substitution (in eqn (6.5))

$$i\langle\bar{\psi}_p(x)\gamma_\mu(1+\lambda\gamma_5)\psi_n(x)\rangle \rightarrow \langle f|\, V_\mu(x)+A_\mu(x)\,|i\rangle \tag{6.14}$$

where f and i represent the final and initial nuclear states, respectively. The vector and axial vector nuclear current are denoted by V_μ and A_μ.

Because of translation invariance, we can, however, write

$$V_\mu(x)+A_\mu(x)=e^{-ipx}\{V_\mu(0)+A_\mu(0)\}e^{ipx} \tag{6.15}$$

where p is the total energy momentum operator. Thus, we obtain

$$\langle f|\, V_\mu(x)+A_\mu(x)\,|i\rangle = e^{i(p_i-p_f)x}\langle f|\, V_\mu(0)+A_\mu(0)\,|i\rangle \tag{6.16}$$

By inserting this in eqn (6.5), we get for the T-matrix instead of eqns. (6.13):

β^--decay

$$T=-\frac{G_\beta}{\sqrt{2}}\langle f|\, V_\mu(0)+A_\mu(0)\,|i\rangle\, i\bar{u}_e\gamma_\mu(1+\gamma_5)v_\nu. \tag{6.17a}$$

β^+-decay

$$T=-\frac{G_\beta}{\sqrt{2}}\langle f|\, V_\mu(0)+A_\mu(0)\,|i\rangle\, i\bar{u}_\nu\gamma_\mu(1+\gamma_5)v_e. \tag{6.17b}$$

electron capture

$$T = -\frac{G_\beta}{\sqrt{2}} \langle f|\, V_\mu(0) + A_\mu(0)\, |i\rangle\, i\bar{u}_\nu \gamma_\mu (1+\gamma_5) u_e. \qquad (6.17c)$$

The Coulomb interaction between electron (positron) and nucleons or nucleus has always been neglected up to now. In order to proceed further we introduce a multipole expansion of this V–A nuclear current. Multipole expansions are often applied in non-relativistic nuclear physics and should, therefore, be familiar to many nuclear physicists.

The advantage of this decomposition of the nuclear current into†

$$\langle f|\, V_\mu(0) + A_\mu(0)\, |i\rangle = \sum_{mn} a^\mu_{mn}(q^2, p_i^2, p_f^2)\, Y_n^m(\hat{\tilde{q}}) \qquad (6.18)$$

is that the kinematical structure of the T-matrix, the nuclear structure model dependent effects and the effect of the electromagnetic interaction between electron and nucleus, discussed in detail later, are clearly separated in the transition amplitude, and this in a nuclear model independent way. The coefficients $a^\mu_{mn}(q^2, p_i^2, p_f^2)$ play the role of form factors.

The special details of nuclear structure and of the beta-decay interaction of nucleons will only influence these form factors but the relations between form factors and observables remain unaltered.

Gross features of different types of beta transitions and relations between different observables can thus be derived and discussed without knowing anything about the special nuclear structure of a nucleus.

Before going further the question remains whether the multipole expansion method can generally be used in relativistic cases too, or is their validity limited to non-relativistic problems only. This means we should be able to construct a multipole expansion in a covariant form.

In fact the existence of such a Lorentz-invariant expansion has been demonstrated by some authors (L. Durand *et al.* 1962; Galindo and Pascual 1969; Delorme 1972 and 1979). The germ-cell of the problem is that both states, initial and final, of the nuclear current are not at rest. Thus one has first to write down explicit relativistic wave functions of a free particle of arbitrary spin J and mass M_N for initial and final nuclear states (if the initial nucleus is at rest, for the final nucleus only). The most usual formalism to do this is the so-called helicity formalism (Jacob and Wick 1959; see also, for example, Durand *et al.* 1962; Perl 1974). Then we get

$$i\rangle = |\mathbf{p}_i J_i M_i\rangle \qquad f\rangle = |\mathbf{p}_f J_f M_f\rangle \qquad (6.19)$$

† $\hat{\tilde{q}} = \tilde{\mathbf{q}}/|\tilde{\mathbf{q}}|$.

where the $2M+1$ helicity states represent plane waves with the $2M+1$ possible values for the projection of the total angular momentum J of the nucleus along its direction of motion.

Let G be an arbitrary proper Lorentz transformation

$$x_\mu \xrightarrow{G} x'_\mu \quad \text{with} \quad x'_\mu = a_{\mu\nu} x_\nu \tag{6.20}$$

(see also Section 5.2).

Then the operators V_μ and A_μ, respectively, transform as a vector or axial vector under Lorentz transformations

$$G(V+A)G^{-1} = (V+A)_G. \tag{6.21}$$

The nuclear current matrix element is related to the transformed matrix element in the following way:

$$\langle \mathbf{p}_f J_f M_f | \, V+A \, | \mathbf{p}_i J_i M_i \rangle = \sum_{M_i' M_f'} D^{J_i}_{M_i' M_i}(R'_g) \times \langle g\mathbf{p}_f J_f M'_f | \, (V+A)_G \, | g\mathbf{p}_i J_i M'_i \rangle \, D^{J_f}_{M'_f M_f}(R_g) \tag{6.22}$$

where D' are the usual representation coefficients of the rotation group in the three dimensions. $g\mathbf{p}JM'\rangle$ are the transformed helicity states (for details see Durand *et al.* 1962). The result given in eqn (6.22) is of importance because it allows us to evaluate the nuclear current matrix in every particular Lorentz frame.

It has been shown that the most suitable reference system for the multipole decomposition is the somewhat unphysical Breit system discussed in Section 5.2 where $\mathbf{p}_i + \mathbf{p}_f = 0$. (Durand *et al.* 1962; Galindo and Pascual 1969; Delorme 1972, 1979).

In that case we can expand the nuclear current matrix element in terms of multipoles

$$Y^M_L(\hat{\tilde{q}})$$

where $\tilde{\mathbf{q}}$ becomes identical with the tridimensional momentum transfer $\mathbf{q}$.

A transformation from the reference system, where the initial nucleus is at rest, to the Breit system, as discussed in Section 5.2, can be performed without difficulties. In the nuclear beta-decay, however, initial and final nuclei are very heavy particles and on the other hand the transition energy Δ is very small. As has been discussed in Section 5.2, the effects of a Lorentz transformation from the Lab-system (initial nucleus at rest) to the Breit system are of the order

$$\frac{|\mathbf{q}|}{2M} \leqslant \frac{\sqrt{(\Delta^2 - m_e^2)}}{M_i + M_f} \leqslant \frac{0{\cdot}01}{A} \tag{6.23}$$

(see eqns (5.8) and (5.37)), i.e. they lie in the range of 10^{-3} to 10^{-5}.

Thus, this point will not be dealt with in greater depth, the reason being that it has so little influence on the matter discussed, and is a little beyond the scope of the problems treated here. We make the assumption in the following, therefore, that the nuclei are infinitely massive, i.e. we accept that the zeroth order of the multipole decomposition is exact only in $\mathbf{q}/(M_i + M_f)$ and $\Delta/(M_i + M_f)$.

Splitting off the nuclear current J in space and time components we obtain

$$J = \begin{pmatrix} \mathbf{J} \\ J_4 \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ \mathrm{i}J_0 \end{pmatrix}$$

The multipole decomposition in the form first proposed by Stech and Schülke (1964) can then explicitly be written as†

$$\langle f|\, V_0(0) + A_0(0)\,|i\rangle = \sum_{LM} (-1)^{J_f - M_f} \sqrt{(2J_i+1)} \times \begin{pmatrix} J_f & L & J_i \\ -M_f & M & M_i \end{pmatrix} \sqrt{(4\pi)}\, \overset{*}{Y}{}^{M}_{L}(\hat{q})\, \frac{(qR)^L}{(2L+1)!!}\, F_L(q^2) \quad (6.24)$$

$$\langle f|\, \mathbf{V}(0) + \mathbf{A}(0)\,|i\rangle = \sum_{KLM} (-1)^{J_f - M_f} \sqrt{(2J_i+1)} \times \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \sqrt{(4\pi)}\, \overset{*}{\mathbf{Y}}{}^{M}_{KL}(\hat{q})\, \frac{(qR)^L}{(2L+1)!!}\, F_{KL}(q^2) \quad (6.25)$$

(see also Donnelly and Walecka 1975; Walecka 1975; Serot 1978).

The $Y^M_L(\hat{q})$ and $\mathbf{Y}^M_{KL}(\hat{q})$ are spherical harmonics and vector spherical harmonics, respectively. The latter ones are defined by (see for example, Edmonds 1964)

$$\mathbf{Y}^M_{KL}(\hat{q}) = \sum_{\mu} C(L1K;\, M-\mu\ \ \mu)\, \mathbf{e}_\mu Y^{M-\mu}_L(\hat{q}) \quad (6.26)$$

where C is a Clebsch–Gordan coefficient and $\mathbf{e}$ a spherical vector with the triad $\mathbf{e}_0 = \mathbf{e}_z$; $\mathbf{e}_{\pm 1} = \mp 1/\sqrt{2}\,(\mathbf{e}_x \pm \mathrm{i}\mathbf{e}_y)$ where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors.

† Because of the orthogonality conditions

$$\int \overset{*}{Y}{}^{M'}_{L'}(\hat{q})\, Y^M_L(\hat{q})\, \mathrm{d}\Omega_q = \delta_{M'M}\delta_{L'L}$$

and

$$\int \overset{*}{\mathbf{Y}}{}^{M'}_{K'L'}(\hat{q})\, \mathbf{Y}^M_{KL}(\hat{q})\, \mathrm{d}\Omega_q = \delta_{K'K}\delta_{L'L}\delta_{M'M}$$

the form factors can immediately be found by multiplying both sides of the eqns. (6.24) and (6.25) by $Y^M_L(\hat{q})$ or $\mathbf{Y}^M_{KL}(\hat{q})$ and integrating over Ω_q.

The form factors $F_L(q^2)$ and $F_{KL}(q^2)$ in eqns. (6.24) and (6.25) play the role of reduced matrix elements. $F_L(q^2)$, for example, describes a transition between two nuclei with the relative momentum $q = |\mathbf{q}|$ and the relative angular momentum L. In the form factor $F_{KL}(q^2)$ the angular momentum L is coupled with the vector operator $\boldsymbol{\sigma}$ or $\mathbf{p}$ to a total angular momentum K. A factor $(qR)^L/(2L+1)!!$ has been extracted (R denotes the nuclear radius) since then the $F_L(q^2)$ and $F_{KL}(q^2)$ are finite for $q \to 0$. This latter point will be eluciated more in a later discussion.

J_i, J_f, M_i and M_f are the spins and magnetic quantum numbers of the relevant nuclear states. Both eqns. (6.24) and (6.25) for time and space components of the nuclear current can also be combined to one decomposition. For this purpose it is helpful to define an irreducible tensor operator $T^M_{KLS}(\hat{q})$ by†

$$T^M_{LL0}(\hat{q}) = \mathrm{i}^L Y^M_L(\hat{q}) \tag{6.27a}$$

$$T^M_{KL1}(\hat{q}) = (-1)^{L-K+1}\mathrm{i}^L \mathbf{Y}^M_{KL}(\hat{q}) \cdot \boldsymbol{\alpha} = \sum_\mu C(1\ L\ K;\ \mu\ M-\mu) \times \mathrm{i}^L \gamma_5 \sigma^\mu Y^{M-\mu}_L(\hat{q}). \tag{6.27b}$$

By making use of these tensor operators one obtains‡

$$(-\mathrm{i})\langle f|\, V_\mu(0) + A_\mu(0)\,|i\rangle\, \gamma_4\gamma_\mu = \sum_{KLMs} (-1)^{J_f - M_f + M} \times (-\mathrm{i})^L \sqrt{(4\pi)}\sqrt{(2J_i+1)} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} T^{-M}_{KLs}(\hat{q}) \frac{(qR)^L}{(2L+1)!!} F_{KLs}(q^2). \tag{6.28}$$

$\gamma_4\gamma_\mu$ have been taken out of the lepton matrix element. Therefore the 4×4 matrix $T^M_{KLS}(\hat{q})$ is acting on the lepton spinors.

The form factors in eqns (6.24) and (6.25) are connected with those in eqn (6.28) by

$$F_{KLs} = \begin{cases} F_L\delta_{KL} & \text{for} \quad s=0 \\ F_{KL} & \text{for} \quad s=1. \end{cases} \tag{6.29}$$

In order to make the more or less bloodless formalism of the form factor decomposition more concrete let us try to calculate them explicitly for some simple special models.

For this purpose we shall first return to the decay of the point like neutron. In that case we had (see eqns (6.13a) and (6.17a))

$$\langle p|\, V_\mu(0) + A_\mu(0)\,|n\rangle = \mathrm{i}\langle \bar{u}_p \gamma_\mu (1+\lambda\gamma_5) u_n\rangle. \tag{6.30}$$

† Note that also

$$\int T^{M'+}_{KL'S'}(\hat{q}) T^M_{KLs}(\hat{q})\, \mathrm{d}\Omega_q = \delta_{K'K}\delta_{L'L}\delta_{S'S}\delta_{M'M}.$$

‡ Note that our metric is different from that of Stech and Schülke (1964).

The usual spinors u_n and u_p, respectively, are explicitly given by (see Chapter 2)

$$u^{(s)} = \sqrt{\frac{(W+M)}{2W}} \begin{pmatrix} -\dfrac{\boldsymbol{\sigma}\mathbf{p}}{W+M}\chi^{\pm 1/2} \\ \chi^{\pm 1/2} \end{pmatrix} \tag{6.31}$$

with the two component spinors

$$\chi^{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi^{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.32}$$

It should be noted that the normalization chosen is

$$(u^{s'}(\mathbf{p}))^{+} u^{s}(\mathbf{p}) = \delta_{s's}. \tag{6.33}$$

As mentioned before this normalization essentially determines the factor N in eqn (5.50). Now eqn (6.30) can be written as

$$\langle p|\ V_4(0) + A_4(0)\ |n\rangle = \mathrm{i}\langle p|\ V_0(0) + A_0(0)\ |n\rangle = \mathrm{i}u_p^{+}\underbrace{\gamma_4\gamma_4}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}(1 + \underbrace{\lambda\gamma_5}_{\lambda\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}})u_n \tag{6.34}$$

and

$$\begin{aligned} \langle p|\ V_0 + A_0\ |n\rangle = {} & \sqrt{\frac{(W_n+M_n)}{2W_n}}\sqrt{\frac{(W_p+M_p)}{2W_p}}\Big\{(\chi_p^{m'})^{+}\chi_n^{m} \\ & + \underbrace{\left(\frac{\boldsymbol{\sigma}\mathbf{p}}{W_p+M_p}\chi_p^{m'}\right)^{+}\frac{\boldsymbol{\sigma}\mathbf{p}}{W_n+M_n}\chi_n^{m}}_{\chi_p^{+}\frac{\boldsymbol{\sigma}\mathbf{p}}{W_p+M_p}\frac{\boldsymbol{\sigma}\mathbf{p}}{W_n+M_n}\chi_n} - \lambda\underbrace{\left[\left(\frac{\boldsymbol{\sigma}\mathbf{p}}{W_p+M_p}\chi_p^{m'}\right)^{+}\chi_n^{m}\right.}_{\chi_p^{+}\frac{\boldsymbol{\sigma}\mathbf{p}}{W_p+M_p}\chi_n} \\ & \left. + (\chi_p^{m'})^{+}\frac{\boldsymbol{\sigma}\mathbf{p}}{W_n+M_n}\chi_n^{m}\right]\Big\}. \end{aligned} \tag{6.35}$$

This can be further simplified by using the relation

$$(\boldsymbol{\sigma}\mathbf{p}_p)(\boldsymbol{\sigma}\mathbf{p}_n) = \mathbf{p}_p\mathbf{p}_n + \mathrm{i}\boldsymbol{\sigma}(\mathbf{p}_p \times \mathbf{p}_n) \tag{6.36}$$

to

$$\begin{aligned} \langle p|\ V_0(0) + A_0(0)\ |n\rangle = {} & \sqrt{\frac{(W_n+M_n}{2W_n}}\sqrt{\frac{W_p+M_p}{2W_p}}\cdot\Big\{(\chi_p^{m'})^{+}\chi_n^{m} \\ & + \frac{\mathbf{p}_p\mathbf{p}_n}{(W_n+M_n)(W_p+M_p)}(\chi_p^{m'})^{+}\chi_n^{m} + \mathrm{i}\frac{\mathbf{p}_p\times\mathbf{p}_n}{(W_n+M_n)(M_p+M_p)}\{(\chi_p^{m'})^{+}\boldsymbol{\sigma}\chi_n^{m}\} \\ & - \lambda\left[\frac{\mathbf{p}_p}{W_p+M_p} + \frac{\mathbf{p}_n}{W_n+M_n}\right]\{(\chi_p^{m'})^{+}\boldsymbol{\sigma}\chi_n^{m}\}\Big\}. \end{aligned} \tag{6.37}$$

Similarly we obtain for the space components

$$\langle p|\,\mathbf{V}+\mathbf{A}\,|n\rangle = iu_p^+\underbrace{\gamma_4\gamma_\mu}_{\begin{pmatrix}0 & i\boldsymbol{\sigma}\\ i\boldsymbol{\sigma} & 0\end{pmatrix}}(1+\underbrace{\lambda\gamma_5}_{\lambda\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}})u_n = \sqrt{\frac{(W_n+M_n)}{2W_n}}\sqrt{\frac{(W_p+M_p}{2W_p}}$$

$$\times\left\{\left(\frac{\boldsymbol{\sigma}\mathbf{p}}{W_p+M_p}\chi_p^{m'}\right)^+\boldsymbol{\sigma}\chi_n^m+(\chi_p^{m'})^+\boldsymbol{\sigma}\frac{\boldsymbol{\sigma}\mathbf{p}}{W_n+M_n}\chi_n^m-\lambda(\chi_p^{m'})^+\boldsymbol{\sigma}\chi_n^m\right.$$

$$\left.-\lambda\left[\left(\frac{\boldsymbol{\sigma}\mathbf{p}}{W_p+M_p}\chi_p^{m'}\right)^+\boldsymbol{\sigma}\frac{\boldsymbol{\sigma}\mathbf{p}}{W_n+M_n}\chi_n^m\right]\right\}. \quad (6.38)$$

This equals to

$$\langle p|\,\mathbf{V}+\mathbf{A}\,|n\rangle = \sqrt{\frac{(W_n+M_n)}{2W_n}}\sqrt{\frac{(W_p+M_p)}{2W_p}}\left\{(\chi_p^{m'})^+\underbrace{\frac{\boldsymbol{\sigma}\mathbf{p}_p}{W_p+M_p}}_{\frac{\mathbf{p}_p+i(\boldsymbol{\sigma}\times\mathbf{p}_p)}{W_p+M_p}}\boldsymbol{\sigma}\chi_n^m\right.$$

$$+(\chi_p^{m'})^+\boldsymbol{\sigma}\underbrace{\frac{\boldsymbol{\sigma}\mathbf{p}_n}{W_n+M_n}}_{\frac{\mathbf{p}_n-i(\boldsymbol{\sigma}\times\mathbf{p}_n)}{W_n+M_n}}\chi_n^m-\lambda(\chi_p^{m'})^+\boldsymbol{\sigma}\chi_n^m$$

$$\left.-\lambda\left[(\chi_p^{m'})^+\underbrace{\frac{\boldsymbol{\sigma}\mathbf{p}_p}{W_p+M_p}\boldsymbol{\sigma}\frac{\boldsymbol{\sigma}\mathbf{p}_n}{W_n+M_n}}_{\frac{-(\mathbf{p}_p\mathbf{p}_n)\boldsymbol{\sigma}+(\boldsymbol{\sigma}\mathbf{p}_p)\mathbf{p}_n+\mathbf{p}_p(\boldsymbol{\sigma}\mathbf{p}_n)-i(\mathbf{p}_p\times\mathbf{p}_n)}{(W_p+M_p)(W_n+M_n)}}\chi_n^m\right]\right\}. \quad (6.39)$$

Finally we obtain for the space components

$$\langle p|\,\mathbf{V}(0)+\mathbf{A}(0)\,|n\rangle = \sqrt{\frac{(W_n+M_n)}{2W_n}}\sqrt{\frac{(W_p+M_p)}{2W_p}}$$

$$\times\left\{\left[\frac{\mathbf{p}_p}{W_p+M_p}+\frac{\mathbf{p}_n}{W_n+M_n}\right](\chi_p^{m'})^+\chi_n^m+(\chi_p^{m'})^+\right.$$

$$\times\left[\frac{i(\boldsymbol{\sigma}\times\mathbf{p}_p)}{W_p+M_p}-\frac{i(\boldsymbol{\sigma}\times\mathbf{p}_n)}{W_n+M_n}\right]\chi_p^m-\lambda(\chi_p^{m'})^+\boldsymbol{\sigma}\chi_n^m$$

$$+\lambda\frac{\mathbf{p}_p\mathbf{p}_n}{(W_p+M_p)(W_n+M_n)}\{(\chi_p^{m'})^+\boldsymbol{\sigma}\chi_n^m\}+\lambda\frac{i(\mathbf{p}_p\times\mathbf{p}_n)}{(W_p+M_p)(W_n+M_n)}$$

$$\left.\times(\chi_p^{m'})^+\chi_n^m-\lambda\left[(\chi_p^{m'})^+\frac{(\boldsymbol{\sigma}\mathbf{p}_p)\mathbf{p}_n+\mathbf{p}_p(\boldsymbol{\sigma}\mathbf{p}_n)}{(W_p+M_p)(W_n+M_n)}\chi_n^m\right]\right\}. \quad (6.40)$$

As mentioned before, the reference system, where the form factor decomposition of the eqns (6.24), (6.25) and (6.28) holds exactly, is the Breit system (see Section 5.2).

Thus we may now consider the decay of the point-like neutron in the Breit system. Then most terms in the eqns (6.37) and (6.40) disappear (because $\mathbf{p}_n = -\mathbf{p}_p$ in this reference system) when we go to the limit $\Delta \to 0$, i.e. to the limit of the elastic case.†

By taking into account the fact that the normalization can be rewritten as

$$\sqrt{\frac{(W+M)}{2W}} = \frac{1}{\sqrt{\left\{1+\left(\frac{p}{W+M}\right)^2\right\}}} \simeq \frac{1}{\sqrt{\left\{1+\left(\frac{p}{2M}\right)^2\right\}}} \tag{6.41}$$

we are, in this case, led to

$$\langle p|\, V_0 + A_0\, |n\rangle = \delta_{m'm}\left(1-\frac{\mathbf{q}^2}{8M_N^2}\right) \tag{6.42a}$$

$$\langle f|\, \mathbf{V}+\mathbf{A}\, |n\rangle = -\lambda(\chi_p^{m'})^+ \boldsymbol{\sigma}\chi_n^m + \frac{\mathrm{i}}{2M_N}(\chi_p^{m'})^+(\boldsymbol{\sigma}\times\mathbf{q})\chi_n^m$$
$$+\frac{\lambda\mathbf{q}}{2(2M_N)^2}(\chi_p^{m'})^+(\boldsymbol{\sigma}\mathbf{q})\chi_n^m. \tag{6.42b}$$

The factor

$$\left(1-\frac{\mathbf{q}^2}{8M_N^2}\right)$$

is a relativistic correction factor (see Friar 1973). Since it is for the following discussion of no relevance we do not consider it further and neglect therefore the term (Darwin term) $\mathbf{q}^2/8M_N^2$ against 1. We have

$$\begin{aligned} -\lambda(\chi_p^{m'})^+\boldsymbol{\sigma}\chi_n^m &= \mp\lambda\overset{*}{\mathbf{e}}_0\delta_{m'm} \quad \text{for} \quad m = m' = \pm\tfrac{1}{2} \\ &= +\lambda\sqrt{2}\,\overset{*}{\mathbf{e}}_{+1} \quad \text{for} \quad m' = +\tfrac{1}{2} \\ &\qquad\qquad\qquad\quad m = -\tfrac{1}{2} \\ &= -\lambda\sqrt{2}\,\overset{*}{\mathbf{e}}_{-1} \quad \text{for} \quad m' = -\tfrac{1}{2} \\ &\qquad\qquad\qquad\quad m = +\tfrac{1}{2} \\ &= -\sum_M (-1)^{\frac{1}{2}-m'}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -m' & M & m \end{pmatrix}\overset{*}{\mathbf{e}}_M \cdot \lambda(\tfrac{1}{2}\,\|\boldsymbol{\sigma}\|\,\tfrac{1}{2}) \end{aligned} \tag{6.42c}$$

This last equation is nothing other than the application of the Wigner–Eckart theorem.

† Since $(\Delta/M) \ll 1$ this approximation is a very good one. $\Delta \neq 0$ is nothing else than an isospin violation caused in an indirect way mainly by the Coulomb interaction.

By comparing the multipole expansion of eqns (6.24) and (6.25) with the above result, we see, if only terms of the lowest order, i.e. terms with $L=0$, are taken into account, in this simple case ($J_i = J_f = \frac{1}{2}$)

$$\langle p|\, V_0 + A_0\, |n\rangle = \sqrt{2}(-1)^{\frac{1}{2}-m'} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -m' & 0 & m \end{pmatrix} \underbrace{\sqrt{(4\pi)}\overset{*}{Y}{}^0_0(\hat{q})}_{1} F_0(q^2) \quad (6.43)$$

and

$$\langle p|\, \mathbf{V} + \mathbf{A}\, |n\rangle = \sum_M \sqrt{2}\,(-1)^{\frac{1}{2}-m'} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -m' & M & m \end{pmatrix} \underbrace{\sqrt{(4\pi)}\overset{*}{\mathbf{Y}}{}^M_{10}(\hat{q})}_{\overset{*}{\mathbf{e}}_M} F_{10}(q^2). \quad (6.44)$$

In the first case the $3j$-symbol times the factor $(-1)^{\frac{1}{2}-m'}$ has the value

$$(-1)^{\frac{1}{2}-m'} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -m' & 0 & m \end{pmatrix} = \frac{1}{\sqrt{2}}\, \delta_{m'm} \quad (6.45)$$

and in the second case for $M=0$

$$(-1)^{\frac{1}{2}-m'} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -m' & 0 & m \end{pmatrix} = \pm\frac{1}{\sqrt{6}}\, \delta_{m'm} \quad (6.46)$$

for $m = m' = \pm\frac{1}{2}$.

The comparison of eqns (6.43) and (6.44) with eqns (6.42) leads then finally to the result

$$F_0(q^2) = F_{000}(q^2) = 1 \quad (6.47)$$

and

$$F_{10}(q^2) = F_{101}(q^2) = -\lambda \frac{1}{\sqrt{2}} (\tfrac{1}{2}|\, |\boldsymbol{\sigma}|\, |\tfrac{1}{2}) = -\lambda\sqrt{3}. \quad (6.48)$$

That is what we expect from the fact that we have assumed point like particles. Under this assumption the form factors $F_{KLS}(q^2)$ should not have any structure, i.e. no q^2-dependence. This fact is also well known from other topics, like, for instance, electron scattering (Rutherford scattering).

We see that one form factor ($F_{000}(q^2)$) is of the vector type and the other one ($F_{101}(q^2)$) of the axial vector type. Later on we shall indicate this fact by a special additional index. The matrix elements

$$\frac{1}{\sqrt{(2J_i+1)}} (\chi_p|\, |1|\, |\chi_n) \quad \text{and} \quad \frac{1}{\sqrt{(2J_i+1)}} (\chi_p|\, |\boldsymbol{\sigma}|\, |\chi_n)$$

belonging to the form factors ${}^{V}F_{000}(q^2)$ and ${}^{A}F_{101}(q^2)$, respectively, are the so called Fermi- and Gamow-Teller-matrix elements, here for the decay of the neutron.

Our attention should also be directed to the point that in this special example we can very easily consider the case of the laboratory reference system too, and that in an exact way. In the lab-system with the initial nucleus at rest we have to put $\mathbf{p}_n = 0$ and $\mathbf{p}_p = \mathbf{q}$ in eqns (6.37) and (6.40). Making then the same approximation $\Delta \to 0$ as before we see immediately the occurrence of the additional terms with $\mathbf{q}/2M$ mentioned before.

In order to give a more detailed picture of the multipole formalism we now consider a second special example which is a rough approximation of the processes occurring in the β^--decay of a nucleus. In this context we treat the nucleus on the one hand as a system consisting of A bound point-like nucleons and on the other hand the nuclear current density as a one-body operator for free single nucleons.

This allows us to write the nuclear current density operator as a sum of individual nucleon contributions

$$J_\mu^+(0) = \sum_{\alpha,\beta} a_p^+(\alpha) \langle p | \, V_\mu(0) - A_\mu(0) \, | n \rangle \, a_n(\beta). \tag{6.49}$$

The annihilation and creation operators $a_n(\beta)$ and $a_p^+(\alpha)$ destroy a neutron in the bound state β and create a proton in the bound state α. $\langle p | \, V_\mu - A_\mu \, | n \rangle$ is taken as the free single neutron transition matrix element given in eqns (6.37) and (6.40).

$$\langle p | \, V_\mu + A_\mu \, | n \rangle = (\chi_p^{m'})^+ O^\mu \chi_n^m \tag{6.50}$$

where†

$$O^0 = \left\{ 1 - \lambda \frac{\mathbf{p}_p + \mathbf{p}_n}{2M_N} \boldsymbol{\sigma} \right\} \tag{6.51a}$$

$$\mathbf{O} = \left\{ -\lambda \boldsymbol{\sigma} + \frac{\mathbf{p}_p + \mathbf{p}_n}{2M_N} \right\} \tag{6.51b}$$

(M_N = nuclear mass). Here equal neutron and proton masses have been assumed, with the momentum operators $\mathbf{p}_p$ and $\mathbf{p}_n$ ($\mathbf{p} = -i\nabla$) acting on the final and initial wave functions for the beta neutron and the alpha proton.

For reasons of simplicity (and also because they can be neglected) terms of order higher than p/M_N have been omitted. That would be the exact free single nuclear transition operator if all the nucleons are placed in one and the same nuclear state, a state without angular momentum l and without any radial quantum number n. In a more realistic model, however, we have to replace $\chi_p^{m'}$ and χ_n^m by single particle wave functions $g_{nljm}(\mathbf{r})$ where j is the spin and μ its third component.

† Exactly in eqn (6.51b) the term $i/2M_N(\boldsymbol{\sigma} \times \mathbf{q})$ has also to be included.

In a spherical symmetric and parity conserving central potential these wave functions have, for example, the form

$$g_{nlj\mu} = g_{nlj}(r)\mathrm{i}^l \sum_m C(l\ \tfrac{1}{2}\ j;\ \mu - m, m)\, Y_l^{\mu-m}\chi^m. \tag{6.52}$$

For the next step we have to go back to the formula given in eqn (6.11*a*) for the S-matrix and to rewrite this equation as follows:

$$S_{\mathrm{fi}} = \frac{G_\beta}{\sqrt{2}}\,\mathrm{i}2\pi\delta(W_\mathrm{p} + W_\mathrm{e} + W_{\bar{\nu}_\mathrm{e}} - W_\mathrm{n}) \cdot \int (\bar{u}_\mathrm{p} \mathrm{e}^{-\mathrm{i}\mathbf{p}_\mathrm{p}\mathbf{r}} \gamma_\mu (1+\lambda\gamma_5)$$
$$\times u_n \mathrm{e}^{\mathrm{i}\mathbf{p}_\mathrm{n}\mathbf{r}}) \cdot (\bar{u}_\mathrm{e}\gamma_\mu(1+\gamma_5)v_\nu)\mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{r}}\,\mathrm{d}^3 r \tag{6.53}$$

($\mathbf{q} = -(\mathbf{p}_\mathrm{e} + \mathbf{p}_{\nu\mathrm{e}})$; see eqn (5.14)).

The δ-function comes from the integration over the fourth component.

In our model, where we are considering the decay of a neutron within a nucleus instead of the decay of one free neutron, we have now to replace

$$\{\bar{u}_\mathrm{p}\mathrm{e}^{-\mathrm{i}\mathbf{p}_\mathrm{p}\mathbf{r}}\gamma_\mu(1+\lambda\gamma_5)u_\mathrm{n}\mathrm{e}^{\mathrm{i}\mathbf{p}_\mathrm{n}\mathbf{r}}\}\{\bar{u}_\mathrm{e}\gamma_\mu(1+\gamma_5)v_\nu\}\mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{r}}$$

$$\to -\sum_{\alpha,\beta} \psi_f(\mathbf{r}_1 \ldots r_\alpha \ldots, \mathbf{r}_\mathrm{A})^+ \langle p|\ V_\mu - A_\mu\ |n\rangle \tag{6.54}$$
$$\times\delta(\mathbf{r}-\mathbf{r}_\alpha)\delta(\mathbf{r}-\mathbf{r}_\beta)\mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{r}} a_\mathrm{p}^+(\alpha)a_\mathrm{n}(\beta)\psi_i(\mathbf{r}_1 \ldots r_\beta \ldots, \mathbf{r}_\mathrm{A})$$
$$\times\mathrm{i}\{\bar{u}_\mathrm{e}\gamma_\mu(1+\gamma_5)v_\nu\}.$$

$\psi_i(r_1 \ldots, r_\mathrm{A})$ is the wave function of the initial nucleus consisting of A-point like nucleons.

The wave function of the final nucleus $\psi_f(r_1 \ldots, r_\mathrm{A})$ is given by

$$\psi_f(r_1 \ldots, r_\mathrm{A}) = \sum_{\alpha,\beta} a_\mathrm{p}^+(\alpha)a_\mathrm{n}(\beta)\psi_i(r_1 \ldots, r_\mathrm{A}). \tag{6.55}$$

It should, however, be noted that the operators $\mathbf{p}$ in eqn (6.55) are only acting on the nuclear wave function and not on the factor $\mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{r}}$.

By introducing the isospin formalism we can, therefore, rewrite the above formula as

$$S_{\mathrm{fi}} = -\frac{G_\beta}{\sqrt{2}}\,\mathrm{i}2\pi\delta(W_\mathrm{p} + W_\mathrm{e} + W_{\bar{\nu}_\mathrm{e}} - W_\mathrm{n}) \sum_{j=1}^{A} \int \psi_f(r_1 \ldots, r_\mathrm{A})^+$$
$$\times O_j^\mu \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{r}}\delta(r - r_j)t_-^j\psi_i(r_1 \ldots, r_\mathrm{A})\,\mathrm{d}^3 r_1 \ldots \mathrm{d}^3 r_\mathrm{A}\, i(\bar{u}_\mathrm{e}\gamma_\mu(1+\gamma_5)v_\nu) \tag{6.56}$$

The index j indicates that the operator is acting on the jth-nucleon. The isospin operators $t_\pm = \frac{1}{2}\{\tau_1 \pm i\tau_2\}$ applied on a nucleon transform a neutron into a proton (t_-) and a proton into a neutron (t_+).

In the usual nuclear models the co-ordinates of the nucleons are measured from the centre of the nucleus, i.e. from the centre of mass of the nucleus. Because of the fundamental property of translational in-

variance the problem can be separated into a centre of mass co-ordinate

$$\mathbf{R} = \frac{1}{A} \sum_{i=1}^{A} \mathbf{r}_i \tag{6.57}$$

and the internal co-ordinates

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}. \tag{6.58a}$$

One also has to make the replacement

$$\mathbf{p}'_i = \mathbf{p}_i - \frac{1}{A}\mathbf{P}. \tag{6.58b}$$

The wave function of the nucleus can then be written as

$$\psi(r_1 \ldots, r_A) = e^{i\mathbf{P}\mathbf{R}} \phi(r'_1 \ldots, r'_{A-1}). \tag{6.59}$$

$\mathbf{P}$ is the total three momentum of the nucleus.

By transforming eqn (6.56) to the co-ordinates $\mathbf{r}'_i$ and $\mathbf{R}$ we then obtain

$$S_{fi} = -\frac{G_\beta}{\sqrt{2}} 2\pi i \delta(W_p + W_e + W_{\tilde{\nu}_e} - W_n) \cdot \sum_{j=1}^{A-1} \phi_f(r'_1 \ldots, r'_{A-1})^+ \times O_j^\mu e^{iqr'} \delta(r' - r'_j) t^i_- \phi_i(r'_1 \ldots, r'_{A-1}) \, d^3r'_1 \ldots d^3r'_{A-1} \times \int d^3R e^{i\{\mathbf{q} - (\mathbf{P}_f - \mathbf{P}_i)\}\mathbf{R}} \, i\{\bar{u}_e \gamma_\mu (1+\gamma_5) v_\nu\}. \tag{6.60}$$

In the operators O_j^μ we have operator terms with $\mathbf{p}_f + \mathbf{p}_i = -i(\nabla_f + \nabla_i)$ which have also to be considered. After introducing internal co-ordinates these terms should be replaced by $\mathbf{p}'_f + \mathbf{p}'_i + (1/A)(\mathbf{P}_f + \mathbf{P}_i)$, i.e. new terms of the order $(\mathbf{P}_i + \mathbf{P}_f)/(AM_N)$ occur. In the Breit frame, however, where $\mathbf{P}_f + \mathbf{P}_i = 0$ the additional terms connected with $\mathbf{P}_i + \mathbf{P}_f = 0$ disappear. Since in the lab-frame with the initial nucleus at rest $\mathbf{P}_i + \mathbf{P}_f = \mathbf{q}$ the difference between both systems is, as mentioned before, of the order $\mathbf{q}/AM_N$.

Since we have

$$(2\pi)^3 \delta(\mathbf{P}_f - \mathbf{P}_i - \mathbf{q}) = \int e^{-i(\mathbf{P}_f - \mathbf{P}_i - \mathbf{q})\mathbf{R}} \, d^3R \tag{6.61}$$

we obtain now for the T-matrix (see eqn (5.46))

$$T = -\frac{G_\beta}{\sqrt{2}} \sum_{j=1}^{A-1} \int \phi_f(r'_1 \ldots, r'_{A-1})^+ \, O_j^\mu e^{iqr'} \delta(r' - r'_j) t^i_- \times \phi_i(r'_1 \ldots, r'_{A-1}) \, d^3r'_1 \ldots d^3r'_A \, i\{\bar{u}_e \gamma_\mu (1+\gamma_5) v_\nu\}. \tag{6.62}$$

By comparing this equation with eqn (6.17a) and integrating the

δ-function we get explicitly

$$\langle f|\, V_\mu(0)+A_\mu(0)\,|i\rangle=\sum_{j=1}^{A-1}\int\phi_f^+(r_1\ldots,r_{A-1})\times O_j^\mu e^{i\mathbf{q}\mathbf{r}_j}t_-^i\phi_i(r_1\ldots,r_{A-1})\,d^3r_j. \quad (6.63)$$

r' will from now on be replaced by r.

In order to show explicitly how the multipole expansion can be derived, we use the well-known plane wave expansion

$$e^{i\mathbf{q}\mathbf{r}}=4\pi\sum_{LM}(i)^L j_L(qr)\,\overset{*}{Y}{}_L^M(\hat{q})\,Y_L^M(\hat{r}) \quad (6.64)$$

together with the Wigner–Eckart theorem and obtain for the time component

$$\langle f|\, V_0(0)+A_0(0)\,|i\rangle=\sum_{LM}(-1)^{J_f-M_f}\begin{pmatrix}J_f & L & J_i\\ -M_f & M & M_i\end{pmatrix}\times\langle\phi_f|\,\Big|\sum_{j=1}^{A-1}i^L j_L(qr_j)\sqrt{(4\pi)}\,Y_L^M(\hat{r}_j)\Big\{1-\lambda\frac{\mathbf{p}_j\cdot\boldsymbol{\sigma}_j}{2M_N}\Big\}t_-^i\,\Big|\,|\phi_i\rangle\sqrt{4\pi}\,\overset{*}{Y}{}_L^M(\hat{q}) \quad (6.65)$$

where $\mathbf{p}=-i(\boldsymbol{\nabla}_f+\boldsymbol{\nabla}_i)\cdot\boldsymbol{\nabla}_f$ and $\boldsymbol{\nabla}_i$ are acting on the final and initial nuclear wave functions, respectively, and therefore

$$F_L(q^2)=\frac{1}{\sqrt{(2J_i+1)}}\frac{(2L+1)!!}{(qR)^L}\langle\phi_f|\,\Big|\sum_{j=1}^{A-1}i^L j_L(qr_j)\sqrt{(4\pi)}\times Y_L^M(\hat{r}_j)\cdot\Big\{1-\lambda\frac{\mathbf{p}_j\boldsymbol{\sigma}_j}{2M_N}\Big\}t_-^i\,\Big|\,|\phi_i\rangle. \quad (6.66)$$

For the space component we get

$$\langle f|\,\mathbf{V}(0)+\mathbf{A}(0)\,|i\rangle=\sum_{Lm}\langle\phi_f|\sum_{j=1}^{A-1}i^L j_L(qr_j)\sqrt{(4\pi)}\,Y_L^m(\hat{r}_j)\times\Big\{-\lambda\boldsymbol{\sigma}_j+\frac{\mathbf{p}_j}{2M_N}\Big\}t_-^i\,|\phi_i\rangle\sqrt{(4\pi)}\,\overset{*}{Y}{}_L^m(\hat{q}). \quad (6.67)$$

Since

$$\boldsymbol{\sigma}=\sum_\lambda\mathbf{e}_\lambda^*\sigma_\lambda,\qquad \mathbf{p}=\sum_\lambda\mathbf{e}_\lambda^*p_\lambda \quad (6.68)$$

and (see Edmonds 1964)

$$\sum_J C(L\ 1\ J;m\lambda)C(L\ 1\ J;m'\lambda')=\delta_{mm'}\delta_{\lambda\lambda'} \quad (6.69)$$

we have

$$\sum_{Lm}\left\{\sum_{K} C(L1K;m'\lambda')\,C(L1K;m'\lambda')\,Y_L^m(\hat{r})\overset{*}{Y}{}_L^m(\hat{q})\sum_{\lambda}\mathbf{e}_\lambda^*\sigma_\lambda\right\}$$

$$=\sum_{LK}\sum_{m'}\sum_{\lambda'} C(L1K;m'\lambda')\,Y_L^{m'}(\hat{r})\sigma'_\lambda\sum_{\lambda}\sum_{m} C(L1K;m\lambda)\,\overset{*}{Y}{}_L^m(\hat{q})\mathbf{e}_\lambda^* \quad (6.70)$$

$$=\sum_{LM}\sum_{K}\cdot\{Y_L(\hat{r})\otimes\boldsymbol{\sigma}\}_M^{(K)}\cdot\overset{*}{\mathbf{Y}}{}_{KL}^M(\hat{q}).$$

This result, of course, does not change when we are replacing $\boldsymbol{\sigma}$ by $\mathbf{p}$. Summing over M instead of m we get finally

$$\langle f|\,\mathbf{V}(0)+\mathbf{A}(0)\,|i\rangle=-\sum_{KLM}(-1)^{J_f-M_f}\begin{pmatrix} J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}$$

$$\times\langle\phi_f|\left|\sum_{j=1}^{A-1} i^L j_L(qr_j)\sqrt{(4\pi)}\left\{\lambda\{Y_L(\hat{r}_j)\otimes\boldsymbol{\sigma}_j\}^{(K)}-\frac{1}{2M_N}\right.\right.$$

$$\left.\left.\times\{Y_L(\hat{r}_j)\otimes\mathbf{p}_j\}^{(K)}\right\}t_-^j\right|\left|\phi_i\right\rangle\sqrt{(4\pi)}\overset{*}{\mathbf{Y}}{}_{KL}^M(\hat{q}). \quad (6.71)$$

and therefore†

$$F_{KL}(q^2)=-\frac{1}{\sqrt{(2J_i+1)}}\frac{(2L+1)!!}{(qR)^L}\langle\phi_f|\left|\sum_{j=1}^{A-1} i^L j_L(qr_j)\sqrt{(4\pi)}\right.$$

$$\left.\times\left\{\{\lambda Y_L(\hat{r}_j)\otimes\boldsymbol{\sigma}_j\}^{(K)}-\frac{1}{2M_N}\{Y_L(\hat{r}_j)\otimes\mathbf{p}_j\}^{(K)}\right\}t_-^j\right|\left|\phi_i\right\rangle. \quad (6.72)$$

One should note that this last multipole expansion has been carried out in the Breit reference system. However this restriction does not play any role since, as explained before, the nuclei can be approximated as infinitely massive.

This simple approach to treating the nucleus and the nuclear beta-decay shows us very clearly, however, that a form factor decomposition, as chosen in eqns (6.24) and (6.25), results directly and explicitly from our weak interaction density. Because the spherical Bessel-function can

† On the right side of eqn (6.51*b*) the quantity $(i/2M_N)(\boldsymbol{\sigma}\times\mathbf{q})$ has been omitted. If it is included an additional term of

$$\frac{1}{\sqrt{(2J_i+1)}}\frac{(2L+1)!!}{(qR)^L}\langle\phi_f|\left|i^L\sqrt{(4\pi)}\left\{\frac{1}{2M_N}\boldsymbol{\sigma}[\nabla\times j_L(qr)\mathbf{Y}_{KL}(\mathbf{r})]\right\}\right|\left|\phi_i\right\rangle$$

on the right side of eqn (6.72) would occur where the gradient operator acts on the Bessel function and on the vector spherical harmonics. Contributions of this type are, however, equal to those of certain induced weak magnetism terms and will therefore be treated in detail in Section 8.1.2.

be expanded as

$$j_L(qr) = \frac{(qr)^L}{(2L+1)!!} \sum_n \frac{(-1)^n (2L+1)!!}{(2n)!!\,(2L+2n+1)!!} (qr)^{2n} \tag{6.73}$$

we see also that an extraction of a factor (if under the integral r is replaced by r/R)

$$\frac{(qR)^L}{(2L+1)!!}$$

out of the form factor is reasonable. In that case the form factors are finite for $q \to 0$.

Beyond that it seems also to be advantageous to expand the form factors themselves in power series of qR since $qR \leqslant \Delta R = W_0 R$ (for $W_0 = 20\,\text{MeV}$ we have $W_0 R \simeq 0{\cdot}4A^{1/3}$), i.e.

$$F_{KLs}(\mathbf{q}^2) = \sum_n \frac{(-1)^n (2L+1)!!}{(2n)!!\,(2L+2n+1)!!} (\mathbf{q}R)^{2n} F^{(n)}_{KLs} \tag{6.74}$$

where the $F^{(n)}_{KLs}$ are usually denoted as the form factor coefficients.

Since we usually have $qR \ll 1$ in nuclear beta-decay, only the first two terms of this expansion are relevant

$$F_{KLs}(\mathbf{q}^2) = F^{(0)}_{KLs} - \frac{(\mathbf{q}R)^2}{2(2L+3)} F^{(1)}_{KLs} + \ldots \tag{6.75}$$

6.3. Coulomb corrections to the weak transition matrix

Beta-decay itself is caused by the weak interaction. The corresponding T-matrix has been discussed in the foregoing chapters, but, besides the weak interaction, we also have the Coulomb interaction acting in the final state between electron (positron) and nucleus (for electron capture in the initial state). Since the electromagnetic interaction is orders of magnitude stronger than the weak one, it is not reasonable to treat it also in first-order perturbation theory. We have, therefore, to start once again in a non-perturbation approach for the whole problem (see Halpern 1970, Holstein 1979*a*). Then, the S-matrix for the beta-decay is given by†

$$S_{fi} = \langle f | P\left[\exp\left(-i\int_{-\infty}^{+\infty} [H_\beta(t) + H_C(t)]\right) \mathrm{d}t\right] | i\rangle. \tag{6.76}$$

† This has to be understood as a more symbolic notation for (see, for example, Roman 1965)

$$S = \langle f | \sum_{n=0}^{\infty} S_n | i \rangle$$

P is the usual time ordering operator (see Roman 1965). $H_\beta(t) = \int H_\beta(\mathbf{x}, t)\, \mathrm{d}^3x$ denotes the weak interaction Hamiltonian density given in eqn (6.1) and eqn (6.5), and $H_C(T)$ the Coulomb interaction which has in our case the form

$$H_C^a(t) = \int \mathrm{d}^3x V_C^a(\mathbf{x}) \bar{\psi}_e(x) \gamma_4 \psi_e(x) \tag{6.77}$$

where $V_C^a = (\alpha Z/R) U(r)$ represents the static external Coulomb potential acting before $(a = i)$ and after $(a = f)$ the beta-decay.

As before, we now treat the S-matrix in first-order perturbation theory, but only as far as the weak interaction $H_\beta(t)$ is concerned. In first order $H_\beta(t)$, eqn (6.76) becomes

$$S_{fi} = \delta_{fi} - i \langle f | \int_{-\infty}^{+\infty} \mathrm{d}t P\left[\exp\left(-i \int_t^{\infty} \mathrm{d}t' H_C^f(t')\right)\right] \times H_\beta(t) P\left[\exp\left(-i \int_{-\infty}^{t} \mathrm{d}t' H_C^i(t')\right)\right] | i \rangle. \tag{6.78}$$

Since the time-dependence of all the factors in eqn (6.78) is explicitly given by

$$A(t) = \mathrm{e}^{\mathrm{i}H_0 t} A(0) \mathrm{e}^{-\mathrm{i}H_0 t}$$

the integration over t can be performed. We then obtain

$$S_{fi} = \delta_{fi} - 2\pi \mathrm{i} \delta(W_f + W_e + W_\nu - W_i) \times \langle f | P\left[\exp\left(-\mathrm{i} \int_0^{\infty} \mathrm{d}t H_C^f(t)\right)\right] H_\beta(0) P\left[\exp\left(-\mathrm{i} \int_{-\infty}^{0} \mathrm{d}t H_C^i(t)\right)\right] | i \rangle. \tag{6.79}$$

This means that beta-decay occurs at $t = 0$, while before and after beta-decay, the Coulomb interaction H_C^i and H_C^f acts on the participating particles.

In principle, the Coulomb potentials $V_C^i(t)$ and $V_C^f(t)$ in initial and final states are different since the atomic number Z of the initial state differs from that in the final state by one unit. In the following, however, we

---

where

$$S_n = \frac{1}{n!} (-\mathrm{i})^n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} P[H(t') \ldots H(t^{(n)})]\, \mathrm{d}t^{(n)} \ldots \mathrm{d}t'.$$

This can be summed up and written as in eqn (6.76). The corresponding successive approximations are

$$S^{(0)} = 1$$
$$S^{(1)} = S_0 + S_1$$
$$S^{(2)} = S_0 + S_1 + S_2$$
$$S^{(n)} = S^{(n-1)} + S_n$$

handle the Coulomb problem in such a way that the Coulomb correction in the simple non-relativistic and first quantization limit is identical with that of the final-state interaction between nucleus and emitted electron (or positron).†

Thus, we make the good approximation to replace $V^i(\mathbf{x})$ by $V^f(\mathbf{x})$. Effects which are introduced by the charge difference of initial and final nucleus have therefore to be included in other correction factors like, for example, the so-called radiative correction or atomic cloud effects which will be discussed later on.

We can then write

$$S_{fi} = \delta_{fi} - 2\pi i \delta(W_f + W_e + W_\nu - W_i) \times \langle (Z+1, A) e^- \nu_e | \, P\left[\exp\left(-i\int_{-\infty}^{+\infty} H_C^f(t)\, dt\right)\right] H_\beta(0) \, |(Z, A)\rangle. \quad (6.80)$$

This result can further be simplified by using the fact that

$$\phi_{e^-}(\mathbf{r}) = \langle e^- | \, P\left[\exp\left(-i\int_{-\infty}^{+\infty} H_C^f(t)\, dt\right)\bar{\psi}_e(\mathbf{r}, 0)\right] |0\rangle. \quad (6.81)$$

$\phi_e(\mathbf{r})$ is the appropriate configuration space wave function for the electron (positron) moving in the potential $V_C^f(\mathbf{x})$, i.e. $\phi_e(\mathbf{r})$ is the solution of the Dirac equation for $V = V_C^f(\mathbf{x}) = V_C^f(r) = (\alpha Z/R) U(r)$ where Z is the atomic number of the daughter nucleus. Hence, the inclusion of the Coulomb interaction reduces to the simple prescription that we have to replace the free particle electron (positron) wave function

$$\bar{u}_e e^{-i\mathbf{p}_{e^-}\cdot\mathbf{r}} (v_e e^{-i\mathbf{p}_{e^+}\cdot\mathbf{r}})$$

by the distorted electron (positron) wave function

$$\phi_{e^-}(\mathbf{r}) \; (\phi_{e^+}(\mathbf{r})).$$

For β^--decay the T-matrix can then be written as

$$(2\pi)^3 \delta(-\mathbf{q}' + \mathbf{p}'_{e^-} + \mathbf{p}_{\bar{\nu}_e}) T = \frac{G_\beta}{\sqrt{2}} i \langle f | \, V_\mu(0) + A_\mu(0) \, | i \rangle \, e^{-i\mathbf{q}'\mathbf{r}} \times \bar{\phi}_{e^-}(\mathbf{r}) \gamma_\mu (1 + \gamma_5) v_\nu e^{-i\mathbf{p}_{\bar{\nu}_e}\mathbf{r}} d^3 r \quad (6.82)$$

† Let us assume the total Hamiltonian can be written in the form

$$H = H^0 + H^1 + H^2$$

where H^0 is the sum of the free systems Hamiltonian's, H^1 the Coulomb interaction and H^2 the weak interaction. Then, following Watson's theorem, the T-matrix can be expressed as

$$T = \langle \chi_f^- | \, H^2 \, | \phi_i \rangle$$

where the χ_f^- is a solution of the Schrödinger equation

$$(H^0 + H^1) \chi^\pm = E_C \chi^\pm$$

(see Roman 1965) and ϕ_i the free particle solution.

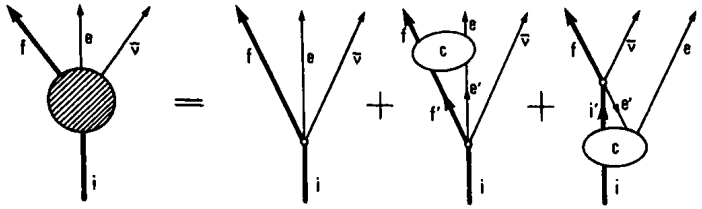


FIG. 6.1. Electromagnetic interaction (represented by c) and beta-decay.

where $\mathbf{q}' = \mathbf{p}_f - \mathbf{p}_i'$ or $\mathbf{q}' = \mathbf{p}_f' - \mathbf{p}_i$. Here, $\mathbf{p}_e'$, $\mathbf{p}_f'$ and p_i' are the corresponding momenta in the intermediate state if it is assumed that the electron is emitted in this intermediate state in a first step and acts then subsequently with the Coulomb field of the nucleus, or vice versa, that in a first step an electron positron pair is created in the Coulomb field of the initial nucleus and then in a second step the positron is absorbed in the intermediate state i'. The situation is illustrated in Fig. 6.1.

In the Breit frame the nuclear current has to be written exactly as

$$\langle f|\, V_\mu(0)+A_\mu(0)\,|i\rangle = \tfrac{1}{2}[\langle f|\, V_\mu(0)+A_\mu(0)\,|i'\rangle + \langle f'|\, V_\mu(0)+A_\mu(0)\,|i\rangle]$$
$$= \tfrac{1}{2}[\langle \mathbf{p}_f|\, V_\mu(0)+A_\mu(0)\,|\mathbf{p}_f-\mathbf{q}'\rangle + \langle \mathbf{p}_i-\mathbf{q}'|\, V_\mu(0)+A_\mu(0)\,|\mathbf{p}_i\rangle]. \quad (6.83)$$

In the foregoing section it has, on the other hand, been shown that the nuclear current expanded in form factors depends on the momentum transfer $\mathbf{q}'$ only. Integrating over all intermediate recoil momenta q' we get finally

$$T = \frac{G_\beta}{\sqrt{2}} \int \langle f|\, V_\mu(0)+A_\mu(0)\,|i\rangle\, L_\mu(\mathbf{q}')\, \mathrm{d}^3q' \quad (6.84)$$

with

$$L_\mu(\mathbf{q}') = \frac{i}{(2\pi)^3} \int \mathrm{e}^{-i\mathbf{q}'\mathbf{r}} \bar{\phi}_{e^-}(\mathbf{r})\gamma_\mu(1+\gamma_5)v_\nu \mathrm{e}^{-i\mathbf{p}_{\bar{\nu}_e}\mathbf{r}}\, \mathrm{d}^3r \quad (6.85)$$
$$= i\bar{\phi}_{e^-}(\mathbf{q}'+\mathbf{p}_{\bar{\nu}_e})\gamma_\mu(1+\gamma_5)v_\nu. \quad (6.86)$$

$L_\mu(\mathbf{q})$ is the lepton current in momentum space† for β^--decay (see also Stech and Schülke 1964).

Similarly we obtain for β^+-decay

$$L_\mu(\mathbf{q}') = i\bar{u}_\nu\gamma_\mu(1+\gamma_5)\phi_{e^+}(\mathbf{q}'+\mathbf{p}_{\nu_e}) \quad (6.87)$$

† Without the Coulomb interaction this equation reduces to

$$L_\mu(\mathbf{q}') = i\bar{u}_e\gamma_\mu(1+\gamma_5)v_\nu\delta(\mathbf{q}'+\mathbf{p}_e+\mathbf{p}_{\bar{\nu}_e}).$$

Since $\mathbf{q} = -(\mathbf{p}_e+\mathbf{p}_{\bar{\nu}_e})$, as can be seen from eqn (5.14), this is identical with eqn (6.17a).

and for electron capture

$$L_\mu(\mathbf{q}') = i\bar{u}_\nu \gamma_\mu (1+\gamma_5)\phi_{e^-}(\mathbf{q}' + \mathbf{p}_{\nu_e}). \tag{6.88}$$

In the last case $\phi_{e^-}(\mathbf{q}' + \mathbf{p}_{\nu_e})$ is, however, the Fourier-transform of a bound electron wave function.

Before closing this section it should be mentioned that the treatment outlined above does not include all effects which are caused by the Coulomb interaction. Other correction terms have to be added which are introduced by Feynman graphs not contained in the treatment of the electrostatic interaction (Sirlin 1978). These so called radiative corrections will, however, be discussed later on.

6.4. Treatment of the lepton current

The lepton current in the T-matrix (see, for examples eqn (6.17a) for β^--decay) is given in the momentum space until now. Since electron and neutrino spinor functions can be explicitly written down (see Chapter 2) it is not difficult to calculate the lepton current similarly as outlined in the case of the point like neutron.

For an exact treatment of the T-matrix it is, however, necessary to take into account the Coulomb interaction between nucleus and emitted electron, i.e. the distortion of the electron wave function by the influence of the nuclear charge. However, in that case it is more practical and usual to handle the electron wave functions in the configuration space. To make the transition from the case of a non-charged nucleus to a charged nucleus easier we will treat the lepton matrix element in configuration space from the beginning.

We have, therefore, to transform the lepton current in eqns (6.17a–c) from momentum to configuration space. To proceed further let us start with the β^--decay and consider the other decay types at the end of Chapter 7. Our procedure follows that outlined in Schülke (1964) and Schopper (1966).

We can rewrite the T-matrix given in eqn (6.17a) in the following way:

$$T = \frac{G_\beta}{\sqrt{2}} \int \langle f |\, V_\mu(0) + A_\mu(0)\, | i \rangle\, L_\mu(\mathbf{q})\, \mathrm{d}^3 q \tag{6.89}$$

where $L_\mu(\mathbf{q})$ is nothing other than the Fourier transform of the lepton current (see also eqn (6.85))

$$L_\mu(\mathbf{q}) = \frac{1}{(2\pi)^3} \int \mathrm{e}^{-i\mathbf{q}\mathbf{r}}\, i \underbrace{\bar{u}_e \mathrm{e}^{-i\mathbf{p}_e\mathbf{r}}}_{\bar{\phi}_e(\mathbf{r})} \gamma_\mu (1+\gamma_5) \underbrace{u_\nu \mathrm{e}^{-i\mathbf{p}_\nu\mathbf{r}}}_{\phi_{\bar{\nu}_e}(\mathbf{r})}\, \mathrm{d}^3 r. \tag{6.90}$$

By expanding the exponential e^{-iqr} in eqn (6.90) in the way given in eqn (6.64) and the nuclear current in the way given in eqn (6.28) we obtain

$$T=\frac{G_\beta}{\sqrt{2}(\pi)^{3/2}}\cdot\sum_{KLsM}\sum_{L'M'}(-1)^{J_f-M_f+M}(-i)^L(i)^{L'}\sqrt{(2J_i+1)}\times\begin{pmatrix}J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\iint\frac{(qR)^L}{(2L+1)!!}F_{KLs}(q^2)j_{L'}(qr)\times\phi_e^+(\mathbf{r})\overset{*}{Y}{}_{L'}^{M'}(\hat{q})Y_{L'}^{M'}(\hat{r})T_{KLs}^{-M}(\hat{q})(1+\gamma_5)\phi_{\bar{\nu}_e}(\mathbf{r})\,d^3r\,d^3q. \quad (6.91)$$

Because of the orthogonality relations of the spherical harmonics we can immediately carry out the integration over $\hat{q}$ and get

$$T=\frac{G_\beta}{\sqrt{2}}\cdot\frac{1}{\pi^{3/2}}\sum_{KLsM}(-1)^{J_f-M_f+M+L}\sqrt{(2J_i+1)}\begin{pmatrix}J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\times\int_0^\infty q^2\,dq\int_0^\infty r^2\,dr\frac{(qR)^L}{(2L+1)!!}j_L(qr)F_{KLs}(q^2)\times\int\phi_e^+(\mathbf{r})T_{KLs}^{-M}(\hat{r})(1+\gamma_5)\phi_{\bar{\nu}_e}(\mathbf{r})\,d\hat{r}. \quad (6.92)$$

Because of the operator $T_{KLs}^{-M}(\hat{r})$ the lepton part of the expression above represents something like a multipole moment. Therefore it is suitable to introduce a spherical basis for electron and neutrino wave functions in order to faciliate further treatment. Thus we try to expand these wave functions in partial waves in the following way (see Rose 1961)

$$\phi_{e^-}(\mathbf{r})=\sum_{\kappa_e\mu_e}a_{\kappa_e\mu_e}\phi_{\kappa_e}^{\mu_e} \quad (6.93)$$

$$\phi_{\nu_e}(\mathbf{r})=\sum_{\kappa_\nu\mu_\nu}b_{\kappa_\nu\mu_\nu}\phi_{\kappa_\nu}^{\mu_\nu} \quad (6.94)$$

where the spherical waves $\phi_\kappa^\mu(\mathbf{r})$ are given by (see eqn (2.64))

$$\phi_\kappa^\mu(\mathbf{r})=\begin{pmatrix}\mathrm{sign}(\kappa)f_\kappa(r)\chi_{-\kappa}^\mu(\mathbf{r})\\ g_\kappa(r)\chi_\kappa^\mu(\mathbf{r})\end{pmatrix} \quad (6.95)$$

with†

$$\chi_\kappa^\mu(\mathbf{r})=i^l\sum_m C(l\ \tfrac{1}{2}\ j;\ \mu-m\ \ m)Y_l^{\mu-m}(\mathbf{r})\chi^m. \quad (6.96)$$

† We have

$$\int\overset{*}{\chi}{}_\kappa^{\mu'}(\mathbf{r})\chi_\kappa^\mu(\mathbf{r})\,d\hat{r}=\delta_{\kappa'\kappa}\delta_{\mu'\mu}.$$

The χ^m are the two-component Pauli spinors discussed before. The index κ is

$$\kappa = \begin{cases} l & \text{for} \quad j = l - \frac{1}{2} \\ -(l+1) & \text{for} \quad j = l + \frac{1}{2}. \end{cases} \tag{6.97}$$

The exact relations between both representations, the expansion in spherical waves and the plane wave solution, can be derived by expanding the plane wave. We have

$$u^{(m)} e^{i\mathbf{pr}} = \sum_{\kappa\mu} a_{\kappa\mu} \phi_\kappa^\mu \tag{6.98}$$

and

$$\sqrt{\frac{(W+m_e)}{2W}} \begin{pmatrix} -\dfrac{\boldsymbol{\sigma}\mathbf{p}}{W+m_e} \chi^m \\ \chi^m \end{pmatrix} e^{i\mathbf{pr}} = \sum_{\kappa\mu} a_{\kappa\mu} \begin{pmatrix} \operatorname{sign}(\kappa) f_\kappa \chi_{-\kappa}^\mu \\ g_\kappa \chi_\kappa^\mu \end{pmatrix}. \tag{6.99}$$

Considering first the large components of the wave function we have

$$\sqrt{\frac{(W+m_e)}{2W}} \chi^m 4\pi \sum_{l'm'} i^{l'} j_{l'}(pr) \overset{*}{Y}{}_{l'}^{m'}(\hat{p}) Y_{l'}^{m'}(\hat{r}) = \sum_{\kappa\mu} a_{\kappa\mu} g_\kappa(r) \chi_\kappa^\mu(\hat{r}). \tag{6.100}$$

Multiplying both sides by $\overset{+}{\chi}{}_\kappa^\mu(\hat{r})$ and integrating over $d\Omega$ we obtain

$$g_\kappa(r) = \sqrt{\frac{(W+m_e)}{W}} j_l(pr) \tag{6.101}$$

and

$$a_{\kappa\mu} = \frac{4\pi}{\sqrt{2}} C(l \ \tfrac{1}{2} \ j; \ \mu - m \ \ m) \overset{*}{Y}{}_l^{\mu-m}(\hat{p}). \tag{6.102}$$

The small component v of the spinor function is related to the large one u by

$$v = -\frac{\boldsymbol{\sigma}\mathbf{p}}{W+m_e} u. \tag{6.103}$$

Thus we have

$$\operatorname{sign}(\kappa) f_\kappa(r) \chi_{-\kappa}^\mu = -\frac{\boldsymbol{\sigma}\mathbf{p}}{W+m_e} g_\kappa(r) \chi_\kappa^\mu. \tag{6.104}$$

Using the identity (see Rose 1961)

$$\boldsymbol{\sigma}\mathbf{p} = \frac{\boldsymbol{\sigma}\mathbf{r}}{r} \left\{ -i \frac{\partial}{\partial r} + \frac{i}{r} \boldsymbol{\sigma}\mathbf{l} \right\} \tag{6.105}$$

and the relations

$$\frac{\boldsymbol{\sigma r}}{r}\chi^{\mu}_{\kappa} = -\mathrm{i}\,\mathrm{sign}(\kappa)\chi^{\mu}_{-\kappa} \tag{6.106}$$

$$\boldsymbol{\sigma l}\chi^{\mu}_{\kappa} = -(\kappa+1)\chi^{\mu}_{\kappa} \tag{6.107}$$

the above equation takes the form

$$\mathrm{sign}(\kappa) f_{\kappa}(r)\chi^{\mu}_{-\kappa} = \frac{1}{\sqrt{\{W(W+m_e)\}}}\,\mathrm{sign}(\kappa)\chi^{\mu}_{-\kappa}\left\{\frac{\partial}{\partial r} j_l(pr) + \frac{\kappa+1}{r} j_l(pr)\right\} \tag{6.108}$$

where

$$\kappa+1 = \begin{cases} l+1 & \kappa>0 \\ -l & \kappa<0. \end{cases} \tag{6.109}$$

Differentiation of the spherical Bessel-functions leads to

$$\frac{\partial}{\partial r} j_l(pr) = \begin{cases} \dfrac{l}{r} j_l(pr) - p j_{l+1}(pr) \\ -\dfrac{l+1}{r} j_l(pr) + p j_{l-1}(pr). \end{cases} \tag{6.110}$$

With $p^2 = \sqrt{(W^2 - m^2)}$ the radial component $f_{\kappa}(r)$ is then given by

$$f_{\kappa}(r) = \mathrm{sign}(\kappa)\sqrt{\frac{(W-m_e)}{W}}\, j_{l(-\kappa)}(pr) \tag{6.111}$$

where

$$\begin{aligned} l(-\kappa) &= l(\kappa)-1 \quad \kappa>0 \\ l(-\kappa) &= l(\kappa)+1 \quad \kappa<0. \end{aligned} \tag{6.112}$$

In the following we denote $l(-\kappa)$ by $\bar{l}$. For the electron radial wave functions $g_{\kappa_e}(r)$ and $f_{\kappa_e}(r)$ it is, however, customary to choose the asymptotic behaviour for $r \to \infty$ as

$$g_{\kappa_e}(r) \xrightarrow[r\to\infty]{} \sqrt{\frac{(W+m_e)}{W}}\frac{1}{r}\cos\left(pr - \frac{l+1}{2}\pi\right) \tag{6.112a}$$

$$f_{\kappa_e}(r) \xrightarrow[r\to\infty]{} -\sqrt{\frac{(W-m_e)}{W}}\frac{1}{r}\sin\left(pr - \frac{l+1}{2}\pi\right). \tag{6.112b}$$

Because

$$j_l(pr) \xrightarrow[r\to\infty]{} \frac{1}{pr}\cos\left(pr - \frac{l+1}{2}\pi\right) \tag{6.113a}$$

$$j_{\bar{l}}(pr) \xrightarrow[r\to\infty]{} -\mathrm{sign}(\kappa)\frac{1}{pr}\sin\left(pr - \frac{l+1}{2}\pi\right) \tag{6.113b}$$

we have, therefore, to multiply $g_\kappa(r)$ and $f_\kappa(r)$ by p and to divide $a_{\kappa\mu}$ by p in the case of the electron. Thus it is

$$g_{\kappa_e} = \sqrt{\frac{(W_e + m_e)}{W_e}}\, p_e j_l(p_e r) \tag{6.114a}$$

$$f_{\kappa_e} = \mathrm{sign}(\kappa_e)\sqrt{\frac{(W_e - m_e)}{W_e}}\, p_e j_{\tilde{l}}(p_e r) \tag{6.114b}$$

and

$$a_{\kappa_e \mu_e} = \frac{4\pi}{\sqrt{2}} \frac{1}{p_e} C(l_e\ \tfrac{1}{2}\ j_e;\ \mu_e - m\ m) \overset{*}{Y}{}^{\mu_e - m}_{l_e}(\hat{p}_e). \tag{6.115a}$$

In the case where the electron radial wave function is distorted by the nuclear charge a phase factor $e^{i\Delta_{\kappa_e}}$ has to be included and we get

$$a_{\kappa_e \mu_e} = \frac{4\pi}{\sqrt{2}} \frac{1}{p_e} C(l_e\ \tfrac{1}{2}\ j_e;\ \mu_e - m\ m) \overset{*}{Y}{}^{\mu_e - m}_{l_e}(\hat{p}_e) e^{i\Delta_{\kappa_e}} \tag{6.115b}$$

(see Section 3.2.3). The asymptotic behaviour of $f(r)$ and $g(r)$ for $r \to \infty$ then reads as given in eqn (3.131).

In the case of the neutrino, on the other hand, the form of eqn (6.101), (6.102) and (6.111) usually is taken. Setting the neutrino mass $m_\nu = 0$ we get†

$$g_{\kappa_\nu} = j_l(p_\nu r) \tag{6.116a}$$

$$f_{\kappa_\nu} = \mathrm{sign}(\kappa_\nu) j_{\tilde{l}}(p_\nu r) \tag{6.116b}$$

$$b_{\kappa_\nu \mu_\nu} = \frac{4\pi}{\sqrt{2}} C(l_\nu\ \tfrac{1}{2}\ j_\nu;\ \mu_\nu - m\ m) \overset{*}{Y}{}^{\mu_\nu - m_\nu}_{l}(\hat{p}_\nu). \tag{6.117}$$

† It is not certain that the neutrino mass is exactly zero. Nowadays, theoretical speculations (Bilenky and Pontecorvo 1978; Wolfenstein 1978) as well as experimental indications for a finite neutrino mass exist (de Rujula *et al.* 1979; Barger *et al.* 1980; Reines *et al.* 1980; Lubimov *et al.* 1980). But even if such a finite mass is found it will be of the order $m_\nu \lesssim 50$ eV. Thus, we then have

$$g_{\kappa_\nu} = b_{-k_\nu} j_l(p_\nu r)$$
$$f_{\kappa_\nu} = b_{+k_\nu}\, \mathrm{sign}(\kappa_\nu) j_{\tilde{l}}(p_\nu r)$$

with

$$b_{-k_\nu} = \sqrt{\left(\frac{W_\nu + m_\nu}{W_\nu}\right)}$$
$$b_{+k_\nu} = \sqrt{\left(\frac{W_\nu - m_\nu}{W_\nu}\right)}$$

Since nuclear beta-decay is usually considered for neutrino energies above 10 keV we are dealing with

$$|b_{\mp k_\nu} - 1| < 2{\cdot}5 \times 10^{-3}$$

These effects are, therefore, so small that they cannot be detected, with very few exceptions (tritium beta-spectrum). Thus, for usual considerations it is entirely justified to ignore them completely.

Up to now we have discussed the electron and neutrino wave functions only. In β^--decay, however, we have to insert the antineutrino wave function (see eqn 6.13*a*) and in β^+-decay the positron wave function. For this reason the corresponding formulae for the antiparticles have to be derived too (it should be noted that this point in particular is not correctly taken into account in many papers).

In order to replace the particle wave function by their antiparticle wave functions we recall that the charge conjugation operator C connects both types of wave functions

$$\psi_{\text{antiparticle}} = C\psi_{\text{particle}} = \tilde{C}\psi^*_{\text{particle}}. \tag{6.118}$$

In our notation $\tilde{C}$ is explicitly given by

$$\tilde{C} = -\gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}. \tag{6.119}$$

As a first example let us apply C on the plane wave solution u_m. Then we find (see Rose 1961)

$$-\gamma_2 u^*_m = \sqrt{\frac{(W+m_e)}{2W}} \begin{pmatrix} i\sigma_2\chi^m \\ i\sigma_2 \dfrac{\boldsymbol{\sigma}^*\mathbf{p}}{W+m_e}\chi^m \end{pmatrix}. \tag{6.120}$$

Now we have

$$i\sigma_2\boldsymbol{\sigma}^*\mathbf{p} = -\boldsymbol{\sigma}\mathbf{p}i\sigma_2 \tag{6.121}$$

and

$$-i\sigma_2\chi^m = (-1)^{m-\frac{1}{2}}\chi^{-m}. \tag{6.122}$$

Combining these results we obtain

$$(-1)^{m-\frac{1}{2}}v_{-m} = -\gamma_2 u^*_m. \tag{6.123}$$

In order to proceed further we have now to consider the expansion in spherical waves discussed before. Going back to eqns (6.93)–(6.95) we have

$$\begin{aligned} \phi(r)_{\text{antiparticle}} &= -\gamma_2\phi^*_{\text{particle}} \\ &= \sum_{\kappa\mu} a^*_{\kappa\mu} \begin{pmatrix} i\sigma_2 g_\kappa(r)\chi^{*\mu}_\kappa \\ -i\sigma_2 \operatorname{sign}(\kappa) f_\kappa(r)\chi^{*\mu}_{-\kappa} \end{pmatrix}. \end{aligned} \tag{6.124}$$

From eqn (6.96) we see that

$$i\sigma_2\chi^{*\mu}_\kappa = (-i)^l \sum_m C(l\ \tfrac{1}{2}\ j;\ \mu-m\ \ m)\, i\sigma_2\chi^m(-1)^{\mu-m}Y_l^{m-\mu}(\hat{r}). \tag{6.125}$$

But using eqn (6.122), replacing the summation index m by $-m$ and remembering that

$$C(l\ \tfrac{1}{2}\ j;\ -\mu-m\ \ m) = (-1)^{l+\frac{1}{2}-j}C(l\ \tfrac{1}{2}\ j;\ \mu+m\ -m) \tag{6.126}$$

we find

$$i\sigma_2 \chi^{*\mu}_{\kappa} = (-1)^{j+\mu} \chi^{-\mu}_{\kappa}. \tag{6.127}$$

Inserting this result in eqn (6.124), we obtain for the positron wave function

$$\phi(r)_{\text{positron}} = \sum_{\kappa_e \mu_e} (-1)^{j_e+\mu_e} a^*_{\kappa_e \mu_e} \phi^{-\mu_e}_{\kappa_e\,\text{positron}} \tag{6.128}$$

with

$$\phi^{-\mu_e}_{\kappa_e\,\text{positron}} = \begin{pmatrix} g_{\kappa_e}(r)\chi^{-\mu_e}_{\kappa_e} \\ -\text{sign}(\kappa_e) f_{\kappa_e}(r)\chi^{-\mu_e}_{-\kappa_e} \end{pmatrix} \tag{6.129}$$

and for the antineutrino wave function

$$\phi(r)_{\text{antineutrino}} = \sum_{\kappa_\nu \mu_\nu} (-1)^{j_\nu+\mu_\nu} b^*_{\kappa_\nu \mu_\nu} \phi^{-\mu_\nu}_{\kappa_\nu\,\text{antineutrino}} \tag{6.130}$$

$$\phi^{-\mu_\nu}_{\kappa_\nu\,\text{antineutrino}} = \begin{pmatrix} j_l(p_\nu r)\chi^{-\mu_\nu}_{\kappa_\nu} \\ -j_{\bar{l}}(p_\nu r)\chi^{-\mu_\nu}_{-\kappa_\nu} \end{pmatrix}. \tag{6.131}$$

It is useful, furthermore, to introduce reduced lepton matrix elements too by applying the Wigner–Eckart theorem. From eqns (6.92), (6.93) and (6.130) we find for the T-matrix

$$\begin{aligned} T_{\beta^-} = {} & \frac{G_\beta}{\sqrt{2}} \frac{1}{\pi^{3/2}} \sum_{KLsM} \sum_{\substack{\kappa_e \mu_e \\ \kappa_\nu \mu_\nu}} (-1)^{J_f - M_f + j_e - \mu_e} \\ & \times (-1)^{L+M+j_\nu+\mu_\nu} \sqrt{(2J_i+1)} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \begin{pmatrix} j_e & K & j_\nu \\ -\mu_e & -M & -\mu_\nu \end{pmatrix} \\ & \times a^*_{\kappa_e \mu_e} b^*_{\kappa_\nu \mu_\nu} \int_0^\infty q^2\,dq \int_0^\infty r^2\,dr \frac{(qR)^L}{(2L+1)!!} j_L(qr) F_{KLs}(q^2) \\ & \times \langle \phi_{\kappa_e\,\text{electron}} | \, | T_{KLs}(1+\gamma_5) | \, | \phi_{\kappa_\nu\,\text{antineutrino}} \rangle. \end{aligned} \tag{6.132}$$

The reduced lepton matrix element in the equation above can, of course, be calculated, since wave functions as well as operators are well-known in this matrix element. Using the Wigner–Eckart theorem and some orthogonality relations of the $3j$-symbols we obtain

$$\langle \chi_{\kappa_e} | \, | \hat{T}_{KLs} | \, | \chi_{\kappa_\nu} \rangle = \sum_{M \mu_e \mu_\nu} (-1)^{j_e - \mu_e} \begin{pmatrix} j_e & K & j_\nu \\ -\mu_e & M & \mu_\nu \end{pmatrix} \int {}^{+}\chi^{\mu_e}_{\kappa_e} \hat{T}^M_{KLs}(\hat{r}) \chi^{\mu_\nu}_{\kappa_\nu}\,d\hat{r} \tag{6.133}$$

where $\hat{T}^M_{KLs}(\hat{r})$ is a 2×2 matrix and related to $T^M_{KLs}(\hat{r})$ by

$$T^M_{KLs}(\hat{r}) = \hat{T}^M_{KLs}(\hat{r}) \begin{cases} 1 & \text{for } s=0 \\ \gamma_5 & \text{for } s=1 \end{cases} \tag{6.134}$$

(1 represents the 4×4 unit matrix).

To proceed further we could choose the direct way and insert eqns (6.27) and (6.96) into eqn (6.133). Then we end up with a highly complicated sum of $3j$-symbols, which may be contracted into a $9j$-symbol, with other factors.

However, we intend to use another more transparent procedure. The operator $\hat{T}^{M}_{KLs}(\hat{r})$ can be expressed as a tensor product, i.e.

$$\hat{T}^{M}_{KLs}(\hat{r}) = \mathrm{i}^{L}\{\Sigma^{(s)} \otimes Y_L\}^{K} = \mathrm{i}^{L}(-1)^{L+s-K}\{Y_L \otimes \Sigma^{(s)}\}^{K} \tag{6.135}$$

with $\Sigma^{(0)} = 1$ and $\Sigma^{(1)} = \boldsymbol{\sigma}$.

The wave function $\chi^{\mu_e}_{\kappa_e}$ is nothing other than a wave function of a fermion where the orbital angular momentum $\mathbf{l}$ and the spin $\mathbf{s}$ is coupled to $\mathbf{j} = \mathbf{l} + \mathbf{s}$. We can, therefore, write

$$\langle \chi_{\kappa_e} | \, | \hat{T}_{KLs} | \, | \chi_{\kappa_\nu} \rangle = (-1)^{L+s-K} \langle l_e \; \tfrac{1}{2} \; j_e | \, | \{Y_L \otimes \Sigma^{(s)}\}^{K} | \, | l_\nu \; \tfrac{1}{2} \; j_\nu \rangle \tag{6.136}$$

and obtain (see Edmonds 1964 or de-Shalit and Talmi 1963)

$$\langle \chi_{\kappa_e} | \, | \hat{T}_{KLs} | \, | \chi_{\kappa_\nu} \rangle = \mathrm{i}^{l_e + l_\nu + L}(-1)^{j_e - j_\nu + L + l_e} \sqrt{\{(2j_e+1)(2j_\nu+1)(2K+1)\}} \times \begin{Bmatrix} K & s & L \\ j_e & \frac{1}{2} & l_e \\ j_\nu & \frac{1}{2} & l_\nu \end{Bmatrix} \langle l_e | \, | Y_L | \, | l_\nu \rangle \langle \tfrac{1}{2} | \, | \Sigma^{(s)} | \, | \tfrac{1}{2} \rangle. \tag{6.137}$$

Since we have (see, for example, Edmonds 1964 or de-Shalit and Talmi 1963)

$$\langle l_e | \, | Y_L | \, l_\nu \rangle = \frac{1}{\sqrt{(4\pi)}} (-1)^{l_\nu} C(l_e \; l_\nu \; L; 00) \sqrt{\{(2l_e+1)(2l_\nu+1)\}} \tag{6.138}$$

$$\langle \tfrac{1}{2} | \, | \Sigma^{(s)} | \, | \tfrac{1}{2} \rangle = \sqrt{\{2(2s+1)\}} \tag{6.139}$$

we finally get the expression

$$\langle \chi_{\kappa_e} | \, | \hat{T}_{KLs} | \, | \chi_{\kappa_\nu} \rangle = \frac{\sqrt{2}}{\sqrt{(4\pi)}} G_{KLs}(\kappa_e, \kappa_\nu) \tag{6.140}$$

with

$$G_{KLs}(\kappa_e, \kappa_\nu) = \mathrm{i}^{l_e + l_\nu + L}(-1)^{j_e - j_\nu} \times \sqrt{\{(2s+1)(2K+1)(2j_e+1)(2j_\nu+1)(2l_e+1)(2l_\nu+1)\}} \times C(l_e \; l_\nu \; L; 00) \begin{Bmatrix} K & s & L \\ j_e & \frac{1}{2} & l_e \\ j_\nu & \frac{1}{2} & l_\nu \end{Bmatrix}. \tag{6.141}$$

These quantities $G_{KLs}(\kappa_e, \kappa_\nu)$ where first introduced by Weidenmüller (1961) (see also Stech and Schülke 1964; Schopper 1966; Behrens and Bühring 1971).

We are now in a position to give a final explicit expression for the

reduced lepton matrix element and obtain

$$\begin{aligned}\begin{Bmatrix}\text{for } s=0\\ \text{for } s=1\end{Bmatrix}&\langle\phi_{\kappa_{e\,\text{electron}}}||T_{KLs}(C_i+C_i'\gamma_5)||\phi_{\kappa_{\nu\,\text{antineutrino}}}\rangle\\ &=\begin{Bmatrix}C_i\\ C_i'\end{Bmatrix}\{\text{sign}(\kappa_e)f_{\kappa_e}(r)g_{\kappa_\nu}(r)\langle\chi_{-\kappa_e}||\hat{T}_{KLs}||\chi_{\kappa_\nu}\rangle\\ &\quad-\text{sign}(\kappa_\nu)f_{\kappa_\nu}(r)g_{\kappa_e}(r)\langle\chi_{\kappa_e}||\hat{T}_{KLs}||\chi_{-\kappa_\nu}\rangle\}\\ &\quad+\begin{Bmatrix}C_i'\\ C_i\end{Bmatrix}\{g_{\kappa_e}(r)g_{\kappa_\nu}(r)\langle\chi_{\kappa_e}||\hat{T}_{KLs}||\chi_{\kappa_\nu}\rangle\\ &\quad-\text{sign}(\kappa_e)\text{sign}(\kappa_\nu)f_{\kappa_e}(r)f_{\kappa_\nu}(r)\langle\chi_{-\kappa_e}||\hat{T}_{KLs}||\chi_{-\kappa_\nu}\rangle\}\end{aligned}\tag{6.142}$$

where $i = V, A$. C_i and C_i' determine the degree of the parity violation in nuclear beta-decay. In our case, where we assume maximal parity violation, we have $C_V = C_V' = 1$ and $C_A = C_A' = 1$ (see van Klinken *et al.* 1978).

That these terms are responsible for the parity violation in beta decay can be easily seen when we are, for instance, considering the simplest case, i.e. $K, L, s = 0$.

The quantities $G_{000}(\kappa_e, \kappa_\nu)$ take then the values

$$G_{000}(1, 1) = G_{000}(-1, -1) = 1$$
$$G_{000}(-1, 1) = G_{000}(1, -1) = 0.$$

This means that we always have in the sum over κ_e (for $\kappa_\nu = -1$) two terms for $\kappa_e = -1$ (multiplied by C_V') and two terms for $\kappa_e = 1$ (multiplied by C_V) and vice versa for $\kappa_\nu = +1$.

Obviously in the sum a mixture of lepton wave functions with a difference in the orbital angular momentum of one unit occurs since in the case of $\kappa_e = -1$ we have $l_e = 0$ and in the case of $\kappa_e = +1$ $l_e = 1$. The parity of a wave function is, however, given by $(-1)^l$ and, therefore, the lepton field has no definite parity, i.e. parity is not a good quantum number. On the other hand, if either C_i or C_i' are zero, terms with either $\kappa_e = -1$ or $\kappa_e = +1$ would remain over. In that case the lepton field would have a definite parity.

Using eqns. (6.101), (6.114*a*), (6.114*b*), (6.116*a*), (6.116*b*), (6.140), (6.141) and (6.142) we see that

$$\begin{aligned}\frac{\sqrt{(4\pi)}}{\sqrt{2}}&\langle\phi_{\kappa_{e\,\text{electron}}}||T_{KLs}(1+\gamma_5)||\phi_{\kappa_{\nu\,\text{antineutrino}}}\rangle\\ &=\sqrt{\frac{(W_e+m_e)}{W_e}}\,p_e j_{l(\kappa_e)}(p_e r)\{j_{l(\kappa_\nu)}(p_\nu r)G_{KLs}(\kappa_e,\kappa_\nu)\\ &\quad-j_{l(-\kappa_\nu)}(p_\nu r)G_{KLs}(\kappa_e,-\kappa_\nu)\}+\sqrt{\frac{(W_e-m_e)}{W_e}}\,p_e j_{l(-\kappa_e)}(p_e r)\\ &\quad\times\{j_{l(\kappa_\nu)}(p_\nu r)G_{KLs}(-\kappa_e,\kappa_\nu)-j_{l(-\kappa_\nu)}(p_\nu r)G_{KLs}(-\kappa_e,-\kappa_\nu)\}.\end{aligned}\tag{6.143}$$

Later on, when we are taking into account the Coulomb interaction between the emitted electron and the residual nucleus, we have to replace the free particle radial wave functions, i.e. the spherical Bessel-functions in eqn (6.143), by the radial wave solutions for an electron moving in the field of the charged nucleus. Thus we are also giving the lepton matrix element in a more general form

$$\begin{aligned}\frac{\sqrt{(4\pi)}}{\sqrt{2}}\langle\phi_{\kappa_{e_{\text{electron}}}}||T_{KLs}(1+\gamma_5)||\phi_{\kappa_{\nu_{\text{antineutrino}}}}\rangle \\ = g_{\kappa_e}(Z)\{j_{l(\kappa_\nu)}(p_\nu r)G_{KLs}(\kappa_e,\kappa_\nu)-j_{l(-\kappa_\nu)}(p_\nu r)G_{KLs}(\kappa_e,-\kappa_\nu)\} \\ +\text{sign}(\kappa_e)f_{\kappa_e}(Z)\{j_{l(\kappa_\nu)}(p_\nu r)G_{KLs}(-\kappa_e,\kappa_\nu) \\ -j_{l(-\kappa_\nu)}(p_\nu r)G_{KLs}(-\kappa_e,-\kappa_\nu)\} \qquad (6.144)\end{aligned}$$

where Z is the atomic number.

So far, we have considered only the β^--decay. Since, however, the other two types of decay, i.e. β^+-decay and electron capture, are as important as the β^--decay, we have now to turn to these two other types.

Going back to eqns (6.17*a–c*) we see that the essential difference between the three decay modes lies in the lepton matrix element (the consequences for the so-called induced terms will be discussed later). More precisely the lepton matrix element for β^--decay contains electron–antineutrino wave functions, that for β^+-decay neutrino–positron wave functions, and that for electron capture neutrino–electron wave functions (see also Bambynek *et al.* 1977).

Combining eqns (6.17*b*), (6.92), (6.116), (6.117), (6.128), and (6.129) we hence obtain for β^+-decay

$$\begin{aligned}T_{\beta^+}=\frac{G_\beta}{\sqrt{2}}\frac{1}{\pi^{3/2}}\sum_{KLsM}\sum_{\substack{\kappa_e\mu_e\\ \kappa_\nu\mu_\nu}}(-1)^{J_f-M_f+j_\nu-\mu_\nu}(-1)^{L+M+j_e+\mu_e} \\ \times\sqrt{(2J_i+1)}\begin{pmatrix}J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix}j_\nu & K & j_e\\ -\mu_\nu & -M & -\mu_e\end{pmatrix}a^*_{\kappa_e\mu_e}b^*_{\kappa_\nu\mu_\nu} \\ \times\int_0^\infty q^2\,\mathrm{d}q\int_0^\infty r^2\,\mathrm{d}r\frac{(qR)^L}{(2L+1)!!}j_L(qr)F_{KLs}(q^2) \\ \times\langle\phi_{\kappa_{\nu_{\text{neutrino}}}}||T_{KLs}(1+\gamma_5)||\phi_{\kappa_{e_{\text{positron}}}}\rangle \qquad (6.145)\end{aligned}$$

and for electron capture

$$\begin{aligned}T_{EC}=\frac{G_\beta}{\sqrt{2}}\frac{1}{\pi^{3/2}}\sum_{KLsM}\sum_{\kappa_\nu\mu_\nu}(-1)^{J_f-M_f+j_\nu-\mu_\nu}(-1)^{L+M}\sqrt{(2J_i+1)} \\ \times\begin{pmatrix}J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix}j_\nu & K & j_x\\ -\mu_\nu & -M & \mu_x\end{pmatrix}b^*_{\kappa_\nu\mu_\nu}\int_0^\infty q^2\,\mathrm{d}q\int_0^\infty r^2\,\mathrm{d}r \\ \times\frac{(qR)^L}{(2L+1)!!}j_L(qr)F_{KLs}(q^2) \\ \times\langle\phi_{\kappa_{\nu_{\text{neutrino}}}}||T_{KLs}(1+\gamma_5)||\phi_{\kappa_{x_{\text{electron}}}}\rangle. \qquad (6.146)\end{aligned}$$

Here, x ($= K, L_1, L_2, M_1 \ldots$) denotes the different shells and subshells of the atomic cloud from which the electron can be captured.

The similarity of eqn (6.132) for β^--decay, eqn (6.145) for β^+-decay and eqn (6.146) for electron capture suggests that we need to derive the final formulae of the observables for only one type of decay (β^-, β^+, or EC) and can hence obtain results for other decay modes. For this purpose, we transform eqns (6.145) and (6.146) into a form that is similar to that of eqn (6.132), by interchanging initial and final states in the reduced lepton matrix elements. From the properties of the coefficients $G_{KLs}(\kappa_f, \kappa_e)$ (see eqn (6.141)) it follows that

$$\langle f||T_{KLs}(1+\gamma_5)||i\rangle = (-1)^{K-s+j_i-j_f}\langle i||T_{KLs}(1+\gamma_5)||f\rangle. \tag{6.147}$$

Thus, we obtain for β^+-decay

$$\begin{aligned} T_{\beta^+} = \frac{G_\beta}{\sqrt{2}}\frac{1}{\pi^{3/2}} \sum_{KLsM} \sum_{\substack{\kappa_e\kappa_\nu \\ \kappa_\nu\mu_\nu}} (-1)^{J_f-M_f+j_e-\mu_e}(-1)^{L-s+M+j_\nu+\mu_\nu+1} \\ \times \sqrt{(2J_i+1)}\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix}\begin{pmatrix} j_e & K & j_\nu \\ -\mu_e & -M & -\mu_\nu \end{pmatrix} a^*_{\kappa_e\mu_e} b^*_{\kappa_\nu\mu_\nu} \\ \times \int_0^\infty q^2\,\mathrm{d}q \int_0^\infty r^2\,\mathrm{d}r \frac{(qR)^L}{(2L+1)!!} j_L(qr) F_{KLs}(q^2) \\ \times \langle \phi_{\kappa_{e_{\text{positron}}}} \| T_{KLs}(1+\gamma_5) \| \phi_{\kappa_{\nu_{\text{neutrino}}}} \rangle \end{aligned} \tag{6.148}$$

and for electron capture

$$\begin{aligned} T_{EC} = \frac{G_\beta}{\sqrt{2}}\frac{1}{\pi^{3/2}} \sum_{KLsM} \sum_{\kappa_x\mu_\nu} (-1)^{J_f-M_f+j_x-\mu_x}(-1)^{L-s+M+j_\nu+\mu_\nu+1} \\ \times \sqrt{(2J_i+1)}\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix}\begin{pmatrix} j_x & K & j_\nu \\ \mu_x & -M & -\mu_\nu \end{pmatrix} b^*_{\kappa_\nu\mu_\nu} \cdot (-1)^{j_x+\mu_x} \\ \times \int_0^\infty q^2\,\mathrm{d}q \int_0^\infty r^2\,\mathrm{d}r \frac{(qR)^L}{(2L+1)!!} j_L(qr) F_{KLs}(q^2) \\ \times \langle \phi_{\kappa_{x_{\text{electron}}}} || T_{KLs}(1+\gamma_5) || \phi_{\kappa_{\nu_{\text{neutrino}}}} \rangle. \end{aligned} \tag{6.149}$$

The corresponding reduced lepton matrix for β^+-decay and electron capture can be calculated in a way analogous to that discussed for β^--decay. It is easy to verify that for β^+-decay

$$\begin{aligned} \frac{\sqrt{(4\pi)}}{\sqrt{2}}\langle \phi_{\kappa_{e_{\text{positron}}}} || T_{KLs}(1+\gamma_5) || \phi_{\kappa_{\nu_{\text{neutrino}}}} \rangle \\ = g_{\kappa_e}(-Z)\{j_{l(\kappa_\nu)}(p_\nu r) G_{KLs}(\kappa_e, \kappa_\nu) + j_{l(-\kappa_\nu)}(p_\nu r) \\ \times G_{KLs}(\kappa_e, -\kappa_\nu)\} - \mathrm{sign}(\kappa_e) f_{\kappa_e}(-Z)\{j_{l(\kappa_\nu)}(p_\nu r) \\ \times G_{KLs}(-\kappa_e, \kappa_\nu) + j_{l(-\kappa_\nu)}(p_\nu r) G_{KLs}(-\kappa_e, -\kappa_\nu)\}. \end{aligned} \tag{6.150a}$$

As mentioned before the positron radial wave functions $g_{\kappa_e}(-Z)$ and $f_{\kappa_e}(-Z)$ have been written in a way where the formula above is of a more general validity even if we introduce wave functions which are distorted by the nuclear charge.

As can be seen in Chapter 2, the positron radial wave solutions of the Dirac radial equation can be simply obtained by replacing the charge Z by $-Z$. We have, therefore, indicated this explicitly by a $-Z$ in brackets.

In the case of the free particle solutions, $g_{\kappa_e}(-Z)$ and $f_{\kappa_e}(-Z)$ are given by eqns (6.114*a*) and (*b*) in the same way as for β^--decay.

For electron capture we have similarly

$$\frac{\sqrt{(4\pi)}}{\sqrt{2}}\langle\phi_{\kappa_{x_{\text{electron}}}}||T_{KLS}(1+\gamma_5)||\phi_{\kappa_{\nu_{\text{neutrino}}}}\rangle$$
$$= g_{\kappa_x}(Z)\{j_{l(\kappa_\nu)}(p_\nu r)G_{KLS}(\kappa_e, \kappa_\nu)+j_{l(-\kappa_\nu)}(p_\nu r)G_{KLS}(\kappa_e, -\kappa_\nu)\}$$
$$+\text{sign}(\kappa_x)f_{\kappa_x}(Z)\{j_{l(\kappa_\nu)}(p_\nu r)G_{KLS}(-\kappa_e, \kappa_\nu)+j_{l(-\kappa_\nu)}(p_\nu r)$$
$$\times G_{KLS}(-\kappa_e, -\kappa_\nu)\}. \qquad (6.150b)$$

6.5. The quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$

Before we perform the explicit calculation of the observables let us introduce some new quantities in order to obtain final formulae with a simpler structure. While at a first glance the following may appear rather formalistic, it will become obvious in due course why this form of presentation has been chosen. The reason is, that a set-up like this allows structure and interrelationship between observables to be portrayed more easily, as, general speaking, this kind of set-up makes the structure more transparent. In the following we restrict our discussion, however, to β^--decay. In a later section we will discuss how the final formulae must be changed to represent β^+-decay and electron capture.

The last part of eqn (6.132) can be written as

$$\frac{2}{\pi}\frac{1}{\sqrt{(2K+1)}}\sum_{Ls}(-1)^{K-L}\int_0^\infty q^2\,dq\int_0^\infty r^2\,dr\frac{(qR)^L}{(2L+1)!!}j_L(qr)$$
$$\times F_{KLS}(q^2)\{g_{\kappa_e}(r)[j_l(p_\nu r)G_{KLS}(\kappa_e, \kappa_\nu)-j_{\tilde{l}}(p_\nu r)G_{KLS}(\kappa_e, -\kappa_\nu)]$$
$$+\text{sign}(\kappa_e)f_{\kappa_e}(r)[j_l(p_\nu r)G_{KLS}(-\kappa_e, \kappa_\nu)-j_{\tilde{l}}(p_\nu r)G_{KLS}(-\kappa_e, -\kappa_\nu)]\}$$
$$=-\text{sign}(\kappa_\nu)\alpha_{\kappa_e}\{M_K(k_e, k_\nu)+\text{sign}(\kappa_e)m_K(k_e, k_\nu)\}. \qquad (6.151)$$

Here, the explicit form of the reduced lepton matrix element has been taken from eqn (6.144).

The quantities α_{κ_e} are the amplitudes or normalization constants, respectively, when the electron radial wave functions are expressed in the

form (see the extensive discussion in Section 4.1)

$$f_{+k_e}(r) = \alpha_{+k_e} \frac{(p_e r)^{k_e - 1}}{(2k_e - 1)!!} \{H_{k_e}(r) + h_{k_e}(r)\} \tag{6.152a}$$

$$g_{-k_e}(r) = \alpha_{-k_e} \frac{(p_e r)^{k_e - 1}}{(2k_e - 1)!!} \{H_{k_e}(r) - h_{k_e}(r)\} \tag{6.152b}$$

$$f_{-k_e}(r) = -\alpha_{-k_e} \frac{(p_e r)^{k_e - 1}}{(2k_e - 1)!!} \frac{r}{R} \{D_{k_e}(r) - d_{k_e}(r)\} \tag{6.152c}$$

$$g_{+k_e}(r) = \alpha_{+k_e} \frac{(p_e r)^{k_e - 1}}{(2k_e - 1)!!} \frac{r}{R} \{D_{k_e}(r) + d_{k_e}(r)\}. \tag{6.152d}$$

k_e is related to κ_e by

$$k_e = |\kappa_e|. \tag{6.153}$$

After substitution of the electron radial wave functions in eqn (6.151) we obtain explicitly for the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$

$$\begin{aligned} M_K(k_e, k_\nu) = \frac{2}{\pi} \frac{1}{\sqrt{(2K+1)}} \sum_{Ls} (-1)^{K-L} \int_0^\infty q^2 \, dq \int_0^\infty r^2 \, dr \frac{(qR)^L}{(2L+1)!!} j_L(qr) \\ \times F_{KLs}(q^2) \frac{(p_e r)^{k_e - 1}}{(2k_e - 1)!!} \{H_{k_e}(r)[j_{k_\nu - 1}(p_\nu r) G_{KLs}(-k_e, -k_\nu) \\ - j_{k_\nu}(p_\nu r) G_{KLs}(-k_e, k_\nu)] + \frac{r}{R} D_{k_e}(r)[j_{k_\nu - 1}(p_\nu r) G_{KLs}(k_e, -k_\nu) \\ - j_{k_\nu}(p_\nu r) G_{KLs}(k_e, k_\nu)]\} \end{aligned} \tag{6.154}$$

and

$$\begin{aligned} m_K(k_e, k_\nu) = \frac{2}{\pi} \frac{1}{\sqrt{(2K+1)}} \sum_{Ls} (-1)^{K-L} \int_0^\infty q^2 \, dq \int_0^\infty r^2 \, dr \frac{(qR)^L}{(2L+1)!!} j_L(qr) \\ \times F_{KLs}(q^2) \frac{(p_e r)^{k_e - 1}}{(2k_e - 1)!!} \{h_{k_e}(r)[j_{k_\nu - 1}(p_\nu r) G_{KLs}(-k_e, -k_\nu) \\ - j_{k_\nu}(p_\nu r) G_{KLs}(-k_e, k_\nu)] + \frac{r}{R} d_{k_e}(r)[j_{k_\nu - 1}(p_\nu r) G_{KLs}(k_e, -k_\nu) \\ - j_{k_\nu}(p_\nu r) G_{KLs}(k_e, k_\nu)]\} \end{aligned} \tag{6.155}$$

where $k_\nu = |\kappa_\nu|$.

It should be noted that the quantities $m_K(k_e, k_\nu)$ are obtained from the quantities $M_K(k_e, k_\nu)$ simply by replacing $H_{k_e}(r)$ by $h_{k_e}(r)$ and $D_{k_e}(r)$ by $d_{k_e}(r)$. The importance of eqn (6.151) lies in the fact that the dependence on the signs of κ_e and κ_ν is now explicitly given.

This has the consequence and the advantage that the summation over

κ_e and κ_ν can be carried out much more easily when we are calculating the formulae for the observables in the next section. At the end the observables are expressed in terms of $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ which are directly related to the nuclear form factors by means of the above equations (see Bühring 1963*a* and *b*).

For further treatment we now have two possibilities. We can either try to take numerical solutions of the electron radial wave equations and, therefore, handle the further calculation of the $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ numerically (for an example see Koshigiri *et al.* 1979), or we can try to obtain some analytical solutions of the electron radial wave function, at least as far as the r-dependence is concerned, and come to some analytic expressions. The latter method has the advantage that Coulomb and non-Coulomb terms can be clearly separated and the energy dependence from electron and neutrino energy is explicitly given. It will, therefore, be applied in our following discussion. The usual analytic treatment of the electron radial wave function is an expansion of these functions as discussed extensively in Section 4.3.4.

In the case of $Z=0$, i.e. no distortion of the electron radial wave functions by the nuclear charge, the Coulomb amplitudes α_{κ_e} and the expansion of the functions $H_{k_e}(r)$, $h_{k_e}(r)$, $D_{k_e}(r)$ and $d_{k_e}(r)$ are given by (see eqns (6.73) and (6.114*a*, and *b*))

$$\alpha_{-k_e} = \sqrt{\frac{(W_e + m_e)}{W_e}}\, p_e \tag{6.156a}$$

$$\alpha_{+k_e} = \sqrt{\frac{(W_e - m_e)}{W_e}}\, p_e \tag{6.156b}$$

and

$$H_{k_e}(r) = \sum_{n=0}^{\infty} (-1)^n \frac{(2k_e-1)!!\,(p_e r)^{2n}}{(2n)!!\,(2n+2k_e-1)!!} \tag{6.157a}$$

$$h_{k_e}(r) = 0 \tag{6.157b}$$

$$D_{k_e}(r) = \frac{W_e R}{2k_e+1} \sum_{n=0}^{\infty} (-1)^n \frac{(2k_e+1)!!\,(p_e r)^{2n}}{(2n)!!\,(2n+2k_e+1)!!} \tag{6.157c}$$

$$d_{k_e}(r) = \frac{m_e R}{2k_e+1} \sum_{n=0}^{\infty} (-1)^n \frac{(2k_e+1)!!\,(p_e r)^{2n}}{(2n)!!\,(2n+2k_e+1)!!}\,. \tag{6.157d}$$

In the case of $Z \neq 0$, i.e. when the influence of the nuclear charge on the electron radial wave functions is taken into account, we have (see the extensive discussion in Section 4.3.4)

$$H_{k_e}(r) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \sum_{\rho=0}^{2\nu} \frac{(2k_e-1)!!}{(2\mu)!!\,(2\mu+2k_e-1)!!} (-1)^\nu \binom{\mu}{\nu} \binom{2\nu}{\rho} \left(\frac{r}{R}\right)^{2\mu} \times I(k_e, 2\mu, 2\nu, \rho; r)(m_e R)^{2\mu-2\nu}(W_e R)^{2\nu-\rho}(\alpha Z)^\rho \tag{6.158a}$$

$$h_{k_e}(r) = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\mu} \sum_{\rho=1}^{2\nu-1} \frac{(2k_e-1)!!}{(2\mu)!!\,(2\mu+2k_e-1)!!} (-1)^{\nu} \binom{\mu}{\nu} \binom{2\nu-1}{\rho} \left(\frac{r}{R}\right)^{2\mu}$$
$$\times I(k_e, 2\mu, 2\nu-1, \rho; r)(m_e R)^{2\mu-2\nu+1}(W_e R)^{2\nu-1-\rho}(\alpha Z)^{\rho} \quad (6.158b)$$

$$D_{k_e}(r) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \sum_{\rho=0}^{2\nu+1} \frac{(2k_e-1)!!}{(2\mu)!!\,(2\mu+2k_e+1)!!} (-1)^{\nu} \binom{\mu}{\nu} \binom{2\nu+1}{\rho} \left(\frac{r}{R}\right)^{2\mu}$$
$$\times I(k_e, 2\mu+1, 2\nu+1, \rho; r)(m_e R)^{2\mu-2\nu}(WR)^{2\nu+1-\rho}(\alpha Z)^{\rho} \quad (6.158c)$$

$$d_{k_e}(r) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \sum_{\rho=0}^{2\nu} \frac{(2k_e-1)!!}{(2\mu)!!\,(2\mu+2k_e+1)!!} (-1)^{\nu} \binom{\mu}{\nu} \binom{2\nu}{\rho} \left(\frac{r}{R}\right)^{2\mu}$$
$$\times I(k_e, 2\mu+1, 2\nu, \rho; r)(m_e R)^{2\mu+1-2\nu}(WR)^{2\nu-\rho}(\alpha Z)^{\rho}. \quad (6.158d)$$

In this case the Coulomb amplitudes can only be calculated numerically as discussed elsewhere in this book (see Section 3.2); of course, for $Z=0$ the eqns (6.158) are identical with eqns (6.157).

By inserting the above eqns (6.157) or (6.158) in eqn (6.154) or in eqn (6.155) and expanding the neutrino Bessel-functions $j_{k_\nu-1}(p_\nu r)$ and $j_{k_\nu}(p_\nu r)$ in powers of r as given in eqn (6.73) we obtain integrals of the form†

$$F^{(N)}_{KLs}(k_e, m, n, \rho) = \int_0^{\infty} J(q) F_{KLs}(q^2) q^2 \, dq \quad (6.159)$$

where

$$J(q) = \frac{2}{\pi} \frac{(qR)^L}{(2L+1)!!} \int_0^{\infty} \left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r) j_L(qr) r^2 \, dr. \quad (6.160)$$

The $F^{(N)}_{KLs}(k_e, m, n, \rho)$ are called form factor coefficients.

Equation (6.160) can also be written as

$$J(q) = (-1)^{L+N} \frac{1}{(2L+1)!!} q^{2L} \left(\frac{1}{q}\frac{d}{dq}\right)^L \frac{1}{q} \left(\frac{1}{R}\frac{d}{dq}\right)^{2N} \tilde{J}(q) \quad (6.161)$$

where

$$\tilde{J}(q) = \frac{2}{\pi} q \int_0^{\infty} I(k_e, m, n, \rho; r) j_0(qr) r^2 \, dr \quad (6.162)$$

† It should be kept in mind that for harmonic oscillator wave functions the q^2-dependence of the form factors $F_{KLs}(q^2)$ can generally be expressed as (see eqn (8.40))

$$F_{KLs}(q^2) = C e^{-(bq/2)^2} \sum_n a_n (bq)^n$$

where b is the oscillator parameter (see eqn (8.31)). The form factors $F_{KLs}(q^2)$ decrease therefore very fast with increasing q^2.

is something like the Hankel transform of order $\frac{1}{2}$ of the function $rI(k_e, m, n, \rho; r)$.

Then one obtains by repeated partial integration

$$F^{(N)}_{KLs}(k_e, m, n, \rho) = \int_0^\infty q\tilde{J}(q)E(q^2)\,dq \tag{6.163}$$

with

$$E(q^2) = (-1)^N \frac{1}{q}\left(\frac{1}{R}\frac{d}{dq}\right)^{2N}\left(\frac{1}{q}\frac{d}{dq}\right)^{L}\left\{\frac{q^{2L+1}}{(2L+1)!!}F_{KLs}(q^2)\right\}. \tag{6.164}$$

Here use has been made of the fact that the boundary terms vanish if the form factors $F_{KLs}(q^2)$ are well-behaved such that the integral (6.159) exists.

In the case of $Z = 0$ we have $I(k_e, m, n, 0; r) = 1$ (see Section 4.3.4) and therefore

$$\begin{aligned}\tilde{J}(q) &= \frac{2}{\pi}q\int_0^\infty j_0(qr)r^2\,dr \\ &= \frac{i}{\pi}\int_{-\infty}^{+\infty} e^{-iqr}r\,dr = -2\delta'(q) \\ &= \frac{2}{q}\delta(q)\end{aligned} \tag{6.165}$$

where $\delta(q)$ and $\delta'(q)$ are the Dirac delta function and its first derivative, respectively.

Then the integral (6.163) can be carried out and we get

$$\begin{aligned}F^{(N)}_{KLs} &= F^{(N)}_{KLs}(k_e, m, n, 0) = E(0) \\ &= \frac{(-1)^N(2N+2L+1)!!\,(2N)!!}{R^{2N}(2L+1)!!\,N!}\left(\frac{d}{dq^2}\right)^N F_{KLs}(q^2)\Big|_{q^2=0}.\end{aligned} \tag{6.166}$$

The form factor coefficients $F^{(N)}_{KLs}$ are, by the way, identical with those in the expansion of eqn (6.74). Thus an expansion of this type, which we have assumed to be reasonable in the foregoing section, follows from our treatment of the lepton current in a natural way.

Let us now combine the foregoing results. Then we obtain in the case $Z = 0$, i.e. in the case of no Coulomb final state interaction

$$\begin{aligned}M_K(k_e, k_\nu) = \frac{1}{\sqrt{(2K+1)}}&\frac{(p_eR)^{k_e-1}}{(2k_e-1)!!}\frac{(p_\nu R)^{k_\nu-1}}{(2k_\nu-1)!!}\sum_{Ls}(-1)^{K-L}\sum_{\mu=0}^{\infty}\sum_{\lambda=0}^{\infty} \\ &\times(-1)^{\mu+\lambda}\frac{(2k_e-1)!!\,(p_eR)^{2\mu}}{(2\mu)!!\,(2\mu+2k_e-1)!!}\frac{(2k_\nu-1)!!\,(p_\nu R)^{2\lambda}}{(2\lambda)!!\,(2\lambda+2k_\nu-1)!!}\end{aligned}$$

$$\times\left\{G_{KLs}(-k_e,-k_\nu)F^{(N)}_{KLs}-\frac{W_eRp_\nu R}{(2\mu+2k_e+1)(2\lambda+2k_\nu+1)}\right.$$
$$\times G_{KLs}(k_e,k_\nu)F^{(N+1)}_{KLs}+\frac{W_eR}{(2\mu+2k_e+1)}G_{KLs}(k_e,-k_\nu)F^{(\tilde N)}_{KLs}$$
$$\left.-\frac{p_\nu R}{(2\lambda+2k_\nu+1)}G_{KLs}(-k_e,k_\nu)F^{(\tilde N)}_{KLs}\right\} \tag{6.167}$$

and

$$m_K(k_e,k_\nu)=\frac{1}{\sqrt{(2K+1)}}\frac{(p_eR)^{k_e-1}}{(2k_e-1)!!}\frac{(p_\nu R)^{k_\nu-1}}{(2k_\nu-1)!!}\sum_{Ls}(-1)^{K-L}\sum_{\mu=0}^{\infty}\sum_{\lambda=0}^{\infty}(-1)^{\mu+\lambda}$$
$$\times\frac{(2k_e-1)!!\,(p_eR)^{2\mu}}{(2\mu)!!(2\mu+2k_e-1)!!}\frac{(2k_\nu-1)!!\,(p_\nu R)^{2\lambda}}{(2\lambda)!!\,(2\lambda+2k_\nu-1)!!}$$
$$\times\left\{-\frac{m_eRp_\nu R}{(2\mu+2k_e+1)(2\lambda+2k_\nu+1)}G_{KLs}(k_e,k_\nu)F^{(N+1)}_{KLs}\right.$$
$$\left.+\frac{m_eR}{(2\mu+2k_e+1)}G_{KLs}(k_e,-k_\nu)F^{(\tilde N)}_{KLs}\right\} \tag{6.168}$$

where

$$N=\mu+\lambda-1+\tfrac{1}{2}(k_e+k_\nu-L) \tag{6.169}$$
$$\tilde N=\mu+\lambda-1+\tfrac{1}{2}(k_e+k_\nu+1-L). \tag{6.170}$$

N and $\tilde N$ are always integers. This is guaranteed by the behaviour of the coefficients $G_{KLs}(k_e,k_\nu)$ which take care that only the corresponding two (one) of the four (two) terms survive.

The form factor coefficients $F^{(N)}_{KLs}$ means either ${}^{V}F^{(N)}_{KLs}$ or ${}^{A}F^{(N)}_{KLs}$ depending on the parity selection rule (see, for example, eqn (7.158)).

We now want to go from the case of $Z=0$ to the more general case where the distortion of the electron radial wave functions by the nuclear charge Z is taken into account. To that aim we give also the complete expressions for $M_K(k_e,k_\nu)$ and $m_K(k_e,k_\nu)$ for $Z\neq 0$. From the formulae given above we get (Behrens and Bühring 1971):

$$M_K(k_e,k_\nu)=\frac{1}{\sqrt{(2K+1)}}\frac{(p_eR)^{k_e-1}}{(2k_e-1)!!}\frac{(p_\nu R)^{k_\nu-1}}{(2k_\nu-1)!!}\sum_{\lambda=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{(2k_e-1)!!}{(2\mu)!!\,(2\mu+2k_e-1)!!}$$
$$\times\frac{(2k_\nu-1)!!\,(p_\nu R)^{2\lambda}}{(2\lambda)!!\,(2\lambda+2k_\nu-1)!!} \tag{6.171}$$

$$\times\sum_{\sigma=0}^{\mu}(-1)^{\lambda+\sigma}\binom{\mu}{\sigma}\sum_{\rho=0}^{2\sigma+1}(m_eR)^{2\mu-2\sigma}(W_eR)^{2\sigma-\rho}(\alpha Z)^{\rho}\sum_{Ls}(-1)^{K-L}$$
$$\times\left\{\binom{2\sigma}{\rho}G_{KLs}(-k_e,-k_\nu)F^{(N)}_{KLs}(k_e,2\mu,2\sigma,\rho)-\frac{p_\nu R}{2\lambda+2k_\nu+1}\frac{W_eR}{2\mu+2k_e+1}\right.$$

$$\times \binom{2\sigma+1}{\rho} G_{KLs}(k_e, k_\nu) F^{(N+1)}_{KLs}(k_e, 2\mu+1, 2\sigma+1, \rho)$$

$$+ \frac{W_e R}{2\mu+2k_e+1} \binom{2\sigma+1}{\rho} G_{KLs}(k_e, -k_\nu) F^{(\tilde{N})}_{KLs}(k_e, 2\mu+1, 2\sigma+1, \rho)$$

$$- \frac{p_\nu R}{2\lambda+2k_\nu+1} \binom{2\sigma}{\rho} G_{KLs}(-k_e, k_\nu) F^{(\tilde{N})}_{KLs}(k_e, 2\mu, 2\sigma, \rho) \Big\}$$

and

$$m_K(k_e, k_\nu) = \frac{1}{\sqrt{(2K+1)}} \frac{(p_e R)^{k_e-1}}{(2k_e-1)!!} \frac{(p_\nu R)^{k_\nu-1}}{(2k_\nu-1)!!} \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{(2k_e-1)!!}{(2\mu)!!\,(2\mu+2k_e-1)!!}$$

$$\times \frac{(2k_\nu-1)!!\,(p_\nu R)^{2\lambda}}{(2\lambda)!!\,(2\lambda+2k_\nu-1)!!} \sum_{\sigma=0}^{\mu} (-1)^{\lambda+\sigma} \binom{\mu}{\sigma} \sum_{\rho=0}^{2\sigma} (m_e R)^{2\mu-2\sigma} (W_e R)^{2\sigma-\rho}$$

$$\times (\alpha Z)^\rho \sum_{Ls} (-1)^{K-L} \Big\{ m_e R (W_e R)^{-1} \Delta \binom{2\sigma-1}{\rho} G_{KLs}(-k_e, -k_\nu)$$

$$\times F^{(N)}_{KLs}(k_e, 2\mu, 2\sigma-1, \rho) - \frac{p_\nu R}{2\lambda+2k_\nu+1} \frac{m_e R}{2\mu+2k_e+1} \binom{2\sigma}{\rho} G_{KLs}(k_e, k_\nu)$$

$$\times F^{(N+1)}_{KLs}(k_e, 2\mu+1, 2\sigma, \rho) + \frac{m_e R}{2\mu+2k_e+1} \binom{2\sigma}{\rho} G_{KLs}(k_e, -k_\nu)$$

$$\times F^{(\tilde{N})}_{KLs}(k_e, 2\mu+1, 2\sigma, \rho) - \frac{p_\nu R}{2\lambda+2k_\nu+1} m_e R (W_e R)^{-1} \Delta \binom{2\sigma-1}{\rho}$$

$$\times G_{KLs}(-k_e, k_\nu) F^{(\tilde{N})}_{KLs}(k_e, 2\mu, 2\sigma-1, \rho) \Big\} \tag{6.172}$$

where

$$\Delta = \begin{cases} 0 & \text{if} \quad \mu = 0 \\ 0 & \text{if} \quad \sigma = 0 \\ 0 & \text{if} \quad \rho = 0 \\ 1 & \text{elsewhere.} \end{cases} \tag{6.173}$$

Here the symbol Δ has been introduced to account for the fact that the sums over μ, σ, ρ in $h_{k_e}(r)$ start with 1 rather than 0. The sum over ρ is automatically made to terminate at the appropriate value of ρ by the binomial coefficients, which vanish if ρ is too large.

Of course, for $Z = 0$ the eqns (6.171) and (6.172) are identical with the eqns (6.167) and (6.168). The main difference between the case where $Z = 0$ and $Z \neq 0$ is the occurrence of additional form factor coefficients $F^{(N)}_{KLs}(k_e, m, n, \rho)$ corresponding to operators which have an additional r-dependence given by the various functions $I(k_e, m, n, \rho; r)$ associated with the terms containing powers of αZ. It should also be noted that the shape of the nuclear charge distribution is of some relevance for this latter type of form factor coefficients since the functions $I(k_e, m, n, \rho; r)$

are sensitive to the shape of the charge distribution (see the discussion in Section 4.4.1).

From the eqns (6.167), (6.168), (6.171) and (6.172) we see also that the relative order of magnitude of $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ is different ($m_eR < W_eR$; for $Z > 3$ we have $m_eR < \alpha Z$). $M_K(k_e, k_\nu)$ is, usually, greater than $m_K(k_e, k_\nu)$ by a factor of at least 2 to 10. This has the consequence that in many cases, as we will see later on, the terms connected with $m_K(k_e, k_\nu)$ can be neglected. This fact is also indicated by notation of these quantities by a capital and a small letter.

The introduction of the expansion, eqns (6.157*a–d*) or (6.158*a–d*) into the eqns (6.154) and (6.155) raises, however, the question of the convergence. In our treatment we have interchanged summation and integration. This procedure is, on the other hand, not automatically allowed if the upper limit of the integral is infinity, as it is in our case. We have, therefore, to be careful and to consider the whole expansion in detail. Such a detailed discussion of this problem is, however, given elsewhere in this book (see Section 4.3.3).

It is shown that in our case the expansion converges and, as a consequence, the usefulness of this method is preserved. By the way that is also the reason why we are not allowed to introduce an expansion of the electron radial wave functions in powers of r for the case $Z \neq 0$. Doing so we would have serious convergence problems (see Section 4.3.3).

It should, however, be mentioned that this latter method, which is not exact in a strong mathematical sense, has been applied in many older publications (see, for example, Bühring 1963; Stech and Schülke 1964; Schopper 1966). Its validity is restricted to the uniform charge distribution of the nucleus and a nuclear model such that the nucleon radial wave functions vanish identically outside the nuclear radius R.

Finally, the T-matrix for β^--decay can now be expressed in terms of $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ as follows:

$$T_{\beta^-} = \frac{G_\beta}{4\pi} \sum_{KM} \sum_{\substack{\kappa_e \mu_e \\ \kappa_\nu \mu_\nu}} (-1)^{J_f - M_f + K + M + j_e - \mu_e + \mu_\nu - l_\nu + \frac{1}{2}}$$
$$\times \sqrt{\{(2J_i + 1)(2K + 1)\}} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \begin{pmatrix} j_e & K & j_\nu \\ -\mu_e & -M & -\mu_\nu \end{pmatrix}$$
$$\times a^*_{\kappa_e \mu_e} b^*_{\kappa_\nu \mu_\nu} \alpha_{\kappa_e} \{M_K(k_e, k_\nu) + \mathrm{sign}(\kappa_e) m_K(k_e, k_\nu)\}. \qquad (6.174)$$

Here use has been made of the relation

$$\mathrm{sign}(\kappa_\nu) = (-1)^{j_\nu - l_\nu + \frac{1}{2}}.$$

7

OBSERVABLES

7.1. Density matrix

THIS SECTION is devoted to a more detailed derivation of formulae for the transition probability. We have seen in Section 5.3 that the transition probability is essentially determined (apart from some kinematical factors) by the square of the T-matrix

$$|T|^2 = T^+T = T^*_{fi}T_{fi}.$$

For a further discussion we have therefore to evaluate this quantity in more detail.

Equation (6.174) involves the expression

$$T^+T = \frac{G_\beta^2}{16\pi^2}\sum_{KM}\sum_{K'M'}\sum_{\substack{\kappa_e\mu_e\\ \kappa'_e\mu'_e}}\sum_{\substack{\kappa_\nu\mu_\nu\\ \kappa'_\nu\mu'_\nu}}(-1)^{2J_f-M_f-M'_f}$$
$$\times(-1)^{K+K'+M+M'+j_e+j'_e-\mu_e-\mu'_e+\mu_\nu+\mu'_\nu-l_\nu-l'_\nu+1}(2J_i+1)\sqrt{\{(2K+1)(2K'+1)\}} \quad (7.1)$$
$$\begin{pmatrix} J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix} J_f & K' & J_i\\ -M'_f & M' & M'_i\end{pmatrix}\begin{pmatrix} j_e & K & j_\nu\\ -\mu_e & -M & -\mu_\nu\end{pmatrix}\begin{pmatrix} j'_e & K' & j'_\nu\\ -\mu'_e & -M' & -\mu'_\nu\end{pmatrix}$$
$$a^*_{\kappa_e\mu_e}a_{\kappa_e\mu_e}b^*_{\kappa_\nu\mu_\nu}b_{\kappa_\nu\mu_\nu}\,\alpha_{\kappa_e}\alpha_{\kappa'_e}\{M_K(k_e,k_\nu)M_{K'}(k'_e,k'_\nu)$$
$$+\mathrm{sign}(\kappa_e)\mathrm{sign}(\kappa'_e)m_K(k_e,k_\nu)m_{K'}(k'_e,k'_\nu)$$
$$+\mathrm{sign}(\kappa_e)M_{K'}(k'_e,k'_\nu)m_K(k_e,k_\nu)$$
$$+\mathrm{sign}(\kappa'_e)M_K(k_e,k_\nu)m_{K'}(k'_e,k'_\nu)\}.$$

The next step is to evaluate the quantities $a^*_{\kappa_e\mu_e}a_{\kappa'_e\mu'_e}$ and $b^*_{\kappa_\nu\mu_\nu}b_{\kappa'_\nu\mu'_\nu}$.

From eqn (6.115) we derive

$$a^*_{\kappa_e\mu_e}a_{\kappa'_e\mu'_e} = \frac{8\pi^2}{p_e^2}\sqrt{\{(2j_e+1)(2j'_e+1)\}}(-1)^{l_e+l'_e+\mu_e+\mu'_e-1}$$
$$\times\begin{pmatrix} l_e & \frac{1}{2} & j_e\\ \mu_e-m_e & m_e & -\mu_e\end{pmatrix}\begin{pmatrix} l'_e & \frac{1}{2} & j'_e\\ \mu'_e-m'_e & m'_e & -\mu'_e\end{pmatrix}$$
$$\times Y^*_{l'_e}{}^{\mu'_e-m'_e}(\hat{p}_e)\,Y_{l_e}{}^{\mu_e-m_e}(\hat{p}_e)e^{-i(\Delta_{\kappa_e}-\Delta_{\kappa'_e})}. \quad (7.2)$$

The product of the two spherical harmonics can be further simplified by

applying the relation (see, for example, Edmonds 1964)

$$\overset{*}{Y}{}^{\mu_e'-m_e'}_{l_e'}Y^{\mu_e-m_e}_{l_e} = (-1)^{\mu_e'-m_e'}Y^{m_e'-\mu_e'}_{l_e'}(\hat{p}_e)\,Y^{\mu_e-m_e}_{l_e}(\hat{p}_e)$$
$$= (-1)^{\mu_e'-m_e'}\sum_{km}\sqrt{\left\{\frac{(2l_e+1)(2l_e'+1)(2k+1)}{4\pi}\right\}}$$
$$\times\begin{pmatrix} l_e & l_e' & k \\ \mu_e-m_e & m_e'-\mu_e' & m\end{pmatrix}\begin{pmatrix} l_e & l_e' & k \\ 0 & 0 & 0\end{pmatrix}\overset{*}{Y}{}^{m}_{k}(\hat{p}_e). \quad (7.3)$$

Thus $a^*_{\kappa_e\mu_e}a_{\kappa_e'\mu_e'}$ becomes

$$a^*_{\kappa_e\mu_e}a_{\kappa_e'\mu_e'} = \frac{4\pi^{3/2}}{p_e^2}(-1)^{l_e+l_e'+\mu_e+m_e'-1}$$
$$\times\sum_{km}\sqrt{\{(2j_e+1)(2j_e'+1)(2l_e+1)(2l_e'+1)(2k+1)\}}\begin{pmatrix} l_e & l_e' & k \\ \mu_e-m_e & m_e'-\mu_e' & m\end{pmatrix}$$
$$\times\begin{pmatrix} l_e & l_e' & k \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_e & \frac{1}{2} & j_e \\ \mu_e-m_e & m_e & -\mu_e\end{pmatrix}\begin{pmatrix} l_e' & \frac{1}{2} & j_e' \\ \mu_e'-m_e' & m_e' & -\mu_e'\end{pmatrix}\overset{*}{Y}{}^{m}_{k}(\hat{p}_e)e^{-i(\Delta_{\kappa_e}-\Delta_{\kappa_e'})}. \quad (7.4)$$

Similarly we obtain for $b^*_{\kappa_\nu\mu_\nu}b_{\kappa_\nu'\mu_\nu'}$ (see eqn (6.117))

$$b^*_{\kappa_\nu\mu_\nu}b_{\kappa_\nu'\mu_\nu'} = 4\pi^{3/2}(-1)^{l_\nu+l_\nu'+\mu_\nu+m_\nu'-1}$$
$$\sum_{k'm'}\sqrt{\{(2j_\nu+1)(2j_\nu'+1)(2l_\nu+1)(2l_\nu'+1)(2k'+1)\}}\begin{pmatrix} l_\nu & l_\nu' & k' \\ \mu_\nu-m_\nu & m_\nu'-\mu_\nu' & m'\end{pmatrix}$$
$$\times\begin{pmatrix} l_\nu & l_\nu' & k' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_\nu & \frac{1}{2} & j_\nu \\ \mu_\nu-m_\nu & m_\nu & -\mu_\nu\end{pmatrix}\begin{pmatrix} l_\nu' & \frac{1}{2} & j_\nu' \\ \mu_\nu'-m_\nu' & m_\nu' & -\mu_\nu'\end{pmatrix}\overset{*}{Y}{}^{m'}_{k'}(\hat{p}_\nu). \quad (7.5)$$

Combining the above equations we find finally

$$T^+T = \frac{G_\beta^2\pi}{p_e^2}\sum_{KM}\sum_{K'M'}\sum_{\substack{\kappa_e\mu_e\\ \kappa_e'\mu_e'}}\sum_{\substack{\kappa_\nu\mu_\nu\\ \kappa_\nu'\mu_\nu'}}\sum_{km}\sum_{k'm'}$$
$$\times(-1)^{2J_f-M_f-M_f'+K+K'+M+M'+j_e+j_e'+l_e+l_e'-1+m_e'+m_\nu'-\mu_e'-\mu_\nu'}(2J_i+1)$$
$$\times\sqrt{\{(2K+1)(2K'+1)(2j_e+1)(2j_e'+1)(2j_\nu+1)(2j_\nu'+1)}$$
$$\times(2l_e+1)(2l_e'+1)(2l_\nu+1)(2l_\nu'+1)(2k+1)(2k'+1)\} \quad (7.6)$$
$$\times\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix} J_f & K' & J_i \\ -M_f' & M' & M_i'\end{pmatrix}\begin{pmatrix} j_e & K & j_\nu \\ -\mu_e & -M & -\mu_\nu\end{pmatrix}$$
$$\times\begin{pmatrix} j_e' & K' & j_\nu' \\ -\mu_e' & -M' & -\mu_\nu'\end{pmatrix}\begin{pmatrix} l_e & \frac{1}{2} & j_e \\ \mu_e-m_e & m_e & -\mu_e\end{pmatrix}\begin{pmatrix} l_e' & \frac{1}{2} & j_e' \\ \mu_e'-m_e' & m_e' & -\mu_e'\end{pmatrix}$$
$$\times\begin{pmatrix} l_e & l_e' & k \\ \mu_e-m_e & m_e'-\mu_e' & m\end{pmatrix}\begin{pmatrix} l_e & l_e' & k \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_\nu & \frac{1}{2} & j_\nu \\ \mu_\nu-m_\nu & m_\nu & -\mu_\nu\end{pmatrix}$$
$$\times\begin{pmatrix} l_\nu' & \frac{1}{2} & j_\nu' \\ \mu_\nu'-m_\nu' & m_\nu' & -\mu_\nu'\end{pmatrix}\begin{pmatrix} l_\nu & l_\nu' & k' \\ \mu_\nu-m_\nu & m_\nu'-\mu_\nu' & m'\end{pmatrix}\begin{pmatrix} l_\nu & l_\nu' & k' \\ 0 & 0 & 0\end{pmatrix}$$

$$\times \overset{*}{Y}{}_k^m(\hat{p}_e)\overset{*}{Y}{}_{k'}^{m'}(\hat{p}_\nu)\alpha_{\kappa_e}\alpha_{\kappa_e'}\{M_K(k_e, k_\nu)M_{K'}(k_e', k_\nu') + \mathrm{sign}(\kappa_e)\mathrm{sign}(\kappa_e')$$
$$\times m_K(k_e, k_\nu)m_{K'}(k_e', k_\nu') + \mathrm{sign}(\kappa_e)M_{K'}(k_e', k_\nu')m_K(k_e, k_\nu) + \mathrm{sign}(\kappa_e')$$
$$\times M_K(k_e, k_\nu)m_{K'}(k_e', k_\nu')\}e^{-i(\Delta_{\kappa_e}-\Delta_{\kappa_e'})}.$$

T^+T is, however, related to the density matrix for beta-decay. It is a function of the directions of beta-particle and neutrino, and is a matrix with respect to each of the magnetic quantum numbers of the particles involved in final and initial states.

We denote this density matrix by†

$$\rho_\beta(M_i\ M_i'\ M_f\ M_f'\ m_e\ m_e'\ m_\nu\ m_\nu') = T(M_i\ M_f\ m_e\ m_\nu)\,T^*(M_i'\ M_f'\ m_e'\ m_\nu'). \quad (7.7)$$

For the more explicit calculation of special observables we have now to take the trace with respect to the unobserved magnetic quantum states and to integrate over unobserved particle directions (under consideration of the kinematics as shown in Section 5.3). In the following sections formulae for special observables will be derived in this way. In addition it should be mentioned that detailed examples of the application of the density matrix formalism for the calculation of observables in radioactive decays are, for example, given in the reviews by Frauenfelder and Steffen (1965) and by Biedenharn and Rose (1953).

Let us now turn to a more detailed discussion of special observables in the following sections.

7.2. Polarization of electron and neutrino not observed

7.2.1. Electron and neutrino observed, general density matrix

In the case in which we do not observe the polarization of the emitted beta-particle and neutrino, we have to sum over $m_e = m_e'$ and $m_\nu = m_\nu'$, i.e.

$$\rho_\beta(M_i, M_i', M_f, M_f') = \sum_{\substack{m_e\ m_\nu \\ m_e'\ m_\nu'}} \sum \rho_\beta(M_i, M_i', M_f, M_f', m_e, m_e', m_\nu, m_\nu')\delta_{m_e m_e'}\delta_{m_\nu m_\nu'}. \quad (7.8)$$

† Exactly the final state density matrix is given by

$$\rho_\beta(M_f\ M_f'\ m_e\ m_e'\ m_\nu\ m_\nu') = \sum_{M_i M_i'} T(M_i\ M_f\ m_e\ m_\nu)T^*(M_i'\ M_f'\ m_e'\ m_\nu')\rho_i(M_i\ M_i')$$

where ρ_i is the density matrix of the initial state. For unpolarized nuclei we have

$$\rho_i(M_i\ M_i') = \delta_{M_i M_i'}$$

and for (in z-direction) polarized nuclei

$$\rho_i(M_i M_i') = a_{M_i}\delta_{M_i M_i'}$$

where the a_{M_i} are the relative population numbers.

For the explicit summation we first collect those factors in the density matrix that contain the magnetic quantum numbers m_e for the electron and those that contain the magnetic quantum numbers m_ν for the neutrino. Then we can make use of a relation between $3j$-symbols and a $6j$-symbol (see Edmonds 1964).

$$\sum_{m_e}(-1)^{l_e+l_e'+\frac{1}{2}+\mu_e+\mu_e'+m_e}\begin{pmatrix} j_e' & l_e' & \frac{1}{2} \\ -\mu_e' & \mu_e'-m_e & m_e\end{pmatrix}\begin{pmatrix} l_e & j_e & \frac{1}{2} \\ m_e-\mu_e & \mu_e & -m_e\end{pmatrix}$$
$$\times\begin{pmatrix} l_e & l_e' & k \\ \mu_e-m_e & m_e-\mu_e' & m\end{pmatrix}=\begin{pmatrix} j_e' & j_e & k \\ -\mu_e' & \mu_e & m\end{pmatrix}\underbrace{\begin{Bmatrix} j_e' & j_e & k \\ l_e & l_e' & \frac{1}{2}\end{Bmatrix}}_{\begin{Bmatrix} l_e & l_e' & k \\ j_e' & j_e & \frac{1}{2}\end{Bmatrix}} \tag{7.9}$$

and

$$\sum_{m_\nu}(-1)^{l_\nu+l_\nu'+\frac{1}{2}+\mu_\nu+\mu_\nu'+m_\nu}\begin{pmatrix} j_\nu' & l_\nu' & \frac{1}{2} \\ -\mu_\nu' & \mu_\nu'-m_\nu & m_\nu\end{pmatrix}\begin{pmatrix} l_\nu & j_\nu & \frac{1}{2} \\ m_\nu-\mu_\nu & \mu_\nu & -m_\nu\end{pmatrix}$$
$$\times\begin{pmatrix} l_\nu & l_\nu' & k' \\ \mu_\nu-m_\nu & m_\nu-\mu_\nu' & m'\end{pmatrix}=\begin{pmatrix} j_\nu' & j_\nu & k' \\ -\mu_\nu' & \mu_\nu & m'\end{pmatrix}\underbrace{\begin{Bmatrix} j_\nu' & j_\nu & k' \\ l_\nu & l_\nu' & \frac{1}{2}\end{Bmatrix}}_{\begin{Bmatrix} l_\nu & l_\nu' & k' \\ j_\nu' & j_\nu & \frac{1}{2}\end{Bmatrix}} \tag{7.10}$$

However, before we are able to use the above relations we have to permutate some columns and to interchange all m by $-m$ in certain $3j$-symbols, but these procedures can easily be carried out taking into account the simple symmetry properties of the $3j$-symbols (Edmonds 1964). Properties of permutation of rows and columns and of interchange of some arguments will, of course, be extensively utilized in the following too, and also for the $6j$-symbols and $9j$-symbols.

By using the above formulae we have now carried out the summation over m_e', m_e, m_ν and m_ν'.

The next step is to perform the sum over the magnetic quantum numbers μ_e', μ_e, μ_ν' and μ_ν. For that purpose we use a relation between $3j$-symbols and a $9j$-symbol (see, for example, de Shalit and Talmi 1963).

$$\sum_{\substack{\mu_e\mu_e' \\ \mu_\nu\mu_\nu'}}\begin{pmatrix} k & j_e' & j_e \\ m & -\mu_e' & \mu_e\end{pmatrix}\begin{pmatrix} k' & j_\nu' & j_\nu \\ m' & -\mu_\nu' & \mu_\nu\end{pmatrix}\begin{pmatrix} j_\nu' & K' & j_e' \\ -\mu_\nu' & -M' & -\mu_e'\end{pmatrix}\begin{pmatrix} j_\nu & K & j_e \\ \mu_\nu & M & \mu_e\end{pmatrix}$$
$$=\sum_{Ns}(2N+1)\begin{pmatrix} k' & N & k \\ m' & s & m\end{pmatrix}\begin{pmatrix} N & K' & K \\ s & -M' & M\end{pmatrix}\underbrace{\begin{Bmatrix} k' & N & k \\ j_\nu' & K' & j_e' \\ j_\nu & K & j_e\end{Bmatrix}}_{\begin{Bmatrix} k & N & k' \\ j_e & K & j_\nu \\ j_e' & K' & j_\nu'\end{Bmatrix}} \tag{7.11}$$

Collecting all the terms of the various summation processes we arrive at the following density matrix

$$\rho_\beta(M_f, M'_f, M_i, M'_i) = \frac{G_\beta^2 \pi}{p_e^2} \sum_{K'M'} \sum_{KM} \sum_{\substack{\kappa_e \\ \kappa'_e}} \sum_{\substack{\kappa_\nu \\ \kappa'_\nu}} \sum_{N} \sum_{s} \sum_{km} \sum_{k'm'}$$
$$\times (-1)^{2J_f - M_f - M'_f + K + K' + M' + j_e + j'_\nu + K' - l_\nu - l'_\nu + 1}$$
$$\times (2J_i + 1)\sqrt{\{(2K+1)(2K'+1)(2j_\nu+1)(2j'_\nu+1)(2j_e+1)(2j'_e+1)}$$
$$\times (2l_\nu+1)(2l'_\nu+1)(2l_e+1)(2l'_e+1)(2k+1)(2k'+1)\}$$
$$\times \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \begin{pmatrix} J_f & K' & J_i \\ -M'_f & M' & M'_i \end{pmatrix} \begin{pmatrix} N & K' & K \\ s & -M' & M \end{pmatrix} \begin{pmatrix} l_e & l'_e & k \\ 0 & 0 & 0 \end{pmatrix} \quad (7.12)$$
$$\times \begin{Bmatrix} l_e & l'_e & k \\ j'_e & j_e & \frac{1}{2} \end{Bmatrix} \begin{pmatrix} l_\nu & l'_\nu & k' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_\nu & l'_\nu & k' \\ j'_\nu & j_\nu & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} k & N & k' \\ j_e & K & j_\nu \\ j'_e & K' & j'_\nu \end{Bmatrix}$$
$$\times (2N+1) \begin{pmatrix} k' & N & k \\ m' & s & m \end{pmatrix} \overset{*}{Y}{}_k^m(\hat{p}_e) \overset{*}{Y}{}_{k'}^{m'}(\hat{p}_\nu) \alpha_{\kappa_e} \alpha_{\kappa'_e} \{M_K(k_e, k_\nu) M_{K'}(k'_e, k'_\nu)$$
$$+ \text{sign}(\kappa_e)\text{sign}(\kappa'_e) m_K(k_e, k_\nu) m_{K'}(k'_e, k'_\nu) + \text{sign}(\kappa_e) M_{K'}(k'_e, k'_\nu)$$
$$\times m_K(k_e, k_\nu) + \text{sign}(\kappa'_e) M_K(k_e, k_\nu) m_{K'}(k'_e, k'_\nu)\} e^{-i(\Delta_{\kappa_e} - \Delta_{\kappa'_e})}.$$

We now decompose the above equation into two factors; a factor which depends only on the beta-transition and a factor which contains the quantum numbers of the nuclear states. For that purpose we introduce the following parameters:

$$a^N_{kk'}(K, K') = \frac{1}{F_0 p_e^2} \sum_{\substack{\kappa_e \\ \kappa'_e}} \sum_{\substack{\kappa_\nu \\ \kappa'_\nu}} g^{kk'N}_{KK'}(\kappa_e, \kappa'_e, \kappa_\nu, \kappa'_\nu) \alpha_{\kappa_e} \alpha_{\kappa'_e}$$
$$\times \{M_K(k_e, k_\nu) M_{K'}(k'_e, k'_\nu) + \text{sign}(\kappa_e)\text{sign}(\kappa'_e) m_K(k_e, k_\nu) m_{K'}(k'_e, k'_\nu)$$
$$+ \text{sign}(\kappa_e) M_{K'}(k'_e, k'_\nu) m_K(k_e, k_\nu)$$
$$+ \text{sign}(\kappa'_e) M_K(k_e, k_\nu) m_{K'}(k'_e, k'_\nu)\} e^{-i(\Delta_{\kappa_e} - \Delta_{\kappa'_e})} \quad (7.13)$$

where

$$g^{kk'N}_{KK'}(\kappa_e, \kappa'_e, \kappa_\nu, \kappa'_\nu) = (-1)^{j_e + j'_\nu + K' - l_\nu - l'_\nu + N - 1}$$
$$\times \sqrt{\{(2K+1)(2K'+1)(2k+1)(2k'+1)(2N+1)(2j_e+1)(2j'_e+1)}$$
$$\times (2j_\nu+1)(2j'_\nu+1)(2l_e+1)(2l'_e+1)(2l_\nu+1)(2l'_\nu+1)\}$$
$$\times \begin{pmatrix} l_e & l'_e & k \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_e & l'_e & k \\ j'_e & j_e & \frac{1}{2} \end{Bmatrix} \begin{pmatrix} l_\nu & l'_\nu & k' \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \begin{Bmatrix} l_\nu & l'_\nu & k' \\ j'_\nu & j_\nu & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} k & N & k' \\ j_e & K & j_\nu \\ j'_e & K' & j'_\nu \end{Bmatrix}. \quad (7.14)$$

The parameters $a^N_{kk'}(K, K')$ are usually called particle parameters (see, for example, Fraunfelder and Steffen 1965). In our case, however, they are of a very general type.

The function F_0 has been† extracted from the particle parameters in order to account for the main influence of the nuclear charge on the electron radial wave equations (see the discussion in Section 4.2).

Equation (7.12) then reads as

$$\rho_\beta(M_f, M'_f, M_i, M'_i) = G_\beta^2 \pi F_0 \sum_{K'M'} \sum_{KM} \sum_{Ns} \sum_{km} \sum_{k'm'} (-1)^{2J_f - M_f - M'_f + N + K + K' + M'}$$
$$\times (2J_i + 1)\sqrt{\{2N+1\}} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \begin{pmatrix} J_f & K' & J_i \\ -M'_f & M' & M'_i \end{pmatrix} \begin{pmatrix} N & K' & K \\ s & -M' & M \end{pmatrix}$$
$$\times a^N_{kk'}(K, K') \begin{pmatrix} k' & N & k \\ m' & s & m \end{pmatrix} \overset{*}{Y}{}^m_k(\hat{p}_e) \overset{*}{Y}{}^{m'}_{k'}(\hat{p}_\nu). \tag{7.15}$$

To proceed further we now assume either the magnetic substates of the initial nucleus are observed only (for example, the initial nucleus is oriented) or the magnetic substates of the final nucleus are observed only (for example, by the detection of a following α- or γ-radiation).

In the first case we have to sum over $M_f = M'_f$, i.e.

$$\rho_\beta(M_i, M'_i) = \sum_{M_f M'_f} \rho_\beta(M_i, M'_i, M_f, M'_f) \delta_{M_f M'_f}. \tag{7.16}$$

Similarly as we have done in eqns (7.9) and (7.10) we can make use of the following relation:

$$\sum_{M_f M M'} (-1)^{K+K'+J_f-M'-M-M_f} \begin{pmatrix} J_i & K & J_f \\ -M_i & -M & M_f \end{pmatrix} \begin{pmatrix} K' & J_i & J_f \\ M' & M'_i & -M_f \end{pmatrix}$$
$$\times \begin{pmatrix} K' & K & N \\ -M' & M & s \end{pmatrix} = \begin{pmatrix} J_i & J_i & N \\ -M_i & M'_i & s \end{pmatrix} \begin{Bmatrix} J_i & J_i & N \\ K' & K & J_f \end{Bmatrix}. \tag{7.17}$$

Then we find

$$\rho_\beta(M_i, M'_i) = G_\beta^2 \pi F_0 \sum_{K'K} \sum_{Ns} \sum_{km} \sum_{k'm'} (-1)^{J_f - J_i - J_i + M_i} (-1)^{2J_f}$$
$$\times (2J_i + 1)\sqrt{(2N+1)} \begin{pmatrix} J_i & J_i & N \\ M_i & -M'_i & s \end{pmatrix} \begin{Bmatrix} J_i & J_i & N \\ K' & K & J_f \end{Bmatrix} a^N_{kk'}(K, K')$$
$$\times \begin{pmatrix} k' & N & k \\ m' & s & m \end{pmatrix} \overset{*}{Y}{}^m_k(\hat{p}_e) \overset{*}{Y}{}^{m'}_{k'}(\hat{p}_\nu). \tag{7.18}$$

† $$F_0 = 4(2p_e R)^{-2(1-\gamma_1)} e^{\pi y} \frac{|\Gamma(\gamma_1 + iy)|^2}{|\Gamma(2\gamma_1 + 1)|^2}$$

where

$$\gamma_k = \sqrt{\{k^2 - (\alpha Z)^2\}}$$
$$y = \alpha Z \frac{W_e}{p_e}.$$

Completely analogous, we obtain in the second case

$$\rho_\beta(M_f, M_f') = \sum_{M_i, M_i'} \rho_\beta(M_i, M_i', M_f, M_f')\delta_{M_i M_i'} = G_\beta^2 \pi F_0 \sum_{KK'} \sum_{Ns} \sum_{km} \sum_{k'm'}$$

$$\times (-1)^{J_f - J_i + J_f - M_f'}(-1)^{N+K+K'-2J_f}(2J_i+1)\sqrt{(2N+1)}\begin{pmatrix} J_f & J_f & N \\ M_f & -M_f' & s \end{pmatrix}$$

$$\times \begin{Bmatrix} J_f & J_f & N \\ K' & K & J_i \end{Bmatrix} a_{kk'}^N(K, K')\begin{pmatrix} k' & N & k \\ m' & s & m \end{pmatrix} Y_k^{m*}(\hat{p}_e) Y_{k'}^{m'*}(\hat{p}_\nu). \quad (7.19)$$

It should be noted that in eqns (7.18) and (7.19) the angle function for $s = 0$

$$P_{Nkk'}(\hat{p}_e, \hat{p}_\nu) = \sqrt{(2N+1)} \sum_m (-1)^m \begin{pmatrix} k' & N & k \\ -m & 0 & m \end{pmatrix} Y_k^{m*}(\hat{p}_e)\, Y_{k'}^m(\hat{p}_\nu) \quad (7.20)$$

is identical (apart from some trivial factors) with the special triple correlation functions introduced by Biedenharn (1960) if one of the three vectors lies along the z-axis.

Equations (7.18) and (7.19) are of a very general character. They allow us to calculate all observables where correlations of electrons, of neutrinos, of electrons and neutrinos with polarized or aligned nuclei before and after the decay are involved (as mentioned at the beginning of this chapter, however, with the restriction that no polarization of electron and neutrino are observed).

Before we start explicit calculations of observables we have, however, also to include the kinematical factors as shown in Section 5.3. Going back to eqn (5.56) and integrating over $\mathbf{p}_f$ we get for that kinematical factor† (recoil energy neglected)

$$dR = \delta\{W_\nu - (\Delta - W_e)\} p_e^2\, dp_e\, d\Omega_e p_\nu^2\, dp_\nu\, d\Omega_\nu \quad (7.21)$$

by which the density matrix of eqns (7.18) and (7.19) has to be multiplied.

In the next section we shall apply these formulae to treat the special cases which are usually observed in beta-decay experiments.

† If the kinetic nuclear recoil energy is taken into account in first order W_e/M we would obtain

$$dR = \left(1 - \frac{W_0 - 3W_e + 3\mathbf{p}_e\hat{\mathbf{p}}_\nu}{M_A}\right)(W_0 - W_e) p_e W_e\, dW_e\, d\Omega_e\, d\Omega_\nu$$

with $M_A = \frac{1}{2}(M_i + M_f)$.

7.2.2. Neutrino not observed

7.2.2.1. Distribution of electrons emitted from oriented nuclei

Since we are interested in the type of observables where the initial nucleus is polarized or aligned we have to go out from the density matrix in eqn (7.18).

In the case where the neutrino is not observed we have to integrate over all neutrino directions. Looking at eqn (7.18) and (7.21) we see that this integration leads to integrals of the form

$$\int_{4\pi} Y_{k'}^{-m'}(\hat{p}_\nu)\, \mathrm{d}\Omega_\nu = \sqrt{(4\pi)}\, \delta_{k'0}\delta_{m'0}, \tag{7.22}$$

which are different from zero only if

$$k' = 0 \qquad m' = 0. \tag{7.23}$$

An inspection of eqns (7.14) and (7.18) then shows that in this case the terms with

$$N = k \qquad s = -m \qquad \kappa'_\nu = \kappa_\nu \tag{7.24}$$

only can give a non-vanishing contribution.

This means that the particle parameters $a^k_{k,0}(K, K')$ are the only ones which are needed.

Since later on in this section we intend to sum over $K \leqslant K'$ only we denote this special type of particle parameters by

$$b^{(k)}_{KK'} = \frac{a^k_{k,0}(K, K') + a^k_{k,0}(K', K)}{4(1+\delta_{KK'})}. \tag{7.25}$$

Explicitly the $a^k_{k,0}(K, K')$ are given by

$$\begin{aligned} a^k_{k,0}(K, K') = {} & \frac{2}{p_e^2 F_0} \sum_{\kappa_e \kappa'_e} \sum_{k_\nu} f^{(k)}_{KK'}(\kappa_e, \kappa'_e; k_\nu)\alpha_{\kappa_e}\alpha_{\kappa'_e} \\ & \times \{M_K(k_e, k_\nu)M_{K'}(k'_e, k_\nu) + \mathrm{sign}(\kappa_e)\mathrm{sign}(\kappa'_e)m_K(k_e, k_\nu)m_{K'}(k'_e, k_\nu) \\ & + \mathrm{sign}(\kappa_e)M_{K'}(k'_e, k_\nu)m_K(k_e, k_\nu) + \mathrm{sign}(\kappa'_e)M_K(k_e, k_\nu)m_{K'}(k'_e, k_\nu)\} \\ & \times e^{-i(\Delta_{\kappa_e} - \Delta_{\kappa'_e})} \end{aligned} \tag{7.26}$$

with†

$$f^{(k)}_{KK'}(\kappa_e, \kappa'_e; k_\nu) = g^{k0k}_{KK'}(\kappa_e, \kappa'_e, \kappa_\nu, \kappa_\nu) = (-1)^{K+K'+j_e+j'_e+j_\nu+\frac{1}{2}}$$

$$\sqrt{\{(2K+1)(2K'+1)(2k+1)(2j_e+1)(2j'_e+1)(2l_e+1)(2l'_e+1)\}} \times \begin{pmatrix} l_e & l'_e & k \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_e & l'_e & k \\ j'_e & j_e & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} K & K' & k \\ j'_e & j_e & j_\nu \end{Bmatrix}. \tag{7.27}$$

These coefficients $f^{(k)}_{KK'}(\kappa_e, \kappa'_e; \kappa_\nu)$ are, by the way, identical with those given in the book by Schopper (1966).

From the above equation we can also immediately derive the following symmetry relations:

$$f^{(k)}_{KK'}(\kappa_e, \kappa'_e; k_\nu) = f^{(k)}_{KK'}(-\kappa_e, -\kappa'_e; k_\nu)$$
$$f^{(k)}_{KK'}(\kappa_e, \kappa'_e; k_\nu) = f^{(k)}_{K'K}(\kappa'_e, \kappa_e; k_\nu). \tag{7.28}$$

We also have

$$f^{(0)}_{KK}(\kappa_e, \kappa_e; k_\nu) = (-1)^K \sqrt{(2K+1)}. \tag{7.29}$$

The definition of the $b^{(k)}_{KK'}$ in the special form of eqn (7.25), has historical reasons. In this manner they are defined as in many earlier papers (Bühring 1963; Schopper 1966; Behrens and Jänecke 1969). The definition is also the same (apart from a factor 2) as in Morita and Morita (1958) and Morita (1973).

The particle parameters $b^{(k)}_{KK'}$ defined by Alder *et al.* (1957) and by Frauenfelder and Steffen (1965) differ, on the other hand, by some factors from those defined above.

For convenience of the reader the most important coefficients $f^{(k)}_{KK'}(\kappa_e, \kappa'_e; k_\nu)$ are listed in Table 7.1.

In this section we are interested in the distribution of electrons emitted by oriented nuclei. We have now, therefore, to look for an adequate description for the orientation of an ensemble of nuclei before the decay. The orientation of the initial nuclei can be characterized by the relative population a_{M_i} of the initial magnetic substates. Usually these relative populations a_{M_i} are chosen to be normalized as

$$\sum_{M_i} a_{M_i} = 1.$$

This means we have

$$a_{M_i} = \frac{1}{2J_i + 1} \tag{7.30}$$

in the case of completely unoriented initial nuclei.

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† In eqn (7.14) we have for $k' = 0$ and $m' = 0$

$$\begin{Bmatrix} k & N & 0 \\ j_e & K & j_\nu \\ j'_e & K' & j'_\nu \end{Bmatrix} = \frac{(-1)^{j'_e + K + j_\nu + k}}{\sqrt{\{(2j_\nu+1)(2k+1)\}}} \begin{Bmatrix} K & K' & k \\ j'_e & j_e & j_\nu \end{Bmatrix} \delta_{Nk} \delta_{j_\nu j'_\nu}$$

$$\begin{Bmatrix} l_\nu & l'_\nu & 0 \\ j'_\nu & j_\nu & \frac{1}{2} \end{Bmatrix} = \frac{(-1)^{j_\nu + l_\nu + \frac{1}{2}}}{\sqrt{\{(2j_\nu+1)(2l_\nu+1)\}}} \delta_{j_\nu j'_\nu} \delta_{l_\nu l'_\nu}$$

$$\begin{pmatrix} l_\nu & l'_\nu & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{(-1)^{l_\nu}}{\sqrt{\{(2l_\nu+1)}} \delta_{l_\nu l'_\nu}$$

(see Edmonds 1964).

TABLE 7.1 $f^{(k)}_{KK'}(\kappa_e, \kappa'_e; k_\nu)$

k	K	K'	κ_e	κ'_e	k_ν	f
0	0	0	1	1	1	1
			−1	−1	1	1
			2	2	2	1
			−2	−2	2	1
0	1	1	1	1	1	$-\sqrt{3}$
			−1	−1	1	$-\sqrt{3}$
			1	1	2	$-\sqrt{3}$
			−1	−1	2	$-\sqrt{3}$
			2	2	1	$-\sqrt{3}$
			−2	−2	1	$-\sqrt{3}$
			2	2	2	$-\sqrt{3}$
			−2	−2	2	$-\sqrt{3}$
0	2	2	1	1	2	$\sqrt{5}$
			−1	−1	2	$\sqrt{5}$
			2	2	1	$\sqrt{5}$
			−2	−2	1	$\sqrt{5}$
			2	2	2	$\sqrt{5}$
			−2	−2	2	$\sqrt{5}$
1	0	1	1	−1	1	1
			−1	1	1	1
			1	2	1	$-\sqrt{2}$
			−1	−2	1	$-\sqrt{2}$
			2	1	2	1
			−2	−1	2	1
			2	−2	2	$\sqrt{\frac{1}{5}}$
			−2	2	2	$\sqrt{\frac{1}{5}}$
1	1	0	1	−1	1	1
			−1	1	1	1
			2	1	1	$-\sqrt{2}$
			−2	−1	1	$-\sqrt{2}$
			1	2	2	1
			−1	−2	2	1
			2	−2	2	$\sqrt{\frac{1}{5}}$
			−2	2	2	$\sqrt{\frac{1}{5}}$
1	1	1	1	−1	1	$\sqrt{2}$
			−1	1	1	$\sqrt{2}$
			1	2	1	1
			−1	−2	1	1
			2	1	1	1
			−2	−1	1	1
			1	−1	2	$-1/\sqrt{2}$
			−1	1	2	$-1/\sqrt{2}$
			2	−2	1	$1/\sqrt{2}$
			−2	2	1	$1/\sqrt{2}$

TABLE 7.1 (*Continued*)

k	K	K'	κ_e	κ'_e	k_ν	f
			1	2	2	$\sqrt{\frac{5}{2}}$
			−1	−2	2	$\sqrt{\frac{5}{2}}$
			2	1	2	$\sqrt{\frac{5}{2}}$
			−2	−1	2	$\sqrt{\frac{5}{2}}$
			2	−2	2	$\frac{1}{5}\sqrt{2}$
			−2	2	2	$\frac{1}{5}\sqrt{2}$
1	1	2	1	−1	2	$-\sqrt{\frac{5}{2}}$
			−1	1	2	$-\sqrt{\frac{5}{2}}$
			1	2	1	$\sqrt{5}$
			−1	−2	1	$\sqrt{5}$
			1	2	2	$\sqrt{\frac{5}{2}}$
			−1	−2	2	$\sqrt{\frac{5}{2}}$
			2	1	2	$-1/\sqrt{2}$
			−2	−1	2	$-1/\sqrt{2}$
			2	−2	1	$-1/\sqrt{10}$
			−2	2	1	$-1/\sqrt{10}$
			2	−2	2	$-\frac{2}{5}\sqrt{2}$
			−2	2	2	$-\frac{2}{5}\sqrt{2}$
1	2	1	1	−1	2	$-\sqrt{\frac{5}{2}}$
			−1	1	2	$-\sqrt{\frac{5}{2}}$
			2	1	1	$\sqrt{5}$
			−2	−1	1	$\sqrt{5}$
			1	2	2	$-1/\sqrt{2}$
			−1	−2	2	$-1/\sqrt{2}$
			2	1	2	$\sqrt{\frac{5}{2}}$
			−2	−1	2	$\sqrt{\frac{5}{2}}$
			2	−2	1	$-\sqrt{\frac{1}{10}}$
			−2	2	1	$-\sqrt{\frac{1}{10}}$
			2	−2	2	$-\frac{2}{5}\sqrt{2}$
			−2	2	2	$-\frac{2}{5}\sqrt{2}$
1	2	2	1	−1	2	$-\sqrt{\frac{5}{2}}$
			−1	1	2	$-\sqrt{\frac{5}{2}}$
			1	2	2	$-\sqrt{\frac{5}{2}}$
			−1	−2	2	$-\sqrt{\frac{5}{2}}$
			2	1	2	$-\sqrt{\frac{5}{2}}$
			−2	−1	2	$-\sqrt{\frac{5}{2}}$
			2	−2	1	$-3\sqrt{\frac{1}{10}}$
			−2	2	1	$-3\sqrt{\frac{1}{10}}$
			2	−2	2	$-2\sqrt{\frac{1}{10}}$
			−2	2	2	$-2\sqrt{\frac{1}{10}}$
2	1	1	1	−2	1	$-\sqrt{3}$
			−1	2	1	$-\sqrt{3}$
			2	−1	1	$-\sqrt{3}$
			−2	1	1	$-\sqrt{3}$
			1	−2	2	$\sqrt{\frac{3}{10}}$

TABLE 7.1 (*Continued*)

k	K	K'	κ_e	κ'_e	k_ν	f
			-1	2	2	$\sqrt{\frac{3}{10}}$
			2	-1	2	$\sqrt{\frac{3}{10}}$
			-2	1	2	$\sqrt{\frac{3}{10}}$
			2	2	1	$\sqrt{\frac{3}{2}}$
			-2	-2	1	$\sqrt{\frac{3}{2}}$
			2	2	2	$-\sqrt{\frac{24}{25}}$
			-2	-2	2	$-\sqrt{\frac{24}{25}}$
2	0	2	1	-2	1	$-\sqrt{2}$
			-1	2	1	$-\sqrt{2}$
			2	-1	2	1
			-2	1	2	1
			2	2	2	-1
			-2	-2	2	-1
2	2	0	2	-1	1	$-\sqrt{2}$
			-2	1	1	$-\sqrt{2}$
			-1	2	2	1
			1	-2	2	1
			2	2	2	-1
			-2	-2	2	-1
2	1	2	1	-2	1	$-\sqrt{3}$
			-1	2	1	$-\sqrt{3}$
			1	-2	2	$\sqrt{\frac{3}{2}}$
			-1	2	2	$\sqrt{\frac{3}{2}}$
			2	-1	2	$3\sqrt{\frac{3}{10}}$
			-2	1	2	$3\sqrt{\frac{3}{10}}$
			2	2	1	$-\sqrt{\frac{3}{2}}$
			-2	-2	1	$-\sqrt{\frac{3}{2}}$
			2	2	2	$-\sqrt{\frac{6}{5}}$
			-2	-2	2	$-\sqrt{\frac{6}{5}}$
2	2	1	2	-1	1	$-\sqrt{3}$
			-2	1	1	$-\sqrt{3}$
			2	-1	2	$\sqrt{\frac{3}{2}}$
			-2	1	2	$\sqrt{\frac{3}{2}}$
			1	-2	2	$3\sqrt{\frac{3}{10}}$
			-1	2	2	$3\sqrt{\frac{3}{10}}$
			2	2	1	$-\sqrt{\frac{3}{2}}$
			-2	-2	1	$-\sqrt{\frac{3}{2}}$
			2	2	2	$-\sqrt{\frac{6}{5}}$
			-2	-2	2	$-\sqrt{\frac{6}{5}}$
2	2	2	1	-2	2	$\sqrt{\frac{7}{2}}$
			-1	2	2	$\sqrt{\frac{7}{2}}$
			2	-1	2	$\sqrt{\frac{7}{2}}$
			-2	1	2	$\sqrt{\frac{7}{2}}$
			2	2	1	$-\sqrt{\frac{7}{2}}$
			-2	-2	1	$-\sqrt{\frac{7}{2}}$

Introducing the above definition we obtain for the distribution of electrons emitted by oriented nuclei

$$\omega(W_e, \theta) = \frac{\sum_{M_i M_i'} \rho_\beta(M_i, M_i') a_{M_i} \delta_{M_i M_i'}}{\frac{1}{2J_i+1} \sum_{M_i M_i'} \rho_\beta(M_i, M_i') \delta_{M_i M_i'}}. \tag{7.31}$$

The factor in the denominator has been introduced in order to have normalized distributions of the form

$$\omega(W_e, \theta) = 1 + \sum_{n \geq 1} A_n P_n(\cos\theta).$$

Then the numerator of eqn (7.31) reads as

$$\sum_{M_i M_i'} \rho_\beta(M_i, M_i') a_{M_i} \delta_{M_i M_i'} = 8\pi^{3/2} G_\beta^2 F_0 \sum_{K \leq K'} \sum_k \sum_{M_i} (2J_i + 1) \times (-1)^{J_f - J_i + J_i - M_i} (-1)^{k+2J_f} \begin{pmatrix} J_i & J_i & k \\ M_i & -M_i & 0 \end{pmatrix} a_{M_i} \times \begin{Bmatrix} J_i & J_i & k \\ K' & K & J_f \end{Bmatrix} b_{KK'}^{(k)} Y_k^{0*}(\hat{p}_e) \tag{7.32}$$

The $6j$-symbol in eqn (7.32) requires that the indices K, K' and k must satisfy the following relations

$$|J_i - J_f| \leq K, K' \leq J_i + J_f \qquad |K - K'| \leq k \leq \min(2J_i, K + K'). \tag{7.33}$$

The degree of orientation can now be described by statistical tensors† (first introduced by Fano 1951).

$$G_k(J_i) = \sum_{M_i} (-1)^{J_i - M_i} C(J_i J_i k; M_i - M_i) a_{M_i} = \sqrt{(2k+1)} \sum_{M_i} (-1)^{J_i - M_i} \begin{pmatrix} J_i & J_i & k \\ M_i & -M_i & 0 \end{pmatrix} a_{M_i}. \tag{7.34}$$

Equation (7.32) can then finally be expressed‡ as

$$\sum_{M_i M_i'} \rho_\beta(M_i, M_i') a_{M_i} \delta_{M_i M_i'} = 4\pi G_\beta^2 F_0 \sum_{K \leq K'} \sum_k (-1)^{J_f - J_i + K + K' + k} \times (2J_i + 1) G_k(J_i) W(J_i\ J_i\ K\ K';\ k\ J_f) b_{KK'}^{(k)} P_k(\cos\theta_e) \tag{7.35}$$

where the $6j$-symbol has been replaced by the Racah coefficient W.

† The orientation parameters $f_k(J_i)$ which are usually used and the parameters $G_k(J_i)$ are related by (de Groot *et al.* 1965)

$$\frac{f_k(J_i)}{G_k(J_i)} = \binom{2k}{k}^{-1} J_i^{-k} \left[\frac{(2J_i + k + 1)!}{(2k+1)(2J_i - k)!}\right]^{1/2}.$$

For the final distribution we also have to consider the denominator in eqn (7.31).

In that case, however, the summation over M_i and M_i' can be carried out immediately because

$$\sum_{M_i} (-1)^{J_i - M_i} \begin{pmatrix} J_i & J_i & k \\ M_i & -M_i & 0 \end{pmatrix} = \sqrt{(2J_i+1)}\,\delta_{k0} \tag{7.36}$$

and we get†

$$\frac{1}{2J_i+1} \sum_{M_i M_i'} \rho_\beta(M_i, M_i')\,\delta_{M_i M_i'} = 4\pi G_\beta^2 F_0 \sum_K (-1)^K \frac{1}{\sqrt{(2K+1)}}\, b_{KK}^{(0)}. \tag{7.37}$$

Finally, we obtain for the angular distribution

$$\omega(\theta_e, W_e) = \frac{\sqrt{(2J_i+1)}}{S(W_e)} \sum_k \tilde{B}^{(k)} G_k(J_i) P_k(\cos\theta_e) \tag{7.38}$$

with

$$\tilde{B}^{(k)} = \sum_{K \leqslant K'} (-1)^{K+K'+k} \Gamma_{KK'}(k, J_f, J_i)\, b_{KK'}^{(k)} \tag{7.39}$$

where

$$\Gamma_{KK'}(k, J_f, J_i) = (-1)^{J_f - J_i} \sqrt{(2J_i+1)}\, W(J_i\ J_i\ K\ K';\ k\ J_f) \tag{7.40}$$

and

$$S(W_e) = \sum_K \frac{(-1)^K}{\sqrt{(2K+1)}}\, b_{KK}^{(0)}. \tag{7.41}$$

The quantities $\tilde{B}^{(k)}$ and the coefficients‡ $\Gamma_{KK'}(k, J_f, J_i)$ are defined in the same way as by Behrens and Jänecke (1969) (see also Schopper 1966).

The coefficients $\Gamma_{KK'}(k, J, J')$ have been tabulated too (Appel 1968).

† We have

$$\begin{Bmatrix} J_i & J_i & 0 \\ K' & K & J_f \end{Bmatrix} = \frac{(-1)^{J_f + J_i + K}}{\sqrt{\{2J_i+1)(2K+1)\}}}\, \delta_{KK'}.$$

‡ A difference exists between the $\Gamma_{KK'}(k, J, J')$ defined in the text and the $\Gamma_{KK'}(k, J, J', J'')$ as defined by Schopper (1966).

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‡ It should be noted that

$$\overset{*}{Y}{}_k^0(\hat{p}_e) = \sqrt{\left(\frac{2k+1}{4\pi}\right)} P_k(\cos\theta_e)$$

where θ_e is the angle between the axis of orientation and the direction of the emitted electron and also

$$W(J_i\ J_i\ K\ K',\ k\ J_f) = (-1)^{2J_i + K + K'} \begin{Bmatrix} J_i & J_i & k \\ K' & K & J_f \end{Bmatrix}.$$

For further calculations it is desirable to have explicit expressions for the most important particle parameters. Since we have always a summation over K, K' and k and in addition over κ_e, κ_e' and k_ν, the question arises, do we have to take into account all terms in this summation or can we restrict it to a very few terms because of a different order of magnitude of the terms.

Fortunately the latter is the case. First let us, therefore, consider the order of magnitude of the different terms. The particle parameters $b_{KK'}^{(k)}$, as introduced in eqns (7.25) and (7.26), consist of a sum over the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$. The order of magnitude of these quantities is, however, determined by the factors in front of the eqns (6.167) and (6.168) or (6.171) and (6.172), respectively, i.e.

$$M_K(k_e, k_\nu) = \frac{(p_e R)^{k_e - 1}(p_\nu R)^{k_\nu - 1}}{(2k_e - 1)!!(2k_\nu - 1)!!} F_M(p_e, p_\nu, Z) \tag{7.42}$$

$$m_K(k_e, k_\nu) = \frac{(p_e R)^{k_e - 1}(p_\nu R)^{k_\nu - 1}}{(2k_e - 1)!!(2k_\nu - 1)!!} F_m(p_e, p_\nu, Z). \tag{7.43}$$

Assuming the nuclear radius to be

$$R = r_0 \cdot A^{1/3} \tag{7.44}$$

where $r_0 = 1{\cdot}2\ \text{fm} \hat{=} 0{\cdot}0031$ (natural units) we see that usually

$$\begin{aligned} 0 &\leqslant p_e R \leqslant 0{\cdot}2, \\ 0 &\leqslant p_\nu R \leqslant 0{\cdot}2. \end{aligned} \tag{7.45}$$

The sum over κ_e, κ_e', k_ν in the particle parameters $b_{KK'}^{(k)}$ always contains terms of (see eqns (7.25) and (7.26))

$$\begin{aligned} &M_K(k_e, k_\nu) M_{K'}(k_e', k_\nu) \\ &m_K(k_e, k_\nu) m_{K'}(k_e', k_\nu) \\ &M_K(k_e, k_\nu) m_{K'}(k_e', k_\nu). \end{aligned} \tag{7.46}$$

The order of magnitude of these terms is, therefore, determined by the exponents of eqns (7.42) and (7.43), i.e. by

$$k_e' + k_e + 2k_\nu - 4. \tag{7.47}$$

It is clear that the most important terms are those with the smallest value of the exponent mentioned above.

TABLE 7.2 *Particle parameters* $b_{KK'}^{(k)}$

$$b_{00}^{(0)}/L_0 = M_0^2(1,1) + m_0^2(1,1) - 2\mu_1\gamma_1 \frac{1}{W_e} M_0(1,1)m_0(1,1)$$

$$b_{11}^{(0)}/L_0 = -\sqrt{3}\left\{M_1^2(1,1) + m_1^2(1,1) - 2\mu_1\gamma_1 \frac{1}{W_e} M_1(1,1)m_1(1,1) + M_1^2(1,2)\right.$$
$$\left. - 2\mu_1\gamma_1 \frac{1}{W_e} M_1(1,2)m_1(1,2) + \lambda_2\left[M_1^2(2,1) - \mu_2\gamma_2 \frac{1}{W_e} M_1(2,1)m_1(2,1)\right]\right\}$$

$$b_{22}^{(0)}/L_0 = \sqrt{5}\left\{M_2^2(1,2) - 2\mu_1\gamma_1 \frac{1}{W_e} M_2(1,2)m_2(1,2)\right.$$
$$\left. + \lambda_2\left[M_2^2(2,1) - \mu_2\gamma_2 \frac{1}{W_e} M_2(2,1)m_2(2,1)\right]\right\}$$

$$b_{01}^{(1)}/L_0 = \pm\left\{2\frac{p_e}{W_e}\Lambda_1[M_0(1,1)M_1(1,1) - m_0(1,1)m_1(1,1)]\right.$$
$$\left. - 2\sqrt{2}\left[\eta_{12}M_0(1,1)M_1(2,1) - \hat{\eta}_{12}\frac{1}{W_e}(M_1(2,1)m_0(1,1) + M_0(1,1)m_1(2,1))\right]\right\}$$

$$b_{11}^{(1)}/L_0 = \pm\left\{\sqrt{2}\frac{p_e}{W_e}\Lambda_1[M_1^2(1,1) - m_1^2(1,1)]\right.$$
$$+ 2\left[\eta_{12}M_1(1,1)M_1(2,1) - \hat{\eta}_{12}\frac{1}{W_e}(M_1(2,1)m_1(1,1) + M_1(1,1)m_1(2,1))\right]$$
$$\left. - \frac{1}{\sqrt{2}}\frac{p_e}{W_e}[\Lambda_1 M_1^2(1,2) - \lambda_2\Lambda_2 M_1^2(2,1)]\right\}$$

$$b_{12}^{(1)}/L_0 = \pm\left\{2\sqrt{5}\left[\eta_{12}M_1(1,1)M_2(2,1) - \hat{\eta}_{12}\frac{1}{W_e}(M_2(2,1)m_1(1,1) + M_1(1,1)m_2(2,1))\right]\right.$$
$$\left. - \sqrt{(10)}\frac{p_e}{W_e}[\Lambda_1 M_1(1,2)M_2(1,2) + \tfrac{1}{5}\lambda_2\Lambda_2 M_1(2,1)M_2(2,1)]\right\}$$

$$b_{22}^{(1)}/L_0 = \mp\left\{\frac{\sqrt{5}}{\sqrt{2}}\frac{p_e}{W_e}[\Lambda_1 M_2^2(1,2) + \tfrac{3}{5}\lambda_2\Lambda_2 M_2^2(2,1)]\right\}$$

$$b_{11}^{(2)}/L_0 = -2\sqrt{3}\left[\nu_{12}\frac{p_e}{W_e}M_1(1,1)M_1(2,1) + \hat{\nu}_{12}\alpha^2 Z^2\frac{1}{p_e}(M_1(2,1)m_1(1,1)\right.$$
$$\left. - M_1(1,1)m_1(2,1))\right] + \frac{\sqrt{3}}{\sqrt{2}}\lambda_2\left[M_1^2(2,1) - \mu_2\gamma_2\frac{1}{W_e}M_1(2,1)m_1(2,1)\right]$$

$$b_{02}^{(2)}/L_0 = -2\sqrt{2}\left[\nu_{12}\frac{p_e}{W_e}M_0(1,1)M_2(2,1) + \hat{\nu}_{12}\alpha^2 Z^2\frac{1}{p_e}\right.$$
$$\left.\times(M_2(2,1)m_0(1,1) - M_0(1,1)m_2(2,1))\right] + 2\sqrt{3}\,\eta_{13}M_0(1,1)M_2(3,1)$$

$$b_{12}^{(2)}/L_0 = -2\sqrt{3}\left[\nu_{12}\frac{p_e}{W_e}M_1(1,1)M_2(2,1) + \hat{\nu}_{12}\alpha^2 Z^2\frac{1}{p_e}\right.$$
$$\left.\times(M_2(2,1)m_1(1,1) - M_1(1,1)m_2(2,1))\right]$$

TABLE 7.2 (*Continued*)

$$-\sqrt{6}\,\lambda_2\left[M_1(2,1)M_2(2,1)-\mu_2\gamma_2\frac{1}{2W_e}(M_1(2,1)m_2(2,1)+M_2(2,1)m_1(2,1))\right]$$

$$-2\sqrt{2}\,\eta_{13}M_1(1,1)M_2(3,1)$$

$$b_{22}^{(2)}/L_0=-\frac{\sqrt{7}}{\sqrt{2}}\lambda_2\left[M_2^2(2,1)-\mu_2\gamma_2\frac{1}{W_e}M_2(2,1)m_2(2,1)\right]$$

$$b_{13}^{(2)}/L_0=-2\sqrt{7}\,\eta_{13}M_1(1,1)M_3(3,1)$$

$$b_{12}^{(3)}/L_0=\pm\frac{6\sqrt{3}}{\sqrt{5}}\lambda_2\Lambda_2\frac{p_e}{W_e}M_1(2,1)M_2(2,1)$$

$$b_{22}^{(3)}/L_0=\pm\frac{3\sqrt{2}}{\sqrt{5}}\lambda_2\Lambda_2\frac{p_e}{W_e}M_2^2(2,1)$$

Upper (lower) sign for β^--decay (β^+-decay)

Returning to eqn (7.27) we note that

$$(k_e'+k_e+2k_\nu-4)_{\min}=\begin{cases}0 & K,K'=0\\ K'-1 & \begin{matrix}K'>0\\ K=0\end{matrix}\\ K-1 & \begin{matrix}K>0\\ K'=0\end{matrix}\\ K'+K-2 & K,K'>0.\end{cases}\tag{7.48}$$

As mentioned earlier (see eqn (7.33)) we have also

$$K'_{\min},K_{\min}=|J_i-J_f|,\tag{7.49}$$

$$k\leqslant K+K'.\tag{7.50}$$

For practical applications it follows then that we have to consider the particle parameters $b_{KK'}^{(k)}$ with the lowest values of K, K' and k. The summation over κ_e, κ_e' and k_ν within the particle parameters also has to be carried out over the lowest possible terms given by eqn (7.48).

The particle parameters of most importance, $b_{KK'}^{(k)}$ for $k, K, K'\leqslant 3$ are listed in Table 7.2 (for the calculation use has been made of the coefficients given in Table 7.1). See also Bühring (1963), Schopper (1966), and Behrens and Jänecke (1969).

The Coulomb functions in Table 7.2 are combinations of the Coulomb amplitudes α_{κ_e} and phases Δ_{κ_e}, i.e.

$$L_0=\frac{\alpha_{-1}^2+\alpha_1^2}{2p_e^2F_0}\tag{7.51a}$$

$$\lambda_{k_e}=\frac{\alpha_{-k_e}^2+\alpha_{k_e}^2}{\alpha_{-1}^2+\alpha_1^2}\tag{7.51b}$$

$$\mu_{k_e} = \frac{k_e W_e}{\gamma_{k_e}} \frac{\alpha_{-k_e}^2 - \alpha_{k_e}^2}{\alpha_{-k_e}^2 + \alpha_{k_e}^2} \tag{7.51c}$$

$$\Lambda_{k_e} = \frac{W_e}{p_e} \frac{2\alpha_{-k_e}\alpha_{k_e}\cos(\Delta_{k_e} - \Delta_{-k_e})}{\alpha_{-k_e}^2 + \alpha_{k_e}^2} \tag{7.51d}$$

$$\nu_{k_e,k_e'} = \frac{W_e}{p_e} \frac{\alpha_{k_e}\alpha_{-k_e'}\cos(\Delta_{k_e} - \Delta_{-k_e'}) + \alpha_{-k_e}\alpha_{k_e'}\cos(\Delta_{-k_e} - \Delta_{k_e'})}{\alpha_{-1}^2 + \alpha_1^2} \tag{7.51e}$$

$$\hat{\nu}_{k_e,k_e'} = \frac{p_e}{(\alpha Z)^2} \frac{\alpha_{k_e}\alpha_{-k_e'}\cos(\Delta_{k_e} - \Delta_{-k_e'}) - \alpha_{-k_e}\alpha_{k_e'}\cos(\Delta_{-k_e} - \Delta_{k_e'})}{\alpha_{-1}^2 + \alpha_1^2} \tag{7.51f}$$

$$\eta_{k_e,k_e'} = \frac{\alpha_{-k_e}\alpha_{-k_e'}\cos(\Delta_{-k_e} - \Delta_{-k_e'}) + \alpha_{k_e}\alpha_{k_e'}\cos(\Delta_{k_e} - \Delta_{k_e'})}{\alpha_{-1}^2 + \alpha_1^2} \tag{7.51g}$$

$$\hat{\eta}_{k_e,k_e'} = W_e \frac{\alpha_{-k_e}\alpha_{-k_e'}\cos(\Delta_{-k_e} - \Delta_{-k_e'}) - \alpha_{k_e}\alpha_{k_e'}\cos(\Delta_{k_e} - \Delta_{k_e'})}{\alpha_{-1}^2 + \alpha_1^2}. \tag{7.51h}$$

The reason for the extraction of some special factors from the combinations of amplitudes and phases has been discussed in Section 4.2.

7.2.2.2. Shape factor of the beta-spectrum, half-life and ft-value

In the case where the beta-spectrum only is observed we also have to integrate over the direction of the emitted beta-particle. Since

$$\int_{4\pi} Y_k^m(\hat{p}_e)\, d\Omega_e = \sqrt{(4\pi)}\delta_{k0}\delta_{m0} \tag{7.52}$$

it follows that terms with $k = 0$ only contribute.

Also, because no orientation of initial and final nuclear states is observed, we must average over the initial substates M_i and sum over the final substates M_f. That means the required formula for the beta-spectrum is given by a special case already treated in the foregoing section (see eqn (7.37)). Including the kinematical factors, i.e. the so-called statistical factor, too, we may write for the transition probability† for the electron energy interval between W_e and $W_e + dW_e$

$$P(W_e)\, dW_e = \frac{G_\beta^2}{2\pi^3} F_0 \sum_K \frac{(-1)^K}{\sqrt{(2K+1)}} b_{KK}^{(0)}\, p_e W_e (W_0 - W_e)^2\, dW_e \tag{7.53}$$

† It is

$$P(W_e)\, dW_e = \frac{1}{(2\pi)^5} \int_{4\pi} \left\{ \sum_{M_i M_i'} \rho(M_i, M_i')\delta_{M_i M_i'} \right\} p_e W_e (W_0 - W_e)^2\, dW_e\, d\Omega_e.$$

(see eqn (5.68) and also eqn (7.21) if we integrate over the neutrino momentum p_ν).

The particle parameters $b_{KK}^{(0)}$ can, however, easily be evaluated (see eqns (7.25), (7.26) and (7.29)). They are given by

$$b_{KK}^{(0)} = (-1)^K \sqrt{(2K+1)} L_0 \sum_{k_e k_\nu} \lambda_{k_e} \{M_K^2(k_e, k_\nu) + m_K^2(k_e, k_\nu) - \frac{2\mu_{k_e}\gamma_{k_e}}{k_e W_e} M_K(k_e, k_\nu) m_K(k_e, k_\nu)\}. \quad (7.54)$$

Equation (7.53) can now be expressed as

$$P(W_e)\,\mathrm{d}W_e = \frac{G_\beta^2}{2\pi^3} F_0 L_0 C(W_e) p_e W_e (W_0 - W_e)^2\,\mathrm{d}W_e \quad (7.55)$$

where $C(W_e)$ is the so-called shape factor† given by

$$C(W_e) = \sum_{\substack{k_e k_\nu \\ K}} \lambda_{k_e} \left\{ M_K^2(k_e, k_\nu) + m_K^2(k_e, k_\nu) - \frac{2\mu_{k_e}\gamma_{k_e}}{k_e W_e} M_K(k_e, k_\nu) m_K(k_e, k_\nu) \right\}. \quad (7.56)$$

This factor‡ takes into account the deviation of the beta-spectrum from the statistical form corrected for the distortion of the electron radial wave function by the nuclear charge. This correction is essentially determined by the so-called Fermi function $F(Z, W_e) = F_0 L_0$ which accounts for the main influence of the nuclear charge on the beta-spectrum shape.

The Coulomb functions λ_{k_e} deviate markedly from 1 only in the case of very low beta-energies W_e or very high Z-values. The behaviour of the functions $F_0 L_0$ and λ_{k_e} is, by the way, extensively discussed in Section 4.2.

The probability of the decay per unit time of one nucleus can now be

† It should be noted that another type of shape factor is often introduced which is given by

$$S(W_e) = C(W_e) L_0.$$

‡ The experimental data for the beta-spectrum shape are often drawn in a special way in order to obtain a straight line which intersects the energy axis at W_0. In this approach, which is usually called a Kurie or Fermi plot, the quantities

$$\sqrt{\left\{\frac{N(W_e)}{p_e W_e F_0 L_0}\right\}} \quad \text{or} \quad \sqrt{\left\{\frac{N(p_e)}{p_e^2 F_0 L_0}\right\}},$$

respectively, are drawn as a function of W_e. Here, $N(W_e)$ or $N(p_e)$ are the number of electrons emitted in the energy or momentum interval $\mathrm{d}W_e$ and $\mathrm{d}p_e$, respectively. If the shape factor $C(W_e)$ reads as $C(W_e) = 1$ a straight line is obtained in this way (see Fig. 7.1). The latter situation is, by the way, the usual one for allowed transitions (see Chapter 14).

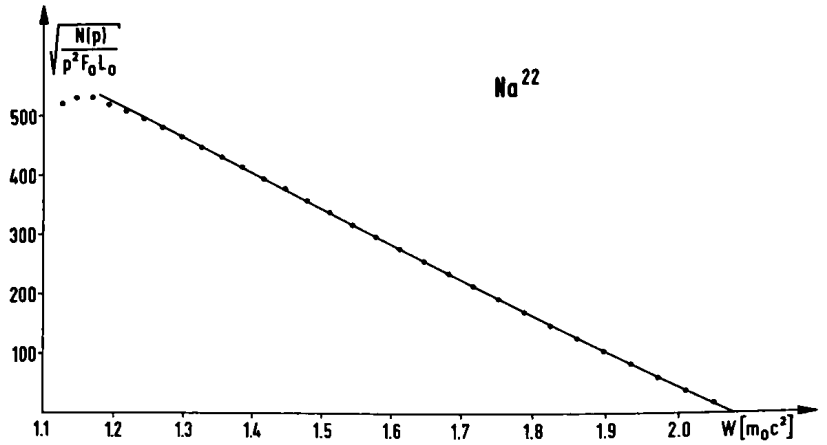

FIG. 7.1. Kurie plot for the beta-spectrum of the ^{22}Na decay (taken from Behrens *et al.* 1972).

expressed as (see Section 5.3)

$$\frac{dW}{dt} = \lambda = \int_1^{W_0} P(W_e)\, dW_e \tag{7.57}$$

i.e. by the integration over all electron energies W_e. The quantity λ is called the decay constant, since in the case of N nuclei the number of decaying nuclei per unit time $N_d = -(dN/dt)$ is given by†

$$-\frac{dN}{dt} = N\frac{dW}{dt} = \lambda N. \tag{7.58}$$

The half-life t is related to the decay constant by

$$t = \frac{\ln 2}{\lambda} \tag{7.59}$$

and is in our case then written as

$$t = \frac{2\pi^3 \ln 2}{G_\beta^2 \overline{C(W_e)} f} \tag{7.60}$$

with

$$f = \int_1^{W_0} p_e W_e (W_0 - W)^2 F(Z, W_e)\, dW_e \tag{7.61}$$

† It should be noted that this differential equation has the well-known solution

$$N = N_0 e^{-\lambda t}.$$

and

$$\overline{C(W_e)} = \frac{\int_1^{W_0} C(W_e) F(Z, W_e) p_e W_e (W_0 - W_e)^2 \, dW_e}{\int_1^{W_0} F(Z, W_e) p_e W_e (W_0 - W_e)^2 \, dW_e}. \tag{7.62}$$

f is usually called the integrated Fermi function.

For $Z = 0$ we have $F(Z, W_e) = 1$ and f can be evaluated analytically (see eqn (5.69)). In the limit $W_0 \gg 1$ we can derive from eqn (5.69) that approximately

$$f \simeq \frac{W_0^5}{30}. \tag{7.63}$$

Because of this strong transition energy dependence of f, and, therefore, of the half-life, this latter quantity is not very characteristic for the dynamic information contained in a special decay. It is more suitable to introduce a special type of reduced half-life, namely†

$$ft = \frac{2\pi^3 \ln 2}{G_\beta^2 \overline{C(W_e)}} \tag{7.64}$$

i.e. the so-called ft-value.‡

In the historical development of the beta-decay treatment it has become customary to quote the ft-values in seconds. We shall, therefore, also comply with that custom and quote the ft-values in the following in seconds and not in the natural time unit $\hbar/m_e c^2$.

Before closing this section we should compare the above results with those obtained in Section 5.3 where we treated the kinematical aspects of the beta-spectrum only. We see that the overall form of the beta-spectrum is essentially determined by the kinematics. Beyond that the

† We use throughout the book natural units ($\hbar = m_e = c = 1$). In order to make the calculation of the half-life or the ft-value, respectively, in other units easier, we would like to mention that

$$ft = \frac{2\pi^3 \ln 2 \, \hbar^7}{m_e^5 c^4 G_\beta^2 \overline{C(W_e)}}$$

when no special choice of units is made from the beginning.

$$\frac{2\pi^3 \ln 2 \, \hbar^7}{m_e^5 c^4} = 1{\cdot}230618 \times 10^{-94} \text{ erg}^2 \text{ cm}^6 \text{ s (cgs units)}$$

$$= 1{\cdot}230618 \times 10^{-120} \text{ J}^2 \text{ m}^6 \text{ s (SI units)}.$$

The above formula is given in a form where f and $\overline{C(W_e)}$ are chosen to be dimensionless, i.e. energy and momentum have to be inserted in these quantities in units $m_e c^2$ and $m_e c$.

‡ In the literature other definitions of the ft-value exist. Here the integration over $C(W_e)$ has been partly included in the definition of f. Since, however, $C(W_e)$ can, in cases of forbidden transitions, be very often not explicitly given, it is more advantageous to have one unique definition for all kinds of beta-transitions.

Coulomb field has a remarkable influence on the momentum distribution of the emitted electrons. As we shall, however, see in detail later on, the dynamic aspects of beta-decay lead to stronger effects only when the transitions are higher forbidden. Nevertheless, even the small deviations of $C(W_e)$ from $C(W_e) = \text{const}$ have been of special interest in the last years, and, therefore, been the aim of many careful experimental investigations of beta-spectra (for a survey see Behrens and Szybisz 1976).

7.2.2.3. *Correlations between the emitted beta-particles and subsequent alpha- or gamma-radiation*

Now we observe the magnetic substates of the final nucleus by means of the observation of a subsequent radiation. Of course, the final nuclear state has here to be an excited or an unstable one. Since the most important cases are those where an alpha-transition or a gamma-transition follows the beta-decay we are particularly interested in treating the beta-alpha and beta-gamma correlations in the following. In this type of correlations the initial nuclear substates are not observed, i.e. we have to average over these states.

Instead of using eqn (7.18), we therefore have to use eqn (7.19). The procedure is, however, completely analogous to that outlined in Section 7.2.2.1, i.e. we integrate over the unobserved neutrino directions and introduce the same particle parameters $b_{KK'}^{(k)}$ as before. Then we arrive at

$$\frac{1}{2J_i+1}\rho_\beta(M_f\ M_f') = 8\pi^{3/2}G_\beta^2 F_0 \sum_{K'\leqslant K}\sum_{km}(-1)^{J_f-J_i-J_f-M_f'+K+K'-m}\begin{pmatrix} J_f & J_f & k \\ M_f & -M_f' & -m\end{pmatrix} \times \begin{Bmatrix} J_f & J_f & k \\ K & K' & J_i\end{Bmatrix} b_{KK'}^{(k)} \overset{*}{Y}{}_k^m(\hat{p}_e). \quad (7.65)$$

Let us now start with the beta-alpha correlations.

In general the correlation function is given by†

$$\omega(\theta, W_e) = \frac{\sum_{M_fM_f'}\rho_\beta(M_fM_f')\rho_\alpha(M_fM_f')}{\frac{1}{2J_i+1}\sum_{M_fM_f'}\rho_\beta(M_fM_f')\delta_{M_fM_f'}\frac{1}{2J_f+1}\sum_{M_fM_f'}\rho_\alpha(M_fM_f')\delta_{M_fM_f'}}. \quad (7.66)$$

† Here it is assumed that the final state of the beta-transition is not changed before the emission of the alpha-particle. If this assumption is not justified (for example in the presence of extranuclear fields) we are speaking of so-called perturbed angular correlations. This type of correlations is, however, beyond the scope of this book. For an extensive discussion of perturbed correlations see Frauenfelder and Steffen (1965).

As before the denominator has been introduced in order to normalize the correlation such that

$$\omega(\theta, W_e) = 1 + \sum_{n \geq 1} a_n P_n(\cos\theta_{\beta\alpha}). \tag{7.67}$$

The density matrix $\rho_\alpha(M_f M_f')$ is the corresponding matrix for the emission of an alpha-particle. It reads as†

$$\rho_\alpha(M_f M_f') = (4\pi)^{3/2} \sum_{L'L} \sum_{km} (-1)^{J_{ff}+J_f+J_f-M_f'}$$
$$\times \sqrt{\{(2L+1)(2L'+1)(2k+1)\}} \begin{pmatrix} J_f & J_f & k \\ M_f & -M_f' & m \end{pmatrix} \begin{Bmatrix} J_f & J_f & k \\ L & L' & J_{ff} \end{Bmatrix} \begin{pmatrix} L & L' & k \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \langle J_{ff} \| L\pi \| J_f \rangle \langle J_{ff} \| L'\pi' \| J_f \rangle . \overset{*}{Y}{}_k^m(\hat{p}_\alpha) \tag{7.68}$$

where J_{ff} is the spin of the final nuclear state after the alpha-transition, L and π angular momentum and parity carried away by the alpha-radiation ($\pi = (-1)^L$), $\langle J_{ff} \| L\pi \| J_f \rangle$ the reduced matrix element for the alpha-transition, and p_α the momentum of the emitted alpha-particle. Since the alpha-transition occurs between states of well-defined parity, the parity selection rule $\pi = (-1)^L$ requires that the L values can differ by only two units, i.e. $L' = L+2$, $L+4$ etc.

This has the consequence that k must be even because the $3j$-symbol vanishes unless $L + L' + k$ is even. A derivation of the above formula is, for example, given in the article by Frauenfelder and Steffen (1965).

To proceed further we now have to carry out the summation over M_f and M_f', first in the numerator of eqn (7.66) and then in the denominator of the same expression.

By applying the relation

$$\sum_{M_f M_f'} \begin{pmatrix} J_f & J_f & k_1 \\ M_f & -M_f' & m_1 \end{pmatrix} \begin{pmatrix} J_f & J_f & k_2 \\ M_f & -M_f' & m_2 \end{pmatrix} = \delta_{k_1 k_2} \delta_{m_1 m_2} (2k_1+1)^{-1} \tag{7.69}$$

(Edmonds 1964) we obtain

$$\frac{1}{2J_i+1} \sum_{M_i M_i'} \rho_\beta(M_f M_f') \rho_\alpha(M_f M_f') = (4\pi)^3 G_\beta^2 F_0 \sum_{K \leq K'} \sum_{km} \sum_{LL'}$$
$$\times (-1)^{J_f - J_i + J_{ff} + J_f} \sqrt{\left\{\frac{(2L+1)(2L'+1)}{2k+1}\right\}} W(J_f J_f K K'; k J_i)$$
$$\times \begin{Bmatrix} J_f & J_f & k \\ L' & L & J_{ff} \end{Bmatrix} \begin{pmatrix} L & L' & k \\ 0 & 0 & 0 \end{pmatrix} \langle J_{ff} \| L\pi \| J_f \rangle \langle J_{ff} \| L'\pi' \| J_f \rangle$$
$$\times b_{KK'}^{(k)} (-1)^m \overset{*}{Y}{}_k^m(\hat{p}_e) \overset{*}{Y}{}_k^{-m}(\hat{p}_\alpha). \tag{7.70}$$

† Here it is assumed that the alpha-particles behave as free particles. However, usually the alpha-particles are charged and move in the Coulomb field of the nucleus, i.e. the plane wave is distorted. In order to take into account this distortion a further factor $\exp i(\Delta_{L'} - \Delta_L)$ is required where Δ_L are the Coulomb phases for the emitted alpha-particle.

The addition theorem for spherical harmonics then asserts that

$$P_k(\cos\theta_{\beta\alpha}) = \frac{4\pi}{(2k+1)} \sum_{m=-k}^{m=+k} Y_k^{m*}(\hat{p}_e) Y_k^m(\hat{p}_\alpha) \tag{7.71}$$

where $\theta_{\beta\alpha}$ is the angle between the emitted α- and β-particle.

For the sums in the denominator we get (see eqns (7.36) and (7.37))

$$\frac{1}{2J_i+1} \sum_{M_i M_i'} \rho(M_f M_f') \delta_{M_i M_i'} = 4\pi G_\beta^2 F_0 \sum_K \frac{(-1)^K}{\sqrt{(2K+1)}} b_{KK}^{(0)} \tag{7.72}$$

and

$$\frac{1}{2J_f+1} \sum_{M_f M_f'} \rho_\alpha(M_f M_f') \delta_{M_f M_f'} = \frac{4\pi}{2J_f+1} \sum_L \langle J_{ff} \| L\pi \| J_f \rangle^2. \tag{7.73}$$

Splitting eqn (7.70) into a part dependent on the properties of the beta-transition and a part dependent on those of the alpha-transition we can write

$$\omega(W_e, \theta_{\beta\alpha}) = \frac{1}{S(W_e)} \sum_k B^{(k)} A_k(\alpha) P_k(\cos\theta_{\beta\alpha}) \tag{7.74}$$

with

$$B^{(k)} = \sum_{K \leqslant K'} b_{KK'}^{(k)} \Gamma_{KK'}(k, J_i, J_f) \tag{7.75}$$

and

$$A_k(\alpha) = \sum_{LL'} (-1)^{J_{ff}+J_f} \sqrt{\{(2L+1)(2L'+1)(2J_f+1)(2k+1)\}} \times \begin{pmatrix} L & L' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J_f & J_f & k \\ L' & L & J_{ff} \end{Bmatrix} \frac{\langle J_{ff} \| L\pi \| J_f \rangle \langle J_{ff} \| L'\pi' \| J_f \rangle}{\sum_L \langle J_{ff} \| L\pi \| J_f \rangle^2}. \tag{7.76}$$

Note that eqn (7.75) corresponds essentially to eqn (7.39), but the spins J_i and J_f in the $\Gamma_{KK'}(k, J_i, J_f)$ coefficient are interchanged and the phase factor $(-1)^{K+K'+k}$ does not occur.

As mentioned before the sum in eqn (7.74) runs over even k only† and from 0 to $\min(2J_f, L+L')$.

The appropriate technique for calculating the formulae for beta–gamma correlations are very similar to those applied before for the beta–alpha correlations.

Thus we have only to replace the density matrix for the alpha-transition $\rho_\alpha(M_f M_f')$ by the density matrix for the gamma-transition $\rho_\gamma(M_f M_f')$ in eqn (7.66).

$\rho_\gamma(M_f M_f')$ is given by

$$\rho_\gamma(M_f M_f') = (4\pi)^{3/2} \sum_{LL'} \sum_{km} (-\tau)^k (-1)^{J_{ff}+J_f-1-J_f+M_f'} \times \sqrt{\{(2L+1)(2L'+1)(2k+1)\}} \begin{pmatrix} J_f & J_f & k \\ M_f & -M_f' & m \end{pmatrix} \begin{pmatrix} L & L' & k \\ 1 & -1 & 0 \end{pmatrix} \times \begin{Bmatrix} J_f & J_f & k \\ L' & L & J_{ff} \end{Bmatrix} \langle J_{ff} \| L\pi \| J_f \rangle \langle J_{ff} \| L'\pi' \| J_f \rangle \overset{*}{Y}{}_k^m(\hat{k}_\gamma) \qquad (7.77)$$

(see, for example, Fraunfelder and Steffen 1965).

The density matrix $\rho_\gamma(M_f, M_f')$ has a very similar structure as the density matrix $\rho_\alpha(M_f, M_f')$. The main difference between both density matrices lies in the occurrence of the different $3j$-symbols

$$\begin{pmatrix} L & L' & k \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L & L' & k \\ 1 & -1 & 0 \end{pmatrix}$$

which comes from the fact that alpha-particle and photon differ in the spin (0 and 1). This latter point leads to the consequence that in the case of the gamma-transition, $L+L'+k$ has not to be even, and, therefore, L values are now allowed to differ by one unit, i.e. $L' = L+1, L+2, L+3 \ldots$

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† As already mentioned in Section 6.2, the assumption has been made that the nuclei are infinitely massive. However, if we are treating the nuclei in initial and final states as particles with finite masses some new purely kinematical terms appear in the above formulae. These terms are caused by the necessary Lorentz-transformation to the lab-frame from the rest frame of the beta-decay daughter nucleus (or from the lab frame to the Breit frame and from the rest frame of the beta-decay daughter nucleus to the Breit frame). Most of these terms are of no importance since they are of the order $p_e/M_A v$. v is here the velocity of the alpha- or gamma-radiation in the centre of mass frame of the beta-decay daughter. In the case of beta–alpha and beta–gamma correlations, however, these kinematical effects have the consequence that additional odd k terms of $P_k(\cos\theta)$ are introduced.

The most important one of these is the term connected with $P_1(\cos\theta)$. Then the correlation $\omega(W_e, \theta)$ reads approximately as

$$\omega(W_e, \theta) = 1 + a_1 P_1(\cos\theta) + \ldots$$

with

$$a_1 = \frac{2p_e}{M_A v}$$

(the exact value of a_1 is also dependent on the spins and form factor coefficients).

For $\beta-\gamma$-correlations the above term is very small (however not unmeasurably small) since v is the speed of light. For beta–alpha correlations, on the other hand, the above term is comparable to the even k terms. An exact treatment of these effects is given by Holstein (1974*a*).

The quantity τ determines the sign of the circular polarization of the gamma-quantum and is equal to ± 1 ($\tau = +1$ right circular, $\tau = -1$ left circular).

The matrix elements $\langle J_{ff} \| L\pi \| J_f \rangle$ are, as before, the reduced transition matrix elements for the 2^L-pole gamma-transition.

Similarly, as for the beta–alpha correlation, we now obtain

$$\frac{1}{2J_i+1}\sum_{M_f M_f'} \rho_\beta(M_f, M_f')\rho_\gamma(M_f, M_f') = (4\pi)^3 G_\beta^2 F_0$$
$$\times \frac{1}{2J_f+1}\sum_{K \leqslant K'}\sum_{km}\sum_{LL'} (-1)^{J_f - J_i} W(J_f J_f K K'; kJ_i)\sqrt{(2J_f+1)}\, b_{KK'}^{(k)} (-\tau)^k$$
$$\times F_k(LL'J_{ff}J_f)\delta_L \delta_{L'} \frac{(-1)^m}{2k+1} \overset{*}{Y}{}_k^m(\hat{p}_e)\overset{*}{Y}{}_k^{-m}(\hat{k}_\gamma). \quad (7.78)$$

As usual, the reduced gamma-transition matrix elements are now denoted by δ_L and normalized according to $\sum_L \delta_L^2 = 1$.

The $F_k(LL'J_{ff}J_f)$ coefficients, well-known from γ-γ correlations, have also been introduced. They are given by

$$F_k(LL'J_{ff}J_f) = (-1)^{J_\pi + J_f - 1}\sqrt{\{(2L+1)(2L'+1)(2k+1)(2J_f+1)\}}$$
$$\times \begin{pmatrix} L & L' & k \\ 1 & -1 & 0 \end{pmatrix} \begin{Bmatrix} J_f & J_f & k \\ L & L' & J_{ff} \end{Bmatrix}$$

(see, for example, Appel 1968).

Analogous to the formulae for the beta–alpha correlation (see eqns (7.66), (7.71), (7.72), (7.73), (7.74) and (7.75)) we now obtain

$$\omega(W_e, \theta_{\beta\gamma}) = \frac{1}{S(W_e)}\sum_k (-\tau)^k B^{(k)} A_k(\gamma) P_k(\cos\theta_{\beta\gamma}) \quad (7.79)$$

where $B^{(k)}$ is given by eqn (7.75).

The $A_k(\gamma)$, which contain all the information about the gamma-transition, have the form

$$A_k(\gamma) = \sum_{LL'} F_k(LL'J_{ff}J_f)\delta_L \delta_{L'}. \quad (7.80)$$

Contrary to the beta–alpha correlations, odd-k terms may also be observed, however, only in the case where the circular polarization τ of the γ-rays is detected. This is demonstrated by the phase factor $(-\tau)^k$.

7.2.3. Electron not observed

The case 'electron not observed' can be treated completely analogously to the case 'neutrino not observed', the reason being that both cases are

symmetric in a certain sense. Starting from eqn (7.18) we have to integrate over all electron directions and obtain then, as shown similarly in eqn (7.22),

$$\int_{4\pi} Y_k^{-m}(\hat{p}_e)\, d\Omega_e = \sqrt{(4\pi)}\delta_{k0}\delta_{m0} \tag{7.81}$$

with the result that $k=0$, $m=0$.

From eqns (7.14) and (7.18) it then follows that

$$N = k' \qquad s = -m' \qquad \kappa_e' = \kappa_e. \tag{7.82}$$

Analogous with what we have done in Section 7.2.2.1, we introduce particle parameters of the type

$$c_{KK'}^{(k')} = \frac{a_{0,k'}^{k'}(K, K') + a_{0,k'}^{k'}(K', K)}{4(1+\delta_{KK'})}. \tag{7.83}$$

These neutrino particle parameters $c_{KK'}^{(k)}$ can explicitly be given by inserting the quantities $a_{0,k'}^{k'}(K, K')$ which have to be derived from eqns (7.13) and (7.14).

For the special cases of eqns (7.81) and (7.82) the particle parameters $a_{0,k'}^{k'}(K, K')$ read as

$$\begin{aligned} a_{0,k'}^{k'}(K, K') = {} & \frac{1}{F_0 p_e^2} \sum_{\kappa_\nu \kappa_\nu'} \sum_{k_e} (-1)^{K+K'+k_\nu+k_\nu'} \\ & \times \operatorname{sign}(\kappa_\nu)\operatorname{sign}(\kappa_\nu') f_{KK'}^{(k')}(\kappa_\nu, \kappa_\nu'; k_e)\{(\alpha_{k_e}^2 + \alpha_{-k_e}^2) \\ & \times [M_K(k_e, k_\nu) M_{K'}(k_e, k_\nu') + m_K(k_e, k_\nu) m_{K'}(k_e, k_\nu')] \\ & + (\alpha_{k_e}^2 - \alpha_{-k_e}^2)[M_{K'}(k_e, k_\nu') m_K(k_e, k_\nu) + M_K(k_e, k_\nu) m_{K'}(k_e, k_\nu')]\}. \end{aligned} \tag{7.84}$$

For comparison with the case 'neutrino not observed' see eqn (7.26). The coefficients $f_{KK'}^{(k)}(\kappa_\nu, \kappa_\nu'; k_e)$ are given by eqn (7.27) where electron and neutrino indices are interchanged (see also Table 7.1).

In analogy to the case 'neutrino not observed' (see Table 7.2) the most important neutrino particle parameters $c_{KK'}^{(k)}$ are listed in Table 7.3.

It should be added that the dominant terms are kept only in that list. Note that for $k=0$ we have $c_{KK}^{(0)} = b_{KK}^{(0)}$.

The neutrino distribution emitted from oriented nuclei, the neutrino shape factor, neutrino–alpha and neutrino–gamma correlations can now easily be calculated by simply replacing $b_{KK'}^{(k)}$ by $c_{KK'}^{(k)}$ and $\hat{p}_e$ by $\hat{p}_\nu$ in all relevant formulae given in Section 7.2.2.

In this context the most important observable is the neutrino spectrum. For convenience of the reader this observable is, therefore, written explicitly down in the following. Taking into account the kinematical factor from eqn (7.21) we obtain for the neutrino spectrum (number of

TABLE 7.3 *Neutrino particle parameters* $c_{KK'}^{(k)}$

$$c_{00}^{(0)}/L_0 = M_0^2(1,1) + m_0^2(1,1) - 2\mu_1\gamma_1 \frac{1}{W_e} M_0(1,1)m_0(1,1)$$

$$c_{11}^{(0)}/L_0 = -\sqrt{3}\left\{M_1^2(1,1) + m_1^2(1,1) - 2\mu_1\gamma_1 \frac{1}{W_e} M_1(1,1)m_1(1,1)\right.$$
$$\left. + M_1^2(1,2) - 2\mu_1\gamma_1 \frac{1}{W_e} M_1(1,2)m_1(1,2) + \lambda_2\left[M_1^2(2,1) - \mu_2\gamma_2 \frac{1}{W_e} M_1(2,1)m_1(2,1)\right]\right\}$$

$$c_{22}^{(0)}/L_0 = \sqrt{5}\left\{M_2^2(1,2) - 2\mu_1\gamma_1 \frac{1}{W_e} M_2(1,2)m_2(1,2)\right.$$
$$\left. + \lambda_2\left[M_2^2(2,1) - \mu_2\gamma_2 \frac{1}{W_e} M_2(2,1)m_2(2,1)\right]\right\}$$

$$c_{01}^{(1)}/L_0 = \pm\left\{2\left[M_0(1,1)M_1(1,1) + m_0(1,1)m_1(1,1)\right.\right.$$
$$\left. - \frac{\mu_1\gamma_1}{W_e}(M_0(1,1)m_1(1,1) + M_1(1,1)m_0(1,1))\right]$$
$$\left. - 2\sqrt{2}\left[M_0(1,1)M_1(1,2) - \frac{\mu_1\gamma_1}{W_e}(M_0(1,1)m_1(1,2) + M_1(1,2)m_0(1,1))\right]\right\}$$

$$c_{11}^{(1)}/L_0 = \mp\left\{\sqrt{2}\left[M_1^2(1,1) + m_1^2(1,1) - 2\mu_1\gamma_1 \frac{1}{W_e} M_1(1,1)m_1(1,1)\right]\right.$$
$$+ 2\left[M_1(1,1)M_1(1,2) - \mu_1\gamma_1 \frac{1}{W_e}(M_1(1,2)m_1(1,1) + M_1(1,1)m_1(1,2))\right]$$
$$\left. - \frac{1}{\sqrt{2}}[\lambda_2 M_1^2(2,1) - M_1^2(1,2)]\right\}$$

$$c_{12}^{(1)}/L_0 = \pm\left\{2\sqrt{5}\left[M_1(1,1)M_2(1,2) - \mu_1\gamma_1 \frac{1}{W_e}(M_1(1,1)m_2(1,2) + M_2(1,2)m_1(1,1))\right]\right.$$
$$\left. - \sqrt{(10)}[\lambda_2 M_1(2,1)M_2(2,1) + \tfrac{1}{5}M_1(1,2)M_2(1,2)]\right\}$$

$$c_{22}^{(1)}/L_0 = \pm\sqrt{\tfrac{5}{2}}[\lambda_2 M_2^2(2,1) + \tfrac{3}{5}M_2^2(1,2)]$$

$$c_{02}^{(2)}/L_0 = -2\sqrt{2}\left[M_0(1,1)M_2(1,2) - \mu_1\gamma_1 \frac{1}{W_e}(M_2(1,2)m_0(1,1) + M_0(1,1)m_2(1,2))\right]$$
$$+ 2\sqrt{3}\, M_0(1,1)M_2(1,3)$$

$$c_{11}^{(2)}/L_0 = -2\sqrt{3}\left[M_1(1,1)M_1(1,2) - \frac{\mu_1\gamma_1}{W_e}(M_1(1,2)m_1(1,1) + M_1(1,1)m_1(1,2))\right]$$
$$+ \sqrt{\left(\frac{3}{2}\right)}\left[M_1^2(1,2) - \frac{\mu_1\gamma_1}{W_e}(M_1(1,2)m_1(1,1) + M_1(1,1)m_1(1,2))\right]$$

$$c_{12}^{(2)}/L_0 = 2\sqrt{3}\left[M_1(1,1)M_2(1,2) - \mu_1\gamma_1 \frac{1}{W_e}(M_2(1,2)m_1(1,1) + m_2(1,2)M_1(1,1))\right]$$
$$+ \sqrt{6}\left[M_1(1,2)M_2(1,2) - \mu_1\gamma_1 \frac{1}{W_e}(M_2(1,2)m_1(1,2) + M_1(1,2)m_2(1,2))\right]$$
$$+ 2\sqrt{2}\, M_1(1,1)M_2(1,3)$$

$$c_{22}^{(2)}/L_0 = -\sqrt{\left(\frac{7}{2}\right)}\left[M_2^2(1,2) - 2\mu_1\gamma_1 \frac{1}{W_e} M_2(1,2)m_2(1,2)\right]$$

$$c_{13}^{(2)}/L_0 = -2\sqrt{7}\, M_1(1,1)M_3(1,3)$$

neutrinos emitted in the momentum interval between p_ν and $p_\nu + dp_\nu$)

$$P(p_\nu)\,dp_\nu = \frac{G_\beta^2}{2\pi^3} F_0 L_0 C(p_\nu) p_\nu^2 (W_0 - W_\nu)\sqrt{\{W_0 - W_\nu)^2 - 1\}}\,dp_\nu \quad (7.85)$$

where the shape factor $C(p_\nu)$ has the same form

$$C(p_\nu) = \sum_{\substack{k_e, k_\nu \\ K}} \lambda_{k_e}\{M_K^2(k_e, k_\nu) + m_K^2(k_e, k_\nu) - \frac{2\mu_{k_e}\gamma_{k_e}}{k_e(W_0 - W_\nu)} M_K(k_e, k_\nu) m_K(k_e, k_\nu)\} \quad (7.86)$$

as in the case of the electron spectrum. Of course, this follows also directly from the fact that neutrino and electron energy are always related by $W_\nu = W_0 - W_e$.

7.2.4. Electron–neutrino correlations

7.2.4.1. Unoriented nuclei

We will consider here the case where electron and neutrino together are observed, but there is no orientation of the nucleus in initial or final state.

We have, therefore, to use eqn (7.18) and average over the initial nuclear substates. The electron–neutrino correlation $\omega(\Theta_{e\nu}, W_e)$ is then written as

$$\omega(\Theta_{e\nu}, W_e) = \frac{\frac{(4\pi)^2}{2J_i+1}\sum_{M_i M_i'} \rho_\beta(M_i, M_i')\delta_{M_i M_i'}}{\frac{1}{2J_i+1}\int_{4\pi}\int_{4\pi} \sum_{M_i M_i'} \rho_\beta(M_i, M_i')\delta_{M_i M_i'}\,d\Omega_e\,d\Omega_\nu} \quad (7.87)$$

where $\Theta_{e\nu}$ is the angle between the emitted electron and neutrino. As before, the normalization† is chosen such that

$$\omega(\Theta_{e\nu}, W_e) = 1 + \sum_{n\geq 1} a_n P_n(\cos\Theta_{e\nu}). \quad (7.88)$$

After inserting eqn (7.18) in the above formula we have now to carry out the summation over M_i and M_i'. This summation, however, leads to

$$\sum_{M_i} (-1)^{J_i - M_i} \begin{pmatrix} J_i & J_i & N \\ M_i & -M_i & 0 \end{pmatrix} = \sqrt{(2J_i+1)}\,\delta_{N0} \quad (7.89)$$

† Note that

$$\frac{1}{2J_i+1}\int_{4\pi}\int_{4\pi} \sum_{M_i M_i'} \rho_\beta(M_i, M_i')\delta_{M_i M_i'}\,d\Omega_e\,d\Omega_\nu = (4\pi)^2 G_\beta^2 F_0 S(W_e).$$

(see also eqn (7.36)) and, therefore, to the consequence that

$$N=0, \qquad s=0. \tag{7.90}$$

As a result, eqn (7.18) can be further simplified† and we find (for the addition theorem of spherical harmonics see eqn (7.71)).

$$\frac{1}{2J_i+1}\sum_{M_iM_i'}\rho_\beta(M_i,M_i')\delta_{M_iM_i'}=\frac{G_\beta^2}{4}F_0\sum_K\sum_k(-1)^{k+K}\times\sqrt{\left(\frac{2k+1}{2K+1}\right)}a_{kk}^0(K,K)P_k(\cos\Theta_{e\nu}) \tag{7.91}$$

where

$$\begin{aligned}a_{kk}^0(K,K)=\frac{(-1)^k}{p_e^2F_0}\sum_{\substack{\kappa_e\kappa_e'\\ \kappa_\nu\kappa_\nu'}}(-1)^{k_\nu+k_\nu'}\,\mathrm{sign}(\kappa_\nu)\mathrm{sign}(\kappa_\nu')\\ \times g_K^{(k)}(\kappa_e,\kappa_e';\kappa_\nu,\kappa_\nu')\alpha_{\kappa_e}\alpha_{\kappa_e'}\{M_K(k_e,k_\nu)M_K(k_e',k_\nu')\\ +\mathrm{sign}(\kappa_e)\mathrm{sign}(\kappa_e')m_K(k_e,k_\nu)m_K(k_e',k_\nu')\\ +\mathrm{sign}(\kappa_e)M_K(k_e',k_\nu')m_K(k_e,k_\nu)\\ +\mathrm{sign}(\kappa_e')M_K(k_e,k_\nu)m_K(k_e',k_\nu')\}\mathrm{e}^{-\mathrm{i}(\Delta_{\kappa_e}-\Delta_{\kappa_e'})}.\end{aligned} \tag{7.92}$$

The geometrical coefficient $g_K^{(k)}(\kappa_e,\kappa_e';\kappa_\nu,\kappa_\nu')$ differs, however, from that obtained in the other cases. It is found to be

$$\begin{aligned}g_K^{(k)}(\kappa_e,\kappa_e';\kappa_\nu,\kappa_\nu')=-\sqrt{\{}(2K+1)(2k+1)(2j_e+1)(2j_e'+1)\\ \times(2j_\nu+1)(2j_\nu'+1)(2l_e+1)(2l_e'+1)(2l_\nu+1)(2l_\nu'+1)\}\begin{pmatrix}l_e & l_e' & k\\ 0 & 0 & 0\end{pmatrix}\\ \times\begin{Bmatrix}l_e & l_e' & k\\ j_e' & j_e & \frac{1}{2}\end{Bmatrix}\begin{pmatrix}l_\nu & l_\nu' & k\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}l_\nu & l_\nu' & k\\ j_\nu' & j_\nu & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix}j_\nu & j_\nu' & k\\ j_e' & j_e & K\end{Bmatrix}.\end{aligned} \tag{7.93}$$

Similarly as in the case of the coefficients $f_{KK'}^{(k)}(\kappa_e,\kappa_e';k_\nu)$ (see eqn (7.28)), we can derive some symmetry relations:

$$g_K^{(k)}(\kappa_e,\kappa_e';\kappa_\nu,\kappa_\nu')=g_K^{(k)}(-\kappa_e,-\kappa_e';\kappa_\nu,\kappa_\nu') \tag{7.94a}$$

$$g_K^{(k)}(\kappa_e,\kappa_e';\kappa_\nu;\kappa_\nu')=g_K^{(k)}(\kappa_e,\kappa_e';-\kappa_\nu,-\kappa_\nu'). \tag{7.94b}$$

In the special case of $k=0$ we have for comparison (see also eqn (7.29))

$$g_K^{(0)}(\kappa_e,\kappa_e;\kappa_\nu,\kappa_\nu)=(-1)^K\sqrt{(2K+1)}. \tag{7.95}$$

† Note that

$$\begin{Bmatrix}J_i & J_i & 0\\ K' & K & J_f\end{Bmatrix}=\frac{(-1)^{J_i+J_f+K}}{\sqrt{\{(2K+1)(2J_i+1)\}}}\delta_{KK'}$$

$$\begin{Bmatrix}k & 0 & k\\ j_e & K & j_\nu\\ j_e' & K' & j_\nu'\end{Bmatrix}=\frac{(-1)^{j_e'+j_\nu+k-K}}{\sqrt{\{(2k+1)(2K+1)\}}}\begin{Bmatrix}j_\nu & j_\nu' & k\\ j_e' & j_e & K\end{Bmatrix}\delta_{kk}\delta_{KK'}$$

$$\begin{pmatrix}k' & 0 & k\\ m' & 0 & m\end{pmatrix}=\frac{(-1)^{k-m}}{\sqrt{(2k+1)}}\delta_{k'k}\delta_{m'-m}$$

Analogous with the treatment in earlier chapters we are now introducing special electron–neutrino particle parameters of the form

$$d_K^{(k)} = \frac{(-1)^{k+K}}{4}\sqrt{\left(\frac{2k+1}{2K+1}\right)}a_{kk}^0(K, K). \tag{7.96}$$

Thus we obtain finally for the electron–neutrino correlation

$$\omega(\Theta_{e\nu}, W_e) = \frac{1}{S(W_e)}\sum_k D^{(k)}P_k(\cos\Theta_{e\nu}) \tag{7.97}$$

with

$$D^{(k)} = \sum_K d_K^{(k)}. \tag{7.98}$$

A list of the most important particle parameters $d_K^{(k)}$ is reproduced in Table 7.4. As before dominant terms have been retained only.

It should be noted that

$$(-1)^K\sqrt{(2K+1)}d_K^{(0)} = b_{KK}^{(0)} = c_{KK}^{(0)} \tag{7.99}$$

In contrast to the distribution of electrons emitted from oriented nuclei or to the beta–alpha and beta–gamma correlations, the electron–neutrino correlation, as with the shape factor, does not contain interference terms of different tensor ranks K, i.e. terms of the form $M_K(k_e, k_\nu)M_{K'}(k_e', k_\nu')$ where $K \neq K'$. This fact has something to do with the connection of the electron–neutrino correlation with another type of spectrum, i.e. the recoil momentum spectrum.

Before closing this section we will therefore also calculate the recoil momentum distribution which is directly determined by the electron–neutrino correlation.

The reason is that the recoil momentum $\mathbf{p}_f$ is related to electron and neutrino momentum by (see eqn (5.6))

$$\mathbf{p}_f = -(\mathbf{p}_e + \mathbf{p}_\nu). \tag{7.100}$$

Therefore, we have

$$\cos\Theta_{e\nu} = \frac{p_f^2 - p_e^2 - p_\nu^2}{2p_e p_\nu} = \frac{p_f^2 + m_e^2 - \Delta^2 + 2W_e(\Delta - W_e)}{2p_e(\Delta - W_e)} \tag{7.101}$$

(since $p_\nu = \Delta - W_e$), i.e. $\cos\Theta_{e\nu}$ can be expressed by the electron energy W_e and the recoil momentum p_f.

In Section 5.3 we have already treated the statistical recoil momentum spectrum. Now we have, however, additionally to take into account the fact that the required transition matrix $|T|^2$ is not a constant. Nevertheless, the calculation of the recoil momentum distribution $P(p_f)\,dp_f$ follows

TABLE 7.4 *Electron–neutrino particle parameters* $d_K^{(k)}$

$$d_0^{(0)}/L_0 = M_0^2(1,1) + m_0^2(1,1) - 2\mu_1\gamma_1 \frac{1}{W_e} M_0(1,1)m_0(1,1)$$

$$d_1^{(0)}/L_0 = M_1^2(1,1) + m_1^2(1,1) - 2\mu_1\gamma_1 \frac{1}{W_e} M_1(1,1)m_1(1,1) + M_1^2(1,2)$$
$$- 2\mu_1\gamma_1 \frac{1}{W_e} M_1(1,2)m_1(1,2) + \lambda_2\left[M_1^2(2,1) - \mu_2\gamma_2 \frac{1}{W_e} M_1(2,1)m_1(2,1)\right]$$

$$d_2^{(0)}/L_0 = M_2^2(1,2) - 2\mu_1\gamma_1 \frac{1}{W_e} M_2(1,2)m_2(1,2) + \lambda_2\left[M_2^2(2,1) - \mu_2\gamma_2 \frac{1}{W_e} M_2(2,1)m_2(2,1)\right]$$

$$d_0^{(1)}/L_0 = \frac{p_e}{W_e}\Lambda_1[M_0^2(1,1) - m_0^2(1,1)] - 2\sqrt{2}\,\eta_{12}M_0(1,1)M_0(2,2)$$

$$d_1^{(1)}/L_0 = -\frac{1}{3}\frac{p_e}{W_e}\Lambda_1[M_1^2(1,1) - m_1^2(1,1)] - \tfrac{4}{3}\sqrt{2}\,\frac{p_e}{W_e}\Lambda_1[M_1(1,1)M_1(1,2) - m_1(1,1)m_1(1,2)]$$
$$- \tfrac{4}{3}\sqrt{2}\,\eta_{12}[M_1(1,1)M_1(2,1) + m_1(1,1)m_1(2,1)]$$
$$+ \tfrac{4}{3}\sqrt{2}\,\frac{1}{W_e}\hat{\eta}_{12}[M_1(1,1)m_1(2,1) + M_1(2,1)m_1(1,1)]$$
$$- \tfrac{2}{3}\sqrt{(10)}\,\eta_{12}M_1(1,1)M_1(2,2) + \tfrac{2}{3}\eta_{12}M_1(2,1)M_1(1,2)$$
$$+ \frac{1}{3}\frac{p_e}{W_e}[\Lambda_1 M_1^2(1,2) + \lambda_2\Lambda_2 M_1^2(2,1)]$$

$$d_2^{(1)}/L_0 = -\frac{1}{5}\frac{p_e}{W_e}[\Lambda_1 M_2^2(1,2) + \lambda_2\Lambda_2 M_2^2(2,1)] + 2\eta_{12}M_2(2,1)M_2(1,2)$$

$$d_0^{(2)}/L_0 = -2\sqrt{2}\,\frac{p_e}{W_e}\nu_{12}M_0(1,1)M_0(2,2)$$

$$d_1^{(2)}/L_0 = 2\sqrt{\left(\frac{2}{5}\right)}\frac{p_e}{W_e}\nu_{12}M_1(1,1)M_1(2,2) + 2\,\frac{p_e}{W_e}\nu_{12}M_1(1,2)M_1(2,1)$$

$$d_2^{(2)}/L_0 = -\frac{2}{5}\frac{p_e}{W_e}\nu_{12}M_2(1,2)M_2(2,1)$$

exactly the same method as shown in eqns (5.59) to (5.70). Then we end up with

$$P(p_f)\,\mathrm{d}p_f = \frac{1}{4\pi^3}\int_{W_{e_1}}^{W_{e_2}} |T|^2(W_f + W_e - M_i)\,W_e\,\mathrm{d}W_e p_f\,\mathrm{d}p_f. \tag{7.102}$$

In this case $|T|^2$ is given by

$$|T|^2 = \frac{1}{2J_i + 1}\sum_{M_i M_i'} \rho_\beta(M_i, M_i')\delta_{M_i M_i'} \tag{7.103}$$

(see eqn (7.87)).

By combining eqns (7.91), (7.96), (7.98), (7.102) and (7.103), we then obtain

$$P(p_t)\,\mathrm{d}p_t = \frac{G_\beta^2}{4\pi^3}\int_{W_{e_1}}^{W_{e_2}} F_0 \sum_k D^{(k)} P_k(\cos\Theta_{e\nu})(\Delta - W_e) W_e\,\mathrm{d}W_e p_t\,\mathrm{d}p_t \tag{7.104}$$

where $W_{e_{1,2}}$ has to be taken from eqn (5.71), $\cos\Theta_{e\nu}$ has to be replaced by eqn (7.101) and the neutrino momentum p_ν in the $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ by $p_\nu = \Delta - W_e$.

In this context the most important terms are, however, those with $k = 0$ and $k = 1$. Neglecting therefore all terms with $k > 1$ we can explicitly write

$$P(p_t)\,\mathrm{d}p_t = \frac{G_\beta^2}{4\pi^3}\int_{W_{e\min}}^{W_{e\max}} F_0\left[D^{(0)}(\Delta - W_e)W_e + \frac{W_e}{p_e} D^{(1)} \frac{p_t^2 + m_e^2 - \Delta^2 + 2W_e(\Delta - W_e)}{2}\right]\mathrm{d}W_e p_t\,\mathrm{d}p_t \tag{7.105}$$

with

$$W_{e_{\min}} = \frac{(\Delta - p_t)^2 + m_e^2}{2(\Delta - p_t)}$$

$$W_{e_{\max}} = \frac{(\Delta + p_t)^2 + m_e^2}{2(\Delta + p_t)}.$$

It should be mentioned that in the above equations the kinetic recoil energy E_R has, as in Section 5.3, been neglected against the transition energy Δ.

Usually the neutrino can not be detected. Electron neutrino correlations can, therefore, also not be measured directly.

As we have demonstrated in this chapter, however, the correlation coefficients (or functions) $D^{(k)}$ can be determined indirectly by investigating the recoil momentum spectrum. This possibility has been used often in the past. Of course, one is also able to measure the electron–neutrino correlation in a more direct way via the electron recoil nucleus correlation. This latter method of a direct determination of the $D^{(k)}$ has the advantage of giving more (i.e. the full) information than is obtained from the recoil spectrum.

7.2.4.2. Oriented nuclei in the initial state

In this section we now turn to the more general situation where the correlation between electrons and neutrinos emitted from oriented nuclei is observed.

In that case we have to start from eqn (7.18) in order to derive further special expressions. In this context, however, we restrict ourselves to the selection of one special example, i.e. to the case

$$k=1, \qquad k'=1, \qquad N=1.$$

Describing the degree of orientation by a statistical tensor $G_1(J_i)$ as shown before in eqn (7.34) we then obtain

$$\sum_{M_i M_i'} \rho_\beta(M_i, M_i') a_{M_i} \delta_{M_i M_i'} = G_\beta^2 \pi F_0 \sum_{KK'} (-1)^{J_f - J_i} \times (2J_i+1) G_1(J_i) (-1)^{2J_f} \begin{Bmatrix} J_i & J_i & 1 \\ K' & K & J_f \end{Bmatrix} a_{11}^1(K, K') \times \sum_m \begin{pmatrix} 1 & 1 & 1 \\ m & -m & 0 \end{pmatrix} \overset{*}{Y}{}_1^m(\hat{p}_e) \overset{*}{Y}{}_1^{-m}(\hat{p}_\nu). \tag{7.106}$$

Before proceeding further we have first to discuss the electron–neutrino angular function in a little more detail.

Since

$$\overset{*}{Y}{}_1^m(\hat{p}_e) = \frac{\sqrt{3}}{\sqrt{4\pi}} \frac{\mathbf{p}_e}{p_e} \mathbf{e}_m^*$$

$$\overset{*}{Y}{}_1^m(\hat{p}_\nu) = \frac{\sqrt{3}}{\sqrt{4\pi}} \frac{\mathbf{p}_\nu}{p_\nu} \mathbf{e}_m^*$$

(the $\mathbf{e}_m$ are the spherical unit vectors; see Section 6.2) and

$$\begin{pmatrix} 1 & 1 & 1 \\ m & -m & 0 \end{pmatrix} = (-1)^{1-m} \frac{m}{\sqrt{6}} \tag{7.108}$$

(see Edmonds 1964), the angular function can be written as

$$\sum_m \begin{pmatrix} 1 & 1 & 1 \\ m & -m & 0 \end{pmatrix} \overset{*}{Y}{}_1^m(\hat{p}_e) \overset{*}{Y}{}_1^{-m}(\hat{p}_\nu) = \frac{1}{4\pi} \sqrt{\left(\frac{3}{2}\right)} \frac{1}{p_e p_\nu} \times \{p_{e_{-1}} p_{\nu_1} + p_{e_1} p_{\nu_{-1}}\}^* = \frac{1}{4\pi} \sqrt{\left(\frac{3}{2}\right)} \frac{i}{p_e p_\nu} \{\mathbf{p}_e \times \mathbf{p}_\nu\}_0.$$

The indices $1, 0, -1$ denote the corresponding components of the vectors in spherical notation.

Equation (7.106) can then be transformed to†

$$\sum_{M_i M_i'} \rho_\beta(M_i, M_i') a_{M_i} \delta_{M_i M_i'} = \frac{\sqrt{3}}{4\sqrt{2}} G_\beta^2 F_0 \sum_{KK'} (-1)^{J_f - J_i} \times (2J_i+1) G_1(J_i) (-1)^{2J_f} \begin{Bmatrix} J_i & J_i & 1 \\ K' & K & J_f \end{Bmatrix} i a_{11}^1(KK') \times \frac{\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)}{J p_e p_\nu} \tag{7.109}$$

† Here, $\mathbf{J}/J$ is a unit vector in the direction of the orientation.

because the polarization of the spin J_i is directed in 0-direction, i.e. in z-direction.

Analogous to other cases treated earlier we are now introducing particle parameters of the type

$$e^{(1)}_{KK'} = \frac{a^1_{11}(KK') + a^1_{11}(K'K)}{4i(1+\delta_{KK'})}. \tag{7.110}$$

Finally we get

$$\sum_{M_i M_i'} \rho_\beta(M_i M_i') a_{M_i} \delta_{M_i M_i} = \sqrt{\tfrac{3}{2}} G_\beta^2 F_0 \sqrt{(2J_i+1)}\, \tilde{C}^{(1)} G_1(J_i) \frac{\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)}{J p_e p_\nu} \tag{7.111}$$

and

$$\tilde{C}^{(1)} = \sum_{K \leqslant K'} (-1)^{K+K'+1} \Gamma_{KK'}(1, J_f, J_i) e^{(1)}_{KK'}. \tag{7.112}$$

Similarly, as in Section 7.2.2.1, we have here introduced the $\Gamma_{KK'}(1, J_f, J_i)$ coefficients (see eqn (7.40)).

It remains to derive special expressions for the particle parameters $e^{(1)}_{KK'}$, but by inserting the special values $k=1$, $k'=1$ and $N=1$ in eqns (7.13) and (7.14) and by applying eqn (7.110) it is a straightforward matter to calculate these particle parameters. The results are shown in Table 7.5. As before the leading terms are listed only.

The Coulomb functions occurring in the particle parameters $e^{(1)}_{KK'}$ are, however, of another type from those introduced before. In this case we have

$$sl_{k_e} = \frac{2\alpha_{-k_e}\alpha_{k_e} \sin(\Delta_{-k_e} - \Delta_{k_e})}{\alpha^2_{-k_e} + \alpha^2_{k_e}}. \tag{7.113a}$$

$$se_{k_e k_e'} = \frac{\alpha_{-k_e}\alpha_{-k_e'} \sin(\Delta_{-k_e} - \Delta_{-k_e'}) + \alpha_{k_e}\alpha_{k_e'} \sin(\Delta_{k_e} - \Delta_{k_e'})}{\alpha^2_{-1} + \alpha^2_1} \tag{7.113b}$$

$$s\hat{e}_{k_e k_e'} = \frac{\alpha_{-k_e}\alpha_{-k_e'} \sin(\Delta_{-k_e} - \Delta_{-k_e'}) - \alpha_{k_e}\alpha_{k_e'} \sin(\Delta_{k_e} - \Delta_{k_e'})}{\alpha^2_{-1} + \alpha^2_1}, \tag{7.113c}$$

i.e. the cosines of the phase differences have to be replaced by the sines of the phases differences. Later on in this chapter we shall discuss this point in more detail in order to make the reason for that difference more evident.

First we have, however, to normalize our correlation such that we obtain a form†

$$\omega_J(\Theta_e, \Theta_\nu, W_e) = 1 + d \frac{\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)}{J p_e p_\nu}. \tag{7.114}$$

† Of course, we have also to include the terms with $k'=0$, $k=1$, $N=1$ and $k'=1$, $k=0$, $N=1$. These terms have, however, been treated in earlier sections. They are, therefore, omitted here.

TABLE 7.5 *Particle parameters $e_{KK'}^{(1)}$ for electron–neutrino correlations of oriented nuclei*

$$
\begin{aligned}
e_{01}^{(1)}/L_0 = 2\{&-\sqrt{\tfrac{2}{3}}\, sl_1[M_1(1,1)m_0(1,1) - M_0(1,1)m_1(1,1)] \\
&-\sqrt{\tfrac{1}{3}}\, sl_1[M_1(1,2)m_0(1,1) - M_0(1,1)m_1(1,2)] \\
&+\sqrt{\tfrac{1}{3}}\, se_{12}[M_0(1,1)M_1(2,1) + m_0(1,1)m_1(2,1)] \\
&-\sqrt{\tfrac{1}{3}}\, s\hat{e}_{12}[M_0(1,1)m_1(2,1) + M_1(2,1)m_0(1,1)]\} \\
e_{11}^{(1)}/L_0 = 2\sqrt{\tfrac{3}{2}}\{&-sl_1[M_1(1,2)m_1(1,1) - M_1(1,1)m_1(1,2)] \\
&-se_{12}[M_1(1,1)M_1(2,1) + m_1(1,1)m_1(2,1)] \\
&+s\hat{e}_{12}[M_1(2,1)m_1(1,1) + M_1(1,1)m_1(2,1)]\} \\
e_{12}^{(1)}/L_0 = 2\sqrt{\tfrac{5}{6}}\{&-sl_1[M_2(1,2)m_1(1,1) - M_1(1,1)m_2(1,2)] \\
&+se_{12}[M_1(1,1)M_2(2,1) + m_1(1,1)m_2(2,1)] \\
&-s\hat{e}_{12}[M_2(2,1)m_1(1,1) + M_1(1,1)m_2(2,1)]\}
\end{aligned}
$$

Then we get

$$
\omega_J(\Theta_e, \Theta_\nu, W_e) = \frac{\sum_{M_i M_i'} \rho_\beta(M_i M_i') a_{M_i} \delta_{M_i M_i'}}{\dfrac{1}{2J_i+1}\dfrac{1}{(4\pi)^2}\displaystyle\int_{4\pi}\int_{4\pi} \sum_{M_i M_i'} \rho_\beta(M_i M_i')\delta_{M_i M_i'}\, \mathrm{d}\Omega_e\, \mathrm{d}\Omega_\nu} \tag{7.115}
$$

and for the coefficient d

$$
d = \frac{\sqrt{(2J_i+1)}}{S(W_e)} \sqrt{\tfrac{3}{2}} \tilde{C}^{(1)} G_1(J_i). \tag{7.116}
$$

Let us now turn briefly to a more general consideration of the correlation $\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)$. Observables of that type are usually investigated if answers to the question of time reversal invariance or non-invariance (in our case of the weak interaction) are tried to be obtained.† The observable $\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)$ changes the sign when a time reversal transformation is applied.‡ There-

† In the case where time reversal invariance of the weak interaction is not fulfilled the form factors or form factor coefficients are not real. This has the consequence that the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ are also not real, i.e. we have for example

$$M_0(1,1) = |M_0(1,1)|e^{i\varphi_1}$$

$$M_1(1,1) = |M_1(1,1)|e^{i\varphi_2}.$$

Then we would obtain one additional term in the particle parameter e_{01} of the form

$$e_{01}^{(1)}/L_0 = \ldots . -2\sqrt{\left(\frac{2}{3}\right)}\frac{p_e}{W_e}\Lambda_1 |M_0(1,1)||M_1(1,1)| \sin(\varphi_1 - \varphi_2).$$

That means our correlation is directly proportional to the sine of this phase difference.

‡ Under a time reversal transformation momentum and angular momentum transform

$$\mathbf{p} \to -\mathbf{p} \qquad \mathbf{J} \to -\mathbf{J}.$$

fore usually it is argued that an occurrence of finite correlations of that type, i.e. the occurrence of a finite value for d, gives an indication for a violation of the time reversal invariance principle.

By inspection of the equations derived above we see, however, that we obtain a finite value for d even if time reversal invariance holds. This is in our case an effect of the Coulomb interaction between beta-particle and charged nucleus. In the case $Z = 0$ the Coulomb phases Δ_{k_e} are zero and, therefore, all the particle parameters $e^{(1)}_{KK'}$ too. That means only in the case where the Coulomb interaction is absent we would have $d = 0$ if time reversal invariance is true. In practice, however, switching off the Coulomb interaction cannot be realized.

Thus a measurement of the correlation $\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)$ does not give an absolute test of the time reversal invariance or non-invariance. We always have to calculate or at least estimate the magnitude of d caused by the Coulomb effects on the basis of the equations derived in this section.

Now it will also become clearer why the Coulomb functions in this observable differ from those in all the observables discussed before. This observable is a time reversal invariance sensitive one, i.e. it is proportional to the sine of a phase φ (or phase difference) characterizing the violation of that principle. Conversely, the other observables already discussed are of the time reversal invariance insensitive type, i.e. they are proportional to the cosine of such a phase φ. This fact is also reflected in the dependence on the Coulomb phases where we have a proportionality either to the sine or to the cosine of the Coulomb phase differences.

7.2.5. Orientation of the nucleus in initial and final state observed, electron and neutrino not observed

As a consequence of beta-decay the relative population of the nuclear magnetic substates is changed. We have, therefore, the possibility to observe the difference in this population between initial and final nuclear states without observing electrons and neutrinos, i.e. a correlation of the type $\mathbf{J}_i \mathbf{J}_f$.

In practice an observable of this type can be measured by starting from a nucleus with a distinct orientation in the initial state and detecting the degree of orientation after the beta-decay by means of a subsequent gamma-radiation, i.e. by measuring the distribution of a gamma-radiation emitted from an oriented nucleus without detecting the intermediate beta-decay.

If beta-particles and neutrinos are not observed we have to integrate over all electron and neutrino directions and energies.

Starting from eqn (7.15) leads first to the following integrals

$$\int_{4\pi} \overset{*}{Y}{}_k^m(\hat{p}_e)\,\mathrm{d}\Omega_e = \sqrt{(4\pi)}\delta_{k0}\delta_{m0} \tag{7.117a}$$

$$\int_{4\pi} \overset{*}{Y}{}_{k'}^{m'}(\hat{p}_\nu)\,\mathrm{d}\Omega_\nu = \sqrt{(4\pi)}\delta_{k'0}\delta_{m'0} \tag{7.117b}$$

and, therefore, to the special case

$$\begin{aligned} k &= 0 \\ k' &= 0 \\ N &= 0. \end{aligned} \tag{7.118}$$

Then eqn (7.15) simplifies to (of course, the kinematical factor has also to be included)

$$\bar{\rho}_\beta(M_f M_f' M_i M_i') = \int_{4\pi}\int_{4\pi}\int_1^{W_0} \rho_\beta(M_f M_f' M_i M_i') p_e W_e$$

$$\times (W_0 - W_e)^2\,\mathrm{d}W_e\,\mathrm{d}\Omega_e\,\mathrm{d}\Omega_\nu = 4\pi^2 G_\beta^2 \sum_{KM} (-1)^{2J_f - M_f - M_f' + M}(2J_i + 1)$$

$$\times \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix}\begin{pmatrix} J_f & K & J_i \\ -M_f' & M & M_i' \end{pmatrix}\begin{pmatrix} 0 & K & K \\ 0 & -M & M \end{pmatrix}$$

$$\times \int_1^{W_0} a_{00}^0(K,K) F_0 p_e W_e (W_0 - W_e)^2\,\mathrm{d}W_e. \tag{7.119}$$

Taking into account the fact that

$$\begin{pmatrix} 0 & K & K \\ 0 & -M & M \end{pmatrix} = (-1)^{K-M}\frac{1}{\sqrt{(2K+1)}} \tag{7.120}$$

and that (see eqn (7.25))

$$b_{KK}^{(0)} = \frac{a_{00}^0(K,K)}{4} \tag{7.121}$$

we can further write

$$\bar{\rho}_\beta(M_f M_f' M_i M_i') = (4\pi)^2 G_\beta^2 \sum_{KM} (-1)^{2J_f - M_f - M_f' + K}$$

$$\times \frac{2J_i + 1}{\sqrt{(2K+1)}}\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix}\begin{pmatrix} J_f & K & J_i \\ -M_f' & M & M_i' \end{pmatrix}\overline{b_{KK}^{(0)}} f \tag{7.122}$$

where

$$\overline{b_{KK}^{(0)}} = \frac{1}{f}\int_1^{W_0} b_{KK}^{(0)} F_0 p_e W_e (W_0 - W_e)^2\,\mathrm{d}W_e \tag{7.123}$$

(f = integrated Fermi function, see eqn (7.61)).

Our gamma-ray distribution emitted from oriented nuclei is now given by

$$\omega_\gamma(\Theta_\gamma)=\frac{\sum_{M_fM_f'}\sum_{M_iM_i'}\bar{\rho}_\beta(M_fM_f'M_iM_i')\rho_\gamma(M_fM_f')a_{M_i}\delta_{M_iM_i'}}{\frac{1}{2J_i+1}\sum_{M_fM_f'}\sum_{M_iM_i'}\bar{\rho}_\beta(M_fM_f'M_iM_i')\delta_{M_fM_f'}\delta_{M_iM_i'}}$$

$$\times\frac{1}{\frac{1}{2J_f+1}\sum_{M_fM_f'}\rho_\gamma(M_fM_f')\delta_{M_fM_f'}}.$$

The density matrix $\rho_\gamma(M_f\ M_f')$ for the gamma-transition in the above equation has to be taken from eqn (7.77). The denominator has been, as before, introduced in order to have normalized distributions, such that

$$\omega_\gamma(\Theta_\gamma)=1+\sum_k h_kP_k(\cos\Theta_\gamma). \tag{7.125}$$

The a_{M_i} characterizes the relative population of the initial nuclear substates in the same way as discussed in Section 7.2.2.1.

For the numerator we then obtain

$$\sum_{M_fM_f'}\sum_{M_iM_i'}\bar{\rho}_\beta(M_fM_f'M_iM_i')\rho_\gamma(M_fM_f')a_{M_i}\delta_{M_iM_i'}=(4\pi)^{7/2}$$

$$\times G_\beta^2\sum_{KM}\sum_{LL'}\sum_{km}\sum_{M_fM_f'}\sum_{M_iM_i'}(-1)^{2J_f+J_\beta-1-M_f+K}$$

$$\times(-\tau)^k\frac{2J_i+1}{\sqrt{(2K+1)}}a_{M_i}\sqrt{\{(2L+1)(2L'+1)(2k+1)\}}$$

$$\times\begin{pmatrix}J_f & J_f & k\\ M_f & -M_f' & m\end{pmatrix}\begin{pmatrix}J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix}J_f & K & J_i\\ -M_f' & M & M_i'\end{pmatrix}$$

$$\times\overline{b_{KK}^{(0)}}f\begin{pmatrix}L & L' & k\\ 1 & -1 & 0\end{pmatrix}\begin{Bmatrix}J_f & J_f & k\\ L' & L & J_{ff}\end{Bmatrix}$$

$$\times\langle J_{ff}\,\|\,L\pi\,\|\,J_f\rangle\langle J_{ff}\,\|\,L'\pi'\,\|\,J_f\rangle\overset{*}{Y}{}_k^m(\hat{k}_\gamma)\delta_{M_iM_i'}. \tag{7.126}$$

Analogous to the treatment in earlier chapters we now proceed further by applying the relation (see Edmonds 1964)

$$\sum_{\substack{M\\M_fM_f'}}(-1)^{2J_f+K+M_f'+M_f+M}\begin{pmatrix}J_i & J_f & K\\ -M_i & M_f & -M\end{pmatrix}\begin{pmatrix}J_f & J_i & K\\ -M_f' & M_i' & M\end{pmatrix}$$

$$\times\begin{pmatrix}J_f & J_f & k\\ M_f' & -M_f & -m\end{pmatrix}=\begin{pmatrix}J_i & J_i & k\\ -M_i & M_i' & -m\end{pmatrix}\begin{Bmatrix}J_i & J_i & k\\ J_f & J_f & K\end{Bmatrix}. \tag{7.127}$$

Then eqn (7.126) can be written as follows:

$$\sum_{M_f M_f'} \sum_{M_i M_i'} \bar{\rho}_\beta(M_f M_f' M_i M_i') \rho_\gamma(M_f M_f') a_{M_i} \delta_{M_i M_i'}$$

$$= (4\pi)^{7/2} G_\beta^2 \sum_K \sum_{LL'} \sum_{km} \sum_{M_i M_i'} (-\tau)^k (-1)^{J_{ff}-1+M_i}$$

$$\times \frac{2J_i+1}{\sqrt{(2K+1)}} \sqrt{\{(2L+1)(2L'+1)(2k+1)\}} \overline{b_{KK}^{(0)}} f \begin{pmatrix} J_i & J_i & k \\ -M_i & M_i' & -m \end{pmatrix}$$

$$\times \begin{Bmatrix} J_i & J_i & k \\ J_f & J_f & K \end{Bmatrix} \begin{pmatrix} L & L' & k \\ 1 & -1 & 0 \end{pmatrix} \begin{Bmatrix} J_f & J_f & k \\ L' & L & J_{ff} \end{Bmatrix}$$

$$\times \delta_L \delta_{L'} Y_k^m(\hat{k}_\gamma) a_{M_i} \delta_{M_i M_i'}. \tag{7.128}$$

Because $M_i = M_i'$ we have $m = 0$. Describing now the degree of orientation by the statistical tensors $G_k(J_i)$ of eqn (7.34) and introducing the F-coefficients of eqn (7.80) we obtain

$$\sum_{M_f M_f'} \sum_{M_i M_i'} \bar{\rho}_\beta(M_f M_f' M_i M_i') \rho_\gamma(M_f M_f') a_{M_i} \delta_{M_i M_i'}$$

$$= (4\pi)^3 G_\beta^2 \sum_K \sum_{LL'} \sum_k (-\tau)^k (-1)^{k+J_i+J_f} \frac{(2J_i+1)}{\sqrt{\{(2K+1)(2J_f+1)\}}}$$

$$\times \overline{b_{KK}^{(0)}} f G_k(J_i) \begin{Bmatrix} J_i & J_i & k \\ J_f & J_f & K \end{Bmatrix} F_k(LL' \; J_{ff} \; J_f)$$

$$\times \langle J_{ff} \| L\pi \| J_f \rangle \langle J_{ff} \| L'\pi' \| J_f \rangle P_k(\cos \Theta_\gamma) \tag{7.129}$$

where Θ_γ is the angle between the direction of the emitted gamma-quantum and the axis of orientation.

It remains to evaluate the denominator in eqn (7.124) in order to normalize the correlation in the right way. Since

$$\frac{1}{2J_i+1} \sum_{M_f M_f'} \sum_{M_i M_i'} \bar{\rho}_\beta(M_f M_f' M_i M_i') \delta_{M_f M_f'} \delta_{M_i M_i'}$$

$$\times \frac{1}{2J_f+1} \sum_{M_f M_f'} \rho_\gamma(M_f M_f') \delta_{M_f M_f'} = (4\pi)^3 G_\beta^2 \sum_K \frac{(-1)^K}{\sqrt{(2K+1)}}$$

$$\times \overline{b_{KK}^{(0)}} f \frac{1}{2J_f+1} \sum_L \langle J_{ff} \| L\pi \| J_f \rangle^2 \tag{7.130}$$

we get finally

$$\omega_\gamma(\Theta_\gamma) = \frac{1}{S(W_e)} \sum_k (-\tau)^k H^{(k)} G_k(J_i) A_k(\gamma) P_k(\cos \Theta_\gamma) \tag{7.131}$$

where

$$H^{(k)} = \sum_K (-1)^{k+J_i+J_f} \frac{(2J_i+1)\sqrt{(2J_f+1)}}{\sqrt{(2K+1)}} \begin{Bmatrix} J_i & J_i & k \\ J_f & J_f & K \end{Bmatrix} \overline{b_{KK}^{(0)}}. \tag{7.132}$$

The quantities $A_k(\gamma)$ which contain all the information about the gamma-transition have been introduced before in eqn (7.80).

At this point it should be mentioned that this type of correlation is one which can also be observed for electron capture, while for most of the other correlations discussed before this is not possible.

7.3. Electron polarization

In the foregoing chapters we have only considered cases where neither the electron nor the neutrino polarization has been observed. In this section, on the other hand, we particularly turn to a discussion of the electron polarization, i.e. to a derivation of the relevant formulae for this observable. For that purpose we have to go back to the general density matrix $\rho_\beta(M_i M_i' M_f M_f' m_e m_e' m_\nu m_\nu')$ given in eqn (7.6) where no summation over any quantum number has been carried out.

The longitudinal polarization P_e, defined as the component of polarization along the direction of motion of the electron, is given by the expectation value of the operator $(\boldsymbol{\sigma}\mathbf{p}_e)/p_e$. It can, therefore, be written as†

$$P_e = \frac{\int_{4\pi} \sum_{M_f M_f'} \sum_{M_i M_i'} \sum_{m_e m_e'} \sum_{m_\nu m_\nu'} \rho_\beta(M_i M_i' M_f M_f' m_e m_e' m_\nu m_\nu')}{\int_{4\pi} \sum_{M_f M_f'} \sum_{M_i M_i'} \sum_{m_e m_e'} \sum_{m_\nu m_\nu'} \rho_\beta(M_i M_i' M_f M_f' m_e m_e' m_\nu m_\nu')}$$

$$\times \frac{\langle \chi_e^{m_e'} \mid \boldsymbol{\sigma}\mathbf{p}_e \mid \chi_e^{m_e} \rangle \delta_{M_f M_f'} \delta_{M_i M_i'} \delta_{m_\nu m_\nu'} \, d\Omega_\nu}{\delta_{M_f M_f'} \delta_{M_i M_i'} \delta_{m_e m_e'} \delta_{m_\nu m_\nu'} \, d\Omega_\nu p_e} = \frac{\int \mathrm{tr}(\rho_\beta \boldsymbol{\sigma}\mathbf{p}_e) \, d\Omega_\nu}{p_e \int \mathrm{tr}(\rho_\beta) \, d\Omega_\nu}. \quad (7.133)$$

Starting with the integration over the directions and with the summation of the quantum states of the unobserved neutrino we get

$$\int_{4\pi} \overset{*}{Y}{}_{k'}^{m'}(\hat{p}_\nu) \, d\Omega_\nu = \sqrt{(4\pi)} \delta_{k'0} \delta_{m'0} \quad (7.134)$$

and then

$$\sum_{\substack{m_\nu \\ m_\nu'}} (-1)^{m_\nu' - \mu_\nu'} \sqrt{\{(2l_\nu + 1)(2l_\nu' + 1)(2j_\nu + 1)(2j_\nu' + 1)\}}$$

$$\times \begin{pmatrix} l_\nu & \frac{1}{2} & j_\nu \\ \mu_\nu - m_\nu & m_\nu & -\mu_\nu \end{pmatrix} \begin{pmatrix} l_\nu' & \frac{1}{2} & j_\nu' \\ \mu_\nu' - m_\nu' & m_\nu' & -\mu_\nu' \end{pmatrix}$$

$$\times \begin{pmatrix} l_\nu & l_\nu' & 0 \\ \mu_\nu - m_\nu & m_\nu' - \mu_\nu' & 0 \end{pmatrix} \begin{pmatrix} l_\nu & l_\nu' & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta_{m_\nu m_\nu'} = \delta_{j_\nu j_\nu'} \delta_{l_\nu l_\nu'} \delta_{\mu_\nu \mu_\nu'}. \quad (7.135)$$

† The wave functions $\chi_e^{m_e}$ are the simple two component spinor functions introduced in eqn (6.32).

Furthermore, the summation over the unobserved nuclear quantum states can easily be carried out

$$\sum_{M_f M_f'} \sum_{M_i M_i'} \sqrt{\{(2K+1)(2K'+1)\}}(-1)^{2J_f - M_f - M_f' + K + K' + M + M'} \times \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \begin{pmatrix} J_f & K' & J_i \\ -M_f' & M' & M_i' \end{pmatrix} \delta_{M_f M_f'} \delta_{M_i M_i'} = \delta_{KK'} \delta_{MM'}. \quad (7.136)$$

Thus the density matrix given in eqn (7.6) can then be simplified to

$$\begin{aligned} &\frac{1}{2J_i+1} \int_{4\pi} \sum_{M_f M_f'} \sum_{M_i M_i'} \sum_{m_\nu m_\nu'} \rho_\beta(M_i M_i' M_f M_f' m_e m_e' m_\nu m_\nu') \delta_{M_f M_f'} \delta_{M_i M_i'} \\ &\quad \times \delta_{m_\nu m_\nu'} \, \mathrm{d}\Omega_\nu = 2\pi^{3/2} G_\beta^2 \sum_{KM} \sum_{\substack{\kappa_e \mu_e \\ \kappa_e' \mu_e'}} \sum_{\kappa_\nu \mu_\nu} \sum_{km} (-1)^{j_e + j_e' + l_e + l_e' - 1 + m_e' - \mu_e'} \frac{1}{p_e^2} \\ &\quad \times \sqrt{\{(2j_e+1)(2j_e'+1)(2l_e+1)(2l_e'+1)(2k+1)\}} \\ &\quad \times \begin{pmatrix} j_e & K & j_\nu \\ -\mu_e & -M & -\mu_\nu \end{pmatrix} \begin{pmatrix} j_e' & K & j_\nu \\ -\mu_e' & -M & -\mu_\nu \end{pmatrix} \\ &\quad \times \begin{pmatrix} l_e & \frac{1}{2} & j_e \\ \mu_e - m_e & m_e & -\mu_e \end{pmatrix} \begin{pmatrix} l_e' & \frac{1}{2} & j_e' \\ \mu_e' - m_e' & m_e' & -\mu_e' \end{pmatrix} \\ &\quad \times \begin{pmatrix} l_e & l_e' & k \\ \mu_e - m_e & m_e' - \mu_e' & m \end{pmatrix} \begin{pmatrix} l_e & l_e' & k \\ 0 & 0 & 0 \end{pmatrix} Y_k^{m*}(\hat{p}_e) \\ &\quad \times \alpha_{\kappa_e} \alpha_{\kappa_e'} \{M_K(k_e, k_\nu) M_K(k_e', k_\nu) + \mathrm{sign}(\kappa_e)\mathrm{sign}(\kappa_e') m_K(k_e, k_\nu) \\ &\quad \times m_K(k_e', k_\nu) + \mathrm{sign}(\kappa_e) M_K(k_e', k_\nu) m_K(k_e, k_\nu) + \mathrm{sign}(\kappa_e') M_K(k_e, k_\nu) \\ &\quad \times m_K(k_e', k_\nu)\} \cdot \mathrm{e}^{-\mathrm{i}(\Delta_{\kappa_e} - \Delta_{\kappa_e'})}. \end{aligned} \quad (7.137)$$

For simplicity we shall denote this simplified density matrix by $\rho_\beta(m_e \; m_e')$ in the following. For the next step we are able to carry out explicitly the summation over μ_ν, M and μ_e since

$$\sum_{M\mu_\nu} \begin{pmatrix} K & j_\nu & j_e \\ -M & -\mu_\nu & -\mu_e \end{pmatrix} \begin{pmatrix} K & j_\nu & j_e' \\ -M & -\mu_\nu & -\mu_e' \end{pmatrix} = \frac{1}{2j_e+1} \delta_{j_e j_e'} \delta_{\mu_e \mu_e'} \quad (7.138)$$

and

$$\begin{aligned} &\sum_{\mu_e} (-1)^{l_e + l_e' + j_e - \mu_e - m_e' - m_e} \\ &\quad \times \begin{pmatrix} \frac{1}{2} & l_e' & j_e \\ m_e' & \mu_e - m_e' & -\mu_e \end{pmatrix} \begin{pmatrix} l_e & \frac{1}{2} & j_e \\ m_e - \mu_e & -m_e & \mu_e \end{pmatrix} \begin{pmatrix} l_e & l_e' & k \\ \mu_e - m_e & m_e' - \mu_e & m \end{pmatrix} \\ &\qquad = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & k \\ m_e' & -m_e & m \end{pmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & k \\ l_e & l_e' & j_e \end{Bmatrix}. \end{aligned} \quad (7.139)$$

Then $\rho_\beta(m_e\ m'_e)$ reads as

$$\rho_\beta(m_e\ m'_e) = 2\pi^{3/2} G_\beta^2 \sum_K \sum_{\substack{\kappa_e \\ \kappa'_e}} \sum_{\kappa_\nu} \sum_{km} (-1)^{j_e + l_e + l'_e + m_e}$$
$$\times \frac{1}{p_e^2} \sqrt{\{(2l_e+1)(2l'_e+1)(2k+1)\}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & k \\ m'_e & -m_e & m \end{pmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & k \\ l_e & l'_e & j_e \end{Bmatrix}$$
$$\times \begin{pmatrix} l_e & l'_e & k \\ 0 & 0 & 0 \end{pmatrix} \alpha_{\kappa_e} \alpha_{\kappa'_e} \{M_K^2(k_e, k_\nu) + \mathrm{sign}(\kappa_e)\mathrm{sign}(\kappa'_e) m_K^2(k_e, k_\nu)$$
$$+ [\mathrm{sign}(\kappa_e) + \mathrm{sign}(\kappa'_e)] M_K(k_e, k_\nu) m_K(k_e, k_\nu)\} e^{-i(\Delta_{\kappa_e} - \Delta_{\kappa'_e})}$$
$$\times \overset{*}{Y}{}_k^m(\hat{p}_e). \tag{7.140}$$

From eqn (7.133) we see that we have to calculate the quantity

$$P_e = \frac{\sum_{m_e m'_e} \rho_\beta(m_e\ m'_e) \langle \chi_e^{m'_e} | \boldsymbol{\sigma} \mathbf{p}_e | \chi_e^{m_e} \rangle}{p_e \sum_{m_e m'_e} \rho_\beta(m_e\ m'_e) \delta_{m_e m'_e}}. \tag{7.141}$$

By application of the Wigner–Eckart theorem we have

$$\langle \chi_e^{m'_e} | \boldsymbol{\sigma} | \chi_e^{m_e} \rangle = \sum_{m'} (-1)^{\frac{1}{2} - m'_e} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -m'_e & m' & m_e \end{pmatrix} (\tfrac{1}{2} \| \boldsymbol{\sigma} \| \tfrac{1}{2}) \overset{*}{\mathbf{e}}_{m'} \tag{7.142}$$

(see eqn (6.42c)), and therefore

$$\langle \chi_e^{m'_e} | \boldsymbol{\sigma} \mathbf{p}_e | \chi_e^{m_e} \rangle = \mathbf{p}_e \langle \chi_e^{m'_e} | \boldsymbol{\sigma} | \chi_e^{m_e} \rangle = \sum_{m'} (-1)^{\frac{1}{2} - m'_e}$$
$$\times \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -m'_e & m' & m_e \end{pmatrix} \sqrt{6}\, \overset{*}{p}_{e_{m'}}. \tag{7.143}$$

Now the possibility occurs to perform the summation over m_e and m'_e. By inserting the above result in eqn (7.141) we come to the following relation

$$\sum_{m_e m'_e} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m'_e & -m_e & m' \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & k \\ m'_e & -m_e & m \end{pmatrix} = \tfrac{1}{3} \delta_{k1} \delta_{m,-m'} \tag{7.144}$$

with the consequence that $k = 1$. Since

$$\overset{*}{Y}{}_1^m(\hat{p}_e) = \sqrt{\left(\frac{3}{4\pi}\right)} \frac{\overset{*}{p}_{e_m}}{p_e} \tag{7.145}$$

and

$$\sum_{m'} (-1)^{m'} \overset{*}{p}_{e_{m'}} \overset{*}{p}_{e_{-m'}} = \mathbf{p}_e^2 \tag{7.146}$$

we obtain then ($\mathbf{p}_e^2 = p_e^2$)

$$\frac{1}{p_e}\sum_{m_e m_e'}\rho_\beta(m_e\ m_e')\langle\chi_e^{m_e'}|\,\boldsymbol{\sigma}\mathbf{p}_e\,|\,\chi_e^{m_e}\rangle = \pi G_\beta^2\sqrt{2}$$
$$\times\sum_K\sum_{\substack{\kappa_e\\ \kappa_e'}}\sum_{\kappa_\nu}(-1)^{j_e+l_e+l_e'+\frac{1}{2}}\sqrt{\{3(2l_e+1)(2l_e'+1)\}}$$
$$\times\begin{pmatrix} l_e & l_e' & 1\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ l_e & l_e' & j_e\end{Bmatrix}\alpha_{\kappa_e}\alpha_{\kappa_e'}\{M_K^2(k_e, k_\nu)$$
$$+\mathrm{sign}(\kappa_e)\mathrm{sign}(\kappa_e')m_K^2(k_e, k_\nu)+[\mathrm{sign}(\kappa_e)+\mathrm{sign}(\kappa_e')]$$
$$\times M_K(k_e, k_\nu)m_K(k_e, k_\nu)\}e^{-i(\Delta_{\kappa_e}-\Delta_{\kappa_e'})}. \qquad (7.147)$$

This expression can, however, be further simplified by evaluating the $3j$- and $6j$-symbol explicitly. Doing so we find that

$$(-1)^{j_e+l_e+l_e'+\frac{1}{2}}\sqrt{\{3(2l_e+1)(2l_e'+1)\}}\begin{pmatrix} l_e & l_e' & 1\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ l_e & l_e' & j_e\end{Bmatrix} = -\frac{1}{\sqrt{2}} \qquad (7.148)$$

if

$$\begin{matrix}\kappa_e = k_e\\ \kappa_e' = -k_e\end{matrix}\quad\text{or}\quad\begin{matrix}\kappa_e = -k_e\\ \kappa_e' = k_e\end{matrix}.$$

Otherwise the expression of eqn (7.148) is equal to zero.

Thus we obtain

$$\frac{1}{p_e}\sum_{m_e m_e'}\rho_\beta(m_e\ m_e')\langle\chi_e^{m_e'}\,|\boldsymbol{\sigma}\mathbf{p}_e|\,\chi_e^{m_e}\rangle = -4\pi G_\beta^2 F_0 L_0$$
$$\times\frac{p_e}{W_e}\sum_K\sum_{k_e k_\nu}\lambda_{k_e}\Lambda_{k_e}[M_K^2(k_e, k_\nu)-m_K^2(k_e, k_\nu)]. \qquad (7.149)$$

On the other hand we have

$$\sum_{m_e m_e'}\rho_\beta(m_e\ m_e')\delta_{m_e m_e'} = 4\pi G_\beta^2 F_0 L_0\sum_K\sum_{k_e k_\nu}\lambda_{k_e}$$
$$\times\{M_K^2(k_e, k_\nu)+m_K^2(k_e, k_\nu)-2\frac{\gamma_{k_e}\mu_{k_e}}{k_e W_e}M_K(k_e, k_\nu)m_K(k_e, k_\nu)\}$$
$$= 4\pi G_\beta^2 F_0 L_0 C(W_e) \qquad (7.150)$$

(see eqn (7.56)).

Therefore we obtain finally

$$P_e = -\frac{1}{C(W_e)}\frac{p_e}{W_e}\sum_K\sum_{k_e k_\nu}\lambda_{k_e}\Lambda_{k_e}[M_K^2(k_e, k_\nu)-m_K^2(k_e, k_\nu)]. \qquad (7.151)$$

By using the same method as above we could, of course, also calculate the polarization of the antineutrino. Since our starting equation (7.6) is

completely symmetrical as far as electron and neutrino quantum numbers are concerned (with exception of the phase factor $(-1)^{j_e+l'_e+l_e+l'_e-1}$) the method of calculation is completely analogous to that outlined above. All we have to do is to interchange electron and neutrino quantum numbers (for example in eqn (7.148)). Then we end up with the result

$$P_{\bar{\nu}} = 1 \tag{7.152}$$

(the sign difference comes from the different phase factor $(-1)^{j_\nu+l'_\nu}=-1$). Of course, the result $|P_{\bar{\nu}}|=1$ follows also trivially from the fact that the antineutrino is massless and therefore always completely polarized.

7.4. Relation between β^--decay and β^+-decay or electron capture, respectively

7.4.1. Relation between β^--decay and β^+-decay

Up to now all the calculations of the observables have been carried out for the β^--decay. We have, therefore, now to demonstrate that this restriction was justified. Thus it is our aim to show what kind of substitutions have to be made in the formulae for the observables when we are going over from β^--decay to β^+-decay.

For that purpose we have first to compare the transition matrices T for both types of decay (see eqns (6.132) and (6.148)).

From eqn (6.148) we see immediately that analogous to eqn (6.151) the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ for β^+-decay are given by

$$\begin{aligned}
&-\mathrm{sign}(\kappa_e)\alpha_{\kappa_e}\{M_K(k_e,k_\nu)+\mathrm{sign}(\kappa_e)m_K(k_e,k_\nu)\}\\
&=\frac{2}{\pi}\frac{1}{\sqrt{(2K+1)}}\sum_{Ls}(-1)^{K-L-s+1}\int_0^\infty q^2\,\mathrm{d}q\int_0^\infty r^2\,\mathrm{d}r\frac{(qR)^L}{(2L+1)!!}j_L(qr)\\
&\quad\times F_{KLs}(q^2)\{g_{\kappa_e}(-Z)[j_l(p_\nu r)G_{KLs}(\kappa_e,\kappa_\nu)+j_{l'}(p_\nu r)\\
&\quad\times G_{KLs}(\kappa_e,-\kappa_\nu)]-\mathrm{sign}(\kappa_e)f_{\kappa_e}(-Z)[j_l(p_\nu r)G_{KLs}(-\kappa_e,\kappa_\nu)\\
&\quad+j_{l'}(p_\nu r)G_{KLs}(-\kappa_e,-\kappa_\nu)]\}.
\end{aligned}\tag{7.153}$$

By comparing this formula with that for β^--decay given in eqn (6.151) we have first to make the following substitution (apart from other substitutions within the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$, which will be treated later on):

$$\begin{aligned}
&\beta^-\text{-decay}\rightarrow\beta^+\text{-decay}\\
&\left.\begin{matrix}M_K(k_e,k_\nu)\\ m_K(k_e,k_\nu)\end{matrix}\right\}\rightarrow\frac{\mathrm{sign}(\kappa_e)}{\mathrm{sign}(\kappa_\nu)}\left\{\begin{matrix}M_K(k_e,k_\nu)\\ m_K(k_e,k_\nu)\end{matrix}\right.
\end{aligned}\tag{7.154}$$

It is

$$\frac{\text{sign}(\kappa_e)}{\text{sign}(\kappa_\nu)} = (-1)^{j_e - i_\nu - l_e + l_\nu} = (-1)^{k_e + k_\nu + l_e + l_\nu}. \tag{7.155}$$

Let us now, however, consider the influence of the phase factors $(-1)^{k_e+k_\nu}$ and $(-1)^{l_e+l_\nu}$ separately. In the density matrix, i.e. in the formulae for the observables, the phase factor $(-1)^{l_e+l_\nu}$ becomes

$$(-1)^{l_e+l'_e+l_\nu+l'_\nu} \tag{7.156}$$

so that all observables, where $l_e + l'_e + l_\nu + l'_\nu$ is odd, change their sign when we are going from β^--decay to β^+-decay. If we now inspect the relevant formulae we find that $3j$-symbols of the form

$$\begin{pmatrix} l_e & l'_e & k \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} l_\nu & l'_\nu & k' \\ 0 & 0 & 0 \end{pmatrix}$$

occur in most cases. These $3j$-symbols are, on the other hand, only different from zero if $l_e + l'_e + k$ and $l_\nu + l'_\nu + k'$ are even. Thus, in the case electron observed and neutrino not observed and, vice versa, neutrino observed and electron not observed, all observables, where k is odd or, vice versa, where k' is odd, change their sign when we are transforming the formulae from β^--decay to β^+-decay.

For example, these are the odd terms in the electron distribution emitted from oriented nuclei and in the beta–gamma (CP) correlation, or the electron polarization. We note that these terms are especially the parity dependent terms, i.e. terms which would be zero in the case where parity conservation holds. However, we see immediately that this phase factor does not influence observables like shape factor, electron neutrino correlations, $\mathbf{J}_i\mathbf{J}_f$ correlations etc. In a first conclusion we can consequently derive the formulae for β^+-decay from those for β^--decay by making the following substitutions (by comparing eqns (6.151) and (7.153):

β^--decay	$\rightarrow$	β^+-decay	
$j_\Gamma(p_\nu r)$	$\rightarrow$	$-j_\Gamma(p_\nu r)$	(7.157*a*)
$f_{\kappa_e}(Z)$	$\rightarrow$	$-f_{\kappa_e}(-Z)$	(7.157*b*)
$F_{KLs}(q^2)$	$\rightarrow$	$(-1)^{1-s}F_{KLs}(q^2)$	(7.157*c*)
$M_K(k_e, k_\nu)$	$\rightarrow$	$(-1)^{k_e+k_\nu}M_K(k_e, k_\nu)$	(7.157*d*)
G	$\rightarrow$	$-G$	(7.157*e*)

where G represents the terms which are due to parity nonconservation.

This prescription, however, looks a little complicated. Thus we will try

to simplify it further. For that purpose we have to consider in a little more detail what kind of form factors or form factor coefficients, and how many, are occurring in the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ (note that we sum over L and s). In Section 6.2 we explicitly calculated the form factors by using some simple nuclear models. In this context (see eqns (6.66) and (6.72)) we found that the form factors contain some special operators acting on the jth-nuclear wave functions (in the impulse approximation). These operators together with the corresponding parity selection rules are listed in the following:†

$$F_{KK0}\begin{cases} Y_K^M(\hat{r}_j) & \pi_i\pi_f = (-1)^K \quad (7.158a) \\ \lambda Y_K^M(\hat{r}_j)\mathbf{p}_j\boldsymbol{\sigma}_j & \pi_i\pi_f = (-1)^{K+1} \quad (7.158b) \end{cases}$$

$$F_{KL1}\begin{cases} \lambda\{Y_L(\hat{r}_j)\otimes\boldsymbol{\sigma}_j\}^{(K)} & \pi_i\pi_f = (-1)^L \quad (7.158c) \\ \{Y_L(\hat{r}_j)\otimes\mathbf{p}_j\}^{(K)} & \pi_i\pi_f = (-1)^{L+1}. \quad (7.158d) \end{cases}$$

π_i and π_f are the parities of the initial and final nuclear states. Thus in every $M_K(k_e, k_\nu)$ or $m_K(k_e, k_\nu)$ we always have a maximum of four different form factors, i.e.

for $\pi_i\pi_f = (-1)^K$

$$^{V}F_{KK0} \quad ^{V}F_{K\,K-1\,1}, \quad ^{A}F_{KK1} \quad ^{V}F_{K\,K+1\,1}; \qquad (7.159)$$

for $\pi_i\pi_f = (-1)^{K+1}$

$$^{A}F_{KK0} \quad ^{A}F_{K\,K-1\,1}, \quad ^{V}F_{KK1} \quad ^{A}F_{K\,K+1\,1}. \qquad (7.160)$$

Next we have to consider the geometrical coefficients $G_{KLs}(\kappa_e, \kappa_\nu)$ in eqn (6.151) or eqn (7.153). In general only two of the four different coefficients $G_{KLs}(\kappa_e, \kappa_\nu)$ are here different from zero for a special set of parameters KLs. Since the $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ are defined in such a way that only the k_e and k_ν are of importance, and not the κ_e and κ_ν, we can assume $\kappa_e = k_e$ and $\kappa_\nu = k_\nu$ for the further discussion without any loss of generality. Then we obtain from eqn (6.141)

$$G_{KLs}\begin{pmatrix} k_e, & k_\nu \\ -k_e, & -k_\nu \end{pmatrix} \neq 0 \quad \text{only for } k_e + k_\nu + L = \text{even}, \qquad (7.161a)$$

$$G_{KLs}\begin{pmatrix} -k_e, & k_\nu \\ k_e, & -k_\nu \end{pmatrix} \neq 0 \quad \text{only for } k_e + k_\nu + L = \text{odd}. \qquad (7.161b)$$

The substitutions, eqns (7.157a) and (7.157b), however, can then be replaced by a change of sign for all terms multiplied by

$$G_{KLs}\begin{pmatrix} k_e, & -k_\nu \\ -k_e, & k_\nu \end{pmatrix}.$$

† Terms multiplied by λ are axial vector terms, the others are vector terms.

This latter point together with the substitutions, eqns (7.157*c*) and (7.157*d*), leads to a new rule which reads as

$$\begin{aligned} \beta^-\text{-decay} &\rightarrow \beta^+\text{-decay} \\ F_{KLs}(q^2) &\rightarrow (-1)^{L-s}F_{KLs}(q^2). \end{aligned} \tag{7.162}$$

By inspection of eqn (7.159) and (7.160) we find on the other hand that this prescription does not differ from the substitution

$$^{V}F_{KLs} \rightarrow {}^{V}F_{KLs} \tag{7.163a}$$

$$^{A}F_{KLs} \rightarrow -{}^{A}F_{KLs} \tag{7.163b}$$

(note that the parity selection rule in the eqns (7.159) and (7.160) guarantees that no overall phase factor $(-1)^K$ survives).

Finally we end up with a simpler relation between β^--decay and β^+-decay which can be written as

$$\beta^-\text{-decay} \rightarrow \beta^+\text{-decay}$$

$$\lambda \rightarrow -\lambda \tag{7.164a}$$

$$Z \rightarrow -Z \tag{7.164b}$$

$$G \rightarrow -G. \tag{7.164c}$$

This is the well-known prescription mentioned in the earlier text books and publications concerning beta-decay (see, for example, Weidenmüller 1961; Schopper 1966; Behrens and Jänecke 1969)

7.4.2. Relation between β^-decay or β^+-decay and electron capture

In the case of electron capture the quantities $M_K(k_x, k_\nu)$ and $m_K(k_x, k_\nu)$ can be derived from eqns (6.149) and (6.151). The result is

$$\begin{aligned} &\beta_{\kappa_x}\{M_K(k_x, k_\nu)+\text{sign}(\kappa_x)m_K(k_x, k_\nu)\} \\ &=\frac{2}{\pi}\frac{1}{\sqrt{(2K+1)}}\sum_{Ls}(-1)^{K-L-s}\int_0^\infty q^2\,dq\int_0^\infty r^2\,dr\frac{(qR)^L}{(2L+1)!!}F_{KLs}(q^2)j_L(qr) \\ &\quad\times\{g_{\kappa_x}(Z')[j_l(p_\nu r)G_{KLs}(\kappa_x, \kappa_\nu)+j_T(p_\nu r)G_{KLs}(\kappa_x, -\kappa_\nu)] \\ &\quad+\text{sign}(\kappa_x)f_{\kappa_x}(Z')[j_l(p_\nu r)G_{KLs}(-\kappa_x, \kappa_\nu)+j_T(p_\nu r) \\ &\quad\times G_{KLs}(-\kappa_x, -\kappa_\nu)]\}. \end{aligned} \tag{7.165}$$

By comparing this equation with eqn (6.151) we see that now the

following substitutions have to be made.

$$\begin{aligned}
\beta^-\text{-decay} &\rightarrow \text{electron capture} \\
j_l(p_\nu r) &\rightarrow j_l(p_\nu r) && (7.166a) \\
j_T(p_\nu r) &\rightarrow -j_T(p_\nu r) && (7.166b) \\
F_{KLs}(q^2) &\rightarrow (-1)^{1-s}F_{KLs}(q^2) && (7.166c) \\
M_K(k_e, k_\nu) &\rightarrow (-1)^{k_\nu+l_\nu}M_K(k_x, k_\nu). && (7.166d)
\end{aligned}$$

As discussed in detail in the foregoing section the effect of the latter substitution, eqn (7.166*d*), can be split off into two factors, the effect of

$$(-1)^{k_\nu+k'_\nu} \quad \text{and} \quad (-1)^{l_\nu+l'_\nu}.$$

The phase factor $(-1)^{l_\nu+l'_\nu}$ leads, as shown before, to a sign change of all parity dependent terms i.e. $G \rightarrow -G$.

Therefore, we have the substitution rules

$$\begin{aligned}
\beta^-\text{-decay} &\rightarrow \text{electron capture} \\
j_l(p_\nu r) &\rightarrow j_l(p_\nu r) && (7.167a) \\
j_T(p_\nu r) &\rightarrow -j_T(p_\nu r) && (7.167b) \\
F_{KLs}(q^2) &\rightarrow (-1)^{1-s}F_{KLs}(q^2). && (7.167c) \\
M_K(k_e, k_\nu) &\rightarrow (-1)^{k_\nu}M_K(k_x, k_\nu) && (7.167d) \\
G &\rightarrow -G. && (7.167e)
\end{aligned}$$

For the electron capture decay constant, which is usually only observed, the first three substitution rules are of importance only.†

In the case where distributions and correlations like neutrino distribution emitted from oriented nuclei, neutrino–gamma (CP) correlations etc. are observed, the last two prescriptions have to be applied too. In all cases, however, one has to start from the formulae derived for those β^-decay observables where the electron is not observed (see Sections 7.2.3 and 7.2.5). In addition to the substitution rules listed above, the Coulomb functions (for example in Table 7.3) have to be replaced by

† For the decay constant the prescription given in eqns (7.167*a*–c) is identical with that mentioned earlier (see Behrens and Jänecke 1969)

$$\begin{aligned}
\beta^-\text{-decay} &\rightarrow \text{electron capture} \\
p_\nu &\rightarrow -p_\nu \\
F_{KLs}(q^2) &\rightarrow (-1)^{K-s}F_{KLs}(q^2).
\end{aligned}$$

other ones, i.e.

$$\begin{aligned} \beta^-\text{-decay} &\rightarrow \text{electron capture} \\ \lambda_{k_e} &\rightarrow \beta^2_{\kappa_x} \\ \frac{\gamma_{k_e}\mu_{k_e}}{k_e W_e} &\rightarrow -\text{sign}(\kappa_x)\beta^2_{\kappa_x} \end{aligned} \tag{7.169}$$

where the quantities $\beta^2_{\kappa_x}$ are the amplitudes of the bound-state electron radial wave functions (see Section 4.6.1) if the electron radial wave functions are defined in the same way as before† (see eqns (4.17*a–d*)). The quantity κ_x denotes here the total angular momentum j_x and the orbital angular momentum l_x of the bound electron in the parent atom.

The values of κ_x ($k_x = |\kappa_x|$) for bound electrons in the different shells (K, L_1, L_2, etc.) from which the electron is captured are as follows:

$$\begin{array}{llll} \text{K}(1s) & \kappa_x = -1 & \text{M}_1(3s) & \kappa_x = -1 \\ \text{L}_1(2s) & \kappa_x = -1 & \text{M}_2(3p_{1/2}) & \kappa_x = +1, \\ \text{L}_2(2p_{1/2}) & \kappa_x = +1 & \text{M}_3(3p_{3/2}) & \kappa_x = -2, \\ \text{L}_3(2p_{3/2}) & \kappa_x = -2 & \text{M}_4(3d_{3/2}) & \kappa_x = +2, \\ & & \text{M}_5(3d_{5/3}) & \kappa_x = -3. \end{array} \tag{7.170}$$

As already discussed in Section 5.3, the phase space in electron capture decay is different from that in β^-- and β^+-decay because we have only two particles in the final state instead of three in the other cases. This has the consequence that the neutrino momentum p_ν has a definite fixed value which is given by (see eqn (5.79))

$$\begin{aligned} p_{\nu_x} &= \Delta + m_e - E'_B \\ &= W_0 + W'_x \end{aligned} \tag{7.171}$$

where W_0 is the maximal total positron decay energy and $W'_x = m_e - |E'_x|$.‡ $E'_B = E'_x$ is the binding energy of the bound electron (in the daughter atom).

We have also to substitute the electron momentum and energy by a fixed value, i.e.

$$\begin{aligned} \beta^-\text{-decay} &\rightarrow \text{electron capture} \\ p_e &\rightarrow p_x \\ W_e &\rightarrow W_x. \end{aligned} \tag{7.172}$$

† That means the amplitudes α_{κ_e} have to be replaced by β_{κ_x}.

‡ If energetically possible we have β^+-decay and electron capture decay as competing processes at the same time.

where $p_x = \sqrt{(W_x^2 - m_e^2)}$ and $W_x = m_e - |E_x|$. E_x is here, however, the binding energy of the bound electron in the parent atom.

As already mentioned the most important observable of electron capture is the decay constant or half-life. Before closing this section we will, therefore, give the explicit expression for this quantity. Starting from eqn (7.86) the analogue to the shape $C(p_\nu)$ in β^--decay can now be written as

$$C_x(p_{\nu_x}) = \sum_{K,\, k_\nu} \{M_K(k_x, k_\nu) + \mathrm{sign}(\kappa_x) m_K(k_x, k_\nu)\}^2. \tag{7.173}$$

It should be mentioned that β_x^2 has been taken out from the shape factor in this case. Taking additionally into account the kinematical aspects discussed in eqns (5.75) to (5.79) we obtain

$$\frac{dW}{dt} = \lambda_x = \frac{\ln 2}{t_x} = \frac{G_\beta^2}{(2\pi)^2} \sum_{K,\, k_\nu} \{M_K(k_x, k_\nu) + \mathrm{sign}(\kappa_x) \times m_K(k_x, k_\nu)\}^2 \beta_x^2 p_{\nu_x}^2. \tag{7.174}$$

Usually this is written as follows:

$$\lambda_{EC} = \frac{\ln 2}{t_{EC}} = \frac{G_\beta^2}{2\pi^3} \sum_x n_x C_x f_x \tag{7.175}$$

where f_x corresponds to the integrated Fermi function of β-decay and reads as†

$$f_x = \frac{\pi}{2} \beta_x^2 p_{\nu_x}^2. \tag{7.176}$$

n_x makes allowance for the fact that the shells in the atomic cloud are not always completely filled. For closed shells n_x equals 1. For partially filled shells, n_x is equal to the relative occupation number of electrons in the shell.

Of course, it is also possible to derive the formulae for electron capture from those for β^+-decay. In that case it follows from the foregoing discussion (for example from a comparison of the eqns (7.157*a–c*) with the eqns (7.167*a–d*) and from the fact that $(-1)^{k_x + k_x'} = (-1)^{2k_x} = 1$) that the relation between β^+-decay and electron capture is established by the

† A complete description of the initial and final states must include the electrons of the atomic cloud. Since the nuclear charge and the number of electrons are different in the initial and final states, the atomic electron wave functions of these two states are not orthogonal, and the overlap between them is not perfect. These so-called exchange and overlap effects in the atomic cloud have been taken into account in Section 4.6.2. These effects, however, influence the electron capture and introduce additional factors in this formula.

substitutions

$$\begin{aligned} \beta^+\text{-decay} &\rightarrow \text{electron capture} \\ f_{\kappa_e}(-Z) &\rightarrow -f_{\kappa_x}(Z') \qquad (7.177a) \\ g_{\kappa_e}(-Z) &\rightarrow g_{\kappa_x}(Z') \qquad (7.177b) \end{aligned}$$

where $g_{\kappa_x}(Z')$ and $f_{\kappa_x}(Z')$ are the large and small components of the bound-state electron radial wave functions with the amplitudes β_x ($Z =$ atomic number of the daughter nucleus, $Z' = Z+1 =$ atomic number of the parent nucleus).

7.5. Summary of the observables

After the derivation of formulae for the most important observables in terms of the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ we are now in a position to discuss

(a) special types of decay modes, i.e. allowed transitions, first forbidden transitions etc.
(b) relations between different observables, i.e. which observables give the same kind of dynamic information and which lead to different results.

That also means we are now able to express the observables for special decay modes in terms of form factors or form factor coefficients, respectively. They determine the outcome of beta-decay and electron capture experiments, and are the only quantities that can be extracted from experimental data.

It should, however, be mentioned that we have not treated all possible observables. Of course, there are others for which formulae can, on the other hand, easily be derived by applying the methods extensively discussed before.

In the description of the decay processes two points, of minor importance, have up to now not been considered.

(1) Effects of the atomic clouds in initial and final states.
(2) Coulomb corrections of higher order (so-called radiative corrections).

These points, however, are discussed in Section 4.6.2 and in Chapter 13.

For explicit theoretical calculation of observables we have to go further and to calculate the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$. The key-point of that calculation is, however, the treatment of the nuclear form factors or form factor coefficients. This topic will, therefore, be discussed in detail in the next chapter.

Before closing this section we would additionally like to mention other reviews where explicit formulae for beta-decay observables can be found.

In many cases, however, the reviews listed in the following contain only formulae for special decay modes, like for example allowed transitions or for a much more restricted number of observables. In that context collections of formulae are contained in the books by Konopinski (1966), Schopper (1966), Behrens and Jänecke (1969), Morita (1973) and in the reviews by Weidenmüller (1961) and Frauenfelder and Steffen (1965).

The review by Bambynek *et al.* (1977) is concerned in detail with electron capture. For allowed transitions allowance should especially be made for the newer reviews and publications by Holstein (1974*a*, 1977*a*, 1977*b*), and Kleppinger *et al.* (1977).

The extension of this treatment to special cases and examples will be given in the following chapters where applications of the above formalism are shown in detail.

8

RELATION BETWEEN FORM FACTORS AND NUCLEAR MATRIX ELEMENTS

8.1. Impulse approximation treatment

8.1.1. Point-like nucleons

8.1.1.1. General relations

THE FORMALISM to express the observables as functions of the form factors or form factor coefficients is of general validity. It is independent of details of the weak and strong interaction, and also of the specific character of the nuclear structure.

However, to compute theoretically beta-decay observables on the basis of a special nuclear model or to extract nuclear structure information from beta-decay experiments, one has to relate the form factors to nuclear transition matrix elements. The form factors or form factor coefficients can, on the other hand, only be expressed in terms of nuclear matrix elements if some approximations are made. Therefore, as already mentioned in Section 6.2, it is first assumed that the nucleons inside the nucleus interact with leptons in the same way as free nucleons do. This approximation is usually called the impulse approximation treatment. This means that the nucleons during the decay process are considered to be independent and to be on their mass shell such that for each the Dirac equation of a free particle can be assumed. Meson exchange and other many body effects are hence for the moment neglected. They will be discussed later on. Next, we will also assume that the nucleons are point like, i.e. that they have no internal structure and the hadron current is not influenced by the presence of strong interactions. Then the hadron current for the decay of a neutron is simply given by (see eqn (6.4))

$$J_\mu^+(x) = i\bar{\psi}_p(x)\gamma_\mu(1+\lambda\gamma_5)\psi_n(x). \tag{8.1}$$

It should be mentioned that in the above formula we have taken a small step beyond the assumption of point-like nucleons by introducing the true free neutron axial vector constant $\lambda = -C_A/C_V$.

For a β^--decay of the jth-neutron within a nucleus the above assumptions then lead to a current density operator

$$O_\mu^+(\mathbf{x}_j) = i a_p^+(\alpha) e^{i\mathbf{q}\mathbf{x}_j} \langle \bar{\phi}_p \gamma_\mu (1+\lambda\gamma_5)\phi_n \rangle a_n(\beta) \tag{8.2}$$

(see the discussions in Sections 6.1 and 6.2) where the annihilation and

creation operators $a_n(\beta)$ and $a_p^+(\alpha)$ destroy a neutron in the bound state β and create a proton in the bound state α.

Because of the indistinguishability of identical particles we have to sum over all nucleons within a nucleus, i.e. over all neutrons and protons. $\phi_p(\alpha)$ and $\phi_n(\beta)$ are the corresponding relativistic single particle wave functions of protons and neutrons in the co-ordinate space.

For the β^--decay of a nucleus, the nuclear current can then be written as†

$$\langle f|\, V_\mu + A_\mu \,|i\rangle = \mathrm{i} \int \ldots \int \sum_{j=1}^{A} \phi_f^+(r_1 \ldots r_j \ldots r_A)\{\gamma_4\gamma_\mu(1+\lambda\gamma_5)t_-\}_j \times \mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{r}_j}\phi_i(r_1 \ldots r_j \ldots r_A)\, \mathrm{d}^3r_1 \ldots \mathrm{d}^3r_A. \quad (8.3)$$

Here ϕ_f and ϕ_i are the nuclear wave functions of the final and initial state respectively, which depend on all co-ordinates of the A nucleons. The sum over j runs over the A nucleons ‡ and all operators are single particle operators operating on the jth nucleon only.

The t_- is the isospin ladder operator which changes a neutron into a proton.

By applying the orthogonality relations for the tensor operators T^M_{KLs} given in eqn (6.27) we can rewrite eqn (6.28) in the following form:§

$$(-1)^{J_f - M_f}\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} F_{KLs}(\mathbf{q}^2) = \frac{(-1)^{K-s}}{\sqrt{\{4\pi(2J_i+1)\}}(-\mathrm{i})^{L-1}} \frac{(2L+1)!!}{(qR)^L} \times \int_{4\pi} \langle f|\, V_\mu + A_\mu \,|i\rangle \gamma_4\gamma_\mu T^M_{KLs}(\hat{q})\, \mathrm{d}\Omega_q. \quad (8.4)$$

If we insert the expression (8.3) for the nuclear current and expand $\mathrm{e}^{\mathrm{i}\mathbf{q}\mathbf{r}_j}$ in spherical harmonics as shown in eqn (6.64) we obtain

$$F_{KLs}(\mathbf{q}^2) = (-1)^{K-L}\mathfrak{M}_{KLs}(\mathbf{q}^2) \quad (8.5)$$

where the nuclear matrix elements $\mathfrak{M}_{KLs}(q^2)$ are now given by (see

† In the isospin formalism $a_p^+ a_n$ can be replaced by the ladder operator t_- which just transforms a neutron into a proton.

‡ Usual nuclear model wave functions are functions of the 3A co-ordinates of the nucleons. In reality, however, there are only $3(A-1)$ independent internal co-ordinates because three of them determine the position of the centre of mass (see the discussion in Section 6.2). This gives a small correction which can, in our case of beta-decay (small q), however, be neglected, (see, for example, de Forest and Walecka 1966).

§ It should be noted that

$$(T^M_{KLs})^+ = (-1)^{K-s+M}T^{-M}_{KLs}.$$

Behrens and Bühring 1971)

$$(-1)^{J_f-M_f}\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix}({}^{V}\mathfrak{M}_{KLs}(\mathbf{q}^2)+\lambda\,{}^{A}\mathfrak{M}_{KLs}(\mathbf{q}^2))$$

$$=\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\frac{(2L+1)!!}{(qR)^L}\int\int\int\phi_f^+(r_1\ldots r_A)$$

$$\times\sum_{j=1}^{A}\{j_L(qr)(1+\lambda\gamma_5)T^M_{KLs}(\hat{r})t_-\}_j\phi_i(r_1\ldots r_A)\,\mathrm{d}^3r_1\ldots\mathrm{d}^3r_A. \quad (8.6)$$

Here use has been made of the fact that

$$\int\sum_{L'M'}\mathrm{i}^{L'}j_{L'}(qr_j)\overset{*}{Y}{}^{M'}_{L'}(\hat{q})\,Y^{M'}_{L'}(\hat{r}_j)T^M_{KLs}(\hat{q})\,\mathrm{d}\Omega_q=\mathrm{i}^L j_L(qr_j)T^M_{KLs}(\hat{r}). \quad (8.7)$$

It should be mentioned that these nuclear matrix elements are reduced with respect to spin but not with respect to isospin.

In most cases, relativistic nuclear wave functions are not known, whence actual calculations must be performed in the context of non-relativistic nuclear models. We have therefore also to derive the non-relativistic analogue to the formula (8.6). This has, by the way, already been done in Section 6.2.

Going back to the eqns (6.50) and (6.51), we see that the transition from the relativistic case to the non-relativistic limit leads to the following replacements for the operators (in the order (v/c) of the nucleons):

relativistic case		non-relativistic case	
1	$\rightarrow$	1	(8.8*a*)
$\lambda\gamma_5$	$\rightarrow$	$-\lambda\dfrac{(\mathbf{p}_f+\mathbf{p}_i)\boldsymbol{\sigma}}{2M_N}$	(8.8*b*)
$\boldsymbol{\alpha}$	$\rightarrow$	$-\dfrac{\mathbf{p}_f+\mathbf{p}_i}{2M_N}$	(8.8*c*)
$\lambda\boldsymbol{\alpha}\gamma_5$	$\rightarrow$	$\lambda\boldsymbol{\sigma}$.	(8.8*d*)

It should be noted that on the left-hand side of the above relations we have 4×4 matrices while the corresponding operators on the right-hand side consist of 2×2 matrices. Usually the above relations are given in the more symbolic form

$$\boldsymbol{\alpha}\rightarrow-\frac{\mathbf{p}}{M} \quad (8.9a)$$

$$\gamma_5\rightarrow-\frac{\boldsymbol{\sigma}\mathbf{p}}{M}. \quad (8.9b)$$

As also already shown in Section 6.2 this procedure† leads to the following replacements for the transition operators in the nuclear matrix elements (see also Stech and Schülke 1964):‡

relativistic case — non-relativistic case

$$(1+\lambda\gamma_5)T^M_{KK0}(\hat{r}) \rightarrow i^K\left(1-\lambda\frac{\mathbf{p}\boldsymbol{\sigma}}{2M_N}\right)Y^M_K(\hat{r}) \tag{8.10a}$$

$$(-1)^{K-L}(1+\lambda\gamma_5)T^M_{KL1}(\hat{r}) \rightarrow i^L\left(\frac{1}{2M_N}\{Y_L(\hat{r})\otimes\mathbf{p}\}^{(K)}_M-\lambda\{Y_L(\hat{r})\otimes\boldsymbol{\sigma}\}^{(K)}_M\right). \tag{8.10b}$$

Here, a simplified notation $\mathbf{p}=-i(\nabla_f+\nabla_i)$ is used. The gradient operator ∇ acts on both the initial and final nuclear wave functions, i.e. $(\phi_f\,|\mathbf{p}|\,\phi_i)=-i\{(\phi_f\,|\,\nabla\phi_i)-(\nabla\phi_f\,|\,\phi_i)\}$ but neither on the spherical harmonics Y^M_L nor on the Bessel functions $j_L(qr)$.

Similar formulae are, for example, given by Walecka (1975) and Delorme (1979).

We now turn to the calculation of the form factor coefficients given in eqns (6.159) and (6.166). By inserting the result of eqns (8.5) and (8.6) in eqn (6.159) we get

$$F^{(N)}_{KLs}(k_e, m, n, \rho)=(-1)^{K-L}\frac{2}{\pi}\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\int_0^\infty\int_0^\infty\left(\frac{r'}{R}\right)^{L+2N}$$
$$\times I(k_e, m, n, \rho; r')j_L(qr')r'^2\,dr'\int\phi^+_f(r_1\ldots r_A)\sum_{j=1}^{A}\{j_L(qr)(1+\lambda\gamma_5)$$
$$\times T_{KLs}(\hat{r})t_-\}_j\phi_i(r_1\ldots r_A)\,d^3r_1\ldots d^3r_A q^2\,dq. \tag{8.11}$$

This expression can be further simplified. Since

$$\int_0^\infty j_L(qr)j_L(qr')q^2\,dq=\frac{\pi}{2}\frac{1}{r^2}\delta(r-r') \tag{8.12}$$

we then obtain

$$F^{(N)}_{KLs}(k_e, m, n, \rho)=(-1)^{K-L}\mathfrak{M}^{(N)}_{KLs}(k_e, m, n, \rho) \tag{8.13}$$

† There is another possibility of going to the non-relativistic limit. By applying the Foldy–Wouthuysen transformation on the total (nuclear plus beta-decay) Hamiltonian, one can construct an effective V–A transition operator that can be used with non-relativistic single particle wave functions (Rose and Osborn 1954).

‡ Exactly, in the non-relativistic limit of eqn (8.10*b*) a third term of the type

$$\frac{1}{2M_N}\frac{\boldsymbol{\sigma}[\nabla\times j_L(qr)\mathbf{Y}^M_{KL}(\hat{r})]}{j_L(qr)}$$

would occur. For a detailed discussion, however, see Section 8.1.2.

where

$$(-1)^{J_f-M_f}\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix}$$
$$\times\{{}^{V}\mathfrak{M}^{(N)}_{KLs}(k_e, m, n, \rho)+\lambda\,{}^{A}\mathfrak{M}^{(N)}_{KLs}(k_e, m, n, \rho)\}=\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}$$
$$\times\int\int\dots\int\phi_f^*(r_1\dots r_A)\sum_{j=1}^{A}\left\{\left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r)(1+\lambda\gamma_5)\right.$$
$$\left.\times T^{M}_{KLs}(\hat{r})t_-\right\}_j\phi_i(r_1\dots r_A)\,d^3r_1\dots d^3r_A. \tag{8.14}$$

It will also be convenient to introduce the more symbolic notation

$${}^{V}\mathfrak{M}^{(N)}_{KLs}(k_e, m, n, \rho)=\int\left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r)T_{KLs} \tag{8.15a}$$

$${}^{A}\mathfrak{M}^{(N)}_{KLs}(k_e, m, n, \rho)=\int\left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r)\gamma_5 T_{KLs}. \tag{8.15b}$$

The other form factor coefficients $F^{(N)}_{KLs}$ can simply be derived from the $F^{(N)}_{KLs}(k_e, m, n, \rho)$ by setting $I(k_e, m, n, \rho; r)=1$.

In summary, we found that in the impulse approximation treatment the nuclear matrix elements related to the form factors always consist of one body operators. Generally these matrix elements are of a relativistic type, but in nuclear models we are usually confronted with non-relativistic models or wave functions, respectively. Therefore we are forced (as shown above) to introduce further approximations beyond the impulse approximation when we are going from the relativistic case to the non-relativistic limit. For the following we should keep in mind that these two approximation steps (impulse approximation treatment and non-relativistic limit) are always different ones and that they should not be confused with one another.

Next we want to calculate these nuclear matrix elements explicitly.

8.1.1.2. Single-particle matrix elements

The simplest case in the context of nuclear models is the motion of a single nucleon in a central field (for example, one nucleon outside a double magic nucleus like ^{208}Pb). The transition matrix elements between these single nucleon bound states (usually called single particle matrix elements) are also of special importance for all more complicated nuclear models since they are, as we see later on, the building stones of all nuclear matrix elements, even in the most complicated situations. It is because of this latter point that we treat the single particle matrix

elements in more detail than would be required by the very few cases of such types of transitions occurring in nature.

The orbits of the nucleons are assumed to have definite angular momentum. In the same notation as used for the electron wave functions (see eqns (2.64) and (6.95)), the relativistic single particle nuclear wave functions can be written

$$\phi_\kappa^\mu(\mathbf{r}) = \begin{Bmatrix} \mathrm{sign}(\kappa) f(r,\kappa)\chi_{-\kappa}^\mu(\hat{r}) \\ g(r,\kappa)\chi_\kappa^\mu(\hat{r}) \end{Bmatrix} \chi_t \tag{8.16a}$$

with

$$\chi_\kappa^\mu(\hat{r}) = \mathrm{i}^l \sum_m C(l\tfrac{1}{2}j;\, \mu - m\; m) Y_l^{\mu-m}(\hat{r})\chi^m. \tag{8.16b}$$

The orbit of a nucleon is identified by the quantum number κ, defined as for leptons (see Table 8.1)

$$|\kappa| = j + \tfrac{1}{2} \qquad \begin{matrix} \kappa > 0 & \text{if } l = j + \frac{1}{2} \\ \kappa < 0 & \text{if } l = j - \frac{1}{2} \end{matrix}. \tag{8.17}$$

The neutron and proton are considered as two states of a nucleon. This is expressed by introducing an isospinor

$$\chi_n = \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_p = \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.18}$$

(see Wilkinson 1969).

Going back to eqn (8.6) we obtain for the single particle matrix

TABLE 8.1 *Quantum number κ for the different nuclear shells*

Shell	κ	Shell	κ
$1s_{1/2}$	−1	$1h_{11/2}$	−6
$1p_{1/2}$	1	$1h_{9/2}$	5
$1p_{3/2}$	−2	$2f_{7/2}$	−4
$1d_{5/2}$	−3	$2f_{5/2}$	3
$2s_{1/2}$	−1	$3p_{3/2}$	−2
$1d_{3/2}$	2	$3p_{1/2}$	1
$1f_{7/2}$	−4	$1i_{13/2}$	−7
$2p_{3/2}$	−2	$2g_{9/2}$	−5
$1f_{5/2}$	3	$3d_{5/2}$	−3
$2p_{1/2}$	1	$1i_{11/2}$	6
$1g_{9/2}$	−5	$2g_{7/2}$	4
$1g_{7/2}$	4	$4s_{1/2}$	−1
$2d_{5/2}$	−3	$3d_{3/2}$	2
$2d_{3/2}$	2	$1j_{15/2}$	−8
$3s_{1/2}$	−1		

elements

$$^{V}\mathfrak{M}_{KLs}(\mathbf{q}^2)+\lambda\,{}^{A}\mathfrak{M}_{KLs}(\mathbf{q}^2)=\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\frac{(2L+1)!!}{(qR)^L}\langle\phi_{\kappa_f}|\,|j_L(qr)\times(1+\lambda\gamma_5)T_{KLs}(\hat{r})|\,|\phi_{\kappa_i}\rangle. \tag{8.19}$$

Here, we have made use of the formula

$$\langle\chi_p|\,t_-\,|\chi_n\rangle=\langle\chi_-|\,t_-\,|\chi_+\rangle=1 \tag{8.20}$$

i.e. the above matrix elements are reduced in the spin space but not in the isospin space.

The relevant formulae for the explicit calculation of the matrix element, eqn (8.19), which is completely analogous to the corresponding lepton matrix element, have, however, been derived before (see eqns (6.133) to (6.142)).

Thus, these nuclear matrix elements can then be written as (following Behrens and Bühring 1971)

$$^{V}\mathfrak{M}_{KK0}(\mathbf{q}^2)=\frac{\sqrt{2}}{\sqrt{(J_i+1)}}\frac{(2K+1)!!}{(qR)^K}\left\{G_{KK0}(\kappa_f,\kappa_i)\times\int_0^\infty g_f(r,\kappa_f)j_K(qr)g_i(r,\kappa_i)r^2\,dr+\text{sign}(\kappa_f)\times\text{sign}(\kappa_i)G_{KK0}(-\kappa_f,-\kappa_i)\int_0^\infty f_f(r,\kappa_f)j_K(qr)f_i(r,\kappa_i)r^2\,dr\right\} \tag{8.21a}$$

$$^{A}\mathfrak{M}_{KL1}(\mathbf{q}^2)=\frac{\sqrt{2}}{\sqrt{(2J_i+1)}}\frac{(2L+1)!!}{(qR)^L}\left\{G_{KL1}(\kappa_f,\kappa_i)\times\int_0^\infty g_f(r,\kappa_f)j_L(qr)g_i(r,\kappa_i)r^2\,dr+\text{sign}(\kappa_f)\text{sign}(\kappa_i)\times G_{KL1}(-\kappa_f,-\kappa_i)\int_0^\infty f_f(r,\kappa_f)j_L(qr)f_i(r,\kappa_i)r^2\,dr\right\} \tag{8.21b}$$

$$^{A}\mathfrak{M}_{KK0}(\mathbf{q}^2)=\frac{\sqrt{2}}{\sqrt{(2J_i+1)}}\frac{(2K+1)!!}{(qR)^K}\left\{\text{sign}(\kappa_i)G_{KK0}(\kappa_f,-\kappa_i)\times\int_0^\infty g_f(r,\kappa_f)j_K(qr)f_i(r,\kappa_i)r^2\,dr+\text{sign}(\kappa_f)G_{KK0}(-\kappa_f,\kappa_i)\times\int_0^\infty f_f(r,\kappa_f)j_K(qr)g_i(r,\kappa_i)r^2\,dr\right\} \tag{8.21c}$$

$$
{}^{V}\mathfrak{M}_{KL1}(\mathbf{q}^2) = \frac{\sqrt{2}}{\sqrt{(2J_i+1)}} \frac{(2L+1)!!}{(qR)^L} \Big\{ \mathrm{sign}(\kappa_i) G_{KL1}(\kappa_f, -\kappa_i) \times \int_0^\infty g_f(r, \kappa_f) j_L(qr) f_i(r, \kappa_i) r^2 \, dr + \mathrm{sign}(\kappa_f) G_{KL1}(-\kappa_f, \kappa_i) \times \int_0^\infty f_f(r, \kappa_f) j_L(qr) g_i(r, \kappa_i) r^2 \, dr \Big\} \tag{8.21d}
$$

(of course, we have here $J_i = |\kappa_i| - \frac{1}{2}$).

The geometrical coefficients $G_{KLs}(\kappa_f, \kappa_i)$ are defined through eqn (6.141). The radial quantum numbers of the orbits are not explicitly indicated.

The matrix elements $\mathfrak{M}^N_{KLs}(k_e, m, n, \rho)$ needed for the calculation of the form factor coefficients (see eqns (8.13) and (8.14)) are simply related to the matrix elements $\mathfrak{M}_{KLs}(\mathbf{q}^2)$ by replacing

$$
\begin{aligned}
\mathfrak{M}_{KLs}(\mathbf{q}^2) &\to \mathfrak{M}^N_{KLs}(k_e, m, n, \rho) \\
j_L(qr) &\to \left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r).
\end{aligned} \tag{8.22}
$$

Because of the importance of the geometrical coefficients $G_{KLs}(\kappa_f, \kappa_i)$ for the calculation of nuclear (and lepton) matrix elements we have listed the most important $G_{KLs}(\kappa_f, \kappa_i)$ in Table 8.2.

If relativistic nuclear wave functions are used (for an example, see Danos and Gillet 1979), the nucleon radial wave functions must be normalized to satisfy the condition

$$
\int_0^\infty g^2(r, \kappa) r^2 \, dr + \int_0^\infty f^2(r, \kappa) r^2 \, dr = 1. \tag{8.23}
$$

In most cases, however, relativistic nucleon wave functions are not available. It is then necessary to determine the small components $f(r, \kappa)$ of the nuclear radial wave functions. For that purpose we remember that the radial wave functions $f(r)$ and $g(r)$ are solutions of the Dirac radial equations (see eqn 2.77 and 3.1). Going to the non-relativistic limit

$$
\begin{aligned}
T &\ll 2M_N \\
V(r) &\ll 2M_N
\end{aligned} \tag{8.24}
$$

we get

$$
-2M_N f(r, \kappa) + \left(\frac{d}{dr} + \frac{\kappa+1}{r}\right) g(r, \kappa) = 0 \tag{8.25a}
$$

$$
(T - V(r)) g(r, \kappa) + \left(\frac{d}{dr} - \frac{\kappa-1}{r}\right) f(r, \kappa) = 0 \tag{8.25b}
$$

TABLE 8.2 *The geometrical coefficients* $G_{KLs}(\kappa_f, \kappa_i)$

K	κ_f	κ_i	$G_{KK0}(\kappa_f, \kappa_i)$	$G_{KK-11}(\kappa_f, \kappa_i)$	$G_{KK1}(\kappa_f, \kappa_i)$	$G_{KK+11}(\kappa_f, \kappa_i)$
0	1	1	1·00000	0	0	0
	−1	1	0	0	0	−1·00000
	1	−1	0	0	0	1·00000
	−1	−1	1·00000	0	0	0
	2	2	1·41421	0	0	0
	−2	2	0	0	0	−1·41421
	2	−2	0	0	0	1·41421
	−2	−2	1·41421	0	0	0
	3	3	1·73205	0	0	0
	−3	3	0	0	0	−1·73205
	3	−3	0	0	0	1·73205
	−3	−3	1·73205	0	0	0
	4	4	2·00000	0	0	0
	−4	4	0	0	0	−2·00000
	4	−4	0	0	0	2·00000
	−4	−4	2·00000	0	0	0
	5	5	2·23607	0	0	0
	−5	5	0	0	0	−2·23607
	5	−5	0	0	0	2·23607
	−5	−5	2·23607	0	0	0
1	1	1	0	−0·57735	0	1·63299
	−1	1	1·00000	0	−1·41421	0
	1	−1	−1·00000	0	−1·41421	0
	−1	−1	0	1·73205	0	0
	1	2	1·41421	0	−1·00000	0
	−1	2	0	0	0	−1·73205
	1	−2	0	1·63299	0	0·57735
	−1	−2	1·41421	0	1·00000	0
	2	1	1·41421	0	1·00000	0
	−2	1	0	−1·63299	0	−0·57735
	2	−1	0	0	0	1·73205
	−2	−1	1·41421	0	−1·00000	0
	2	2	0	−1·09544	0	1·54919
	−2	2	0·63246	0	−1·78885	0
	2	−2	−0·63246	0	−1·78885	0
	−2	−2	0	1·82574	0	−0·51640
	2	3	1·89736	0	−1·34164	0
	−2	3	0	0	0	−2·32379
	2	−3	0	2·19089	0	0·77460
	−2	−3	1·89736	0	1·34164	0
	3	2	1·89736	0	1·34164	0
	−3	2	0	−2·19089	0	−0·77460
	3	−2	0	0	0	2·32379
	−3	−2	1·89736	0	−1·34164	0
	3	3	0	−1·46385	0	1·65616
	−3	3	0·50709	0	−2·15141	0
	3	−3	−0·50709	0	−2·15141	0
	−3	−3	0	2·04939	0	−0·82808
	3	4	2·26778	0	−1·60357	0
	−3	4	0	0	0	−2·77746
	3	−4	0	2·61861	0	0·92582
	−3	−4	2·26779	0	1·60357	0

TABLE 8.2 (*Continued*)

K	κ_f	κ_i	$G_{KK0}(\kappa_f, \kappa_i)$	$G_{KK-11}(\kappa_f, \kappa_i)$	$G_{KK1}(\kappa_f, \kappa_i)$	$G_{KK+11}(\kappa_f, \kappa_i)$
1	4	3	2·26778	0	1·60357	0
	−4	3	0	−2·61861	0	−0·92582
	4	−3	0	0	0	2·77746
	−4	−3	2·26779	0	−1·60357	0
	4	4	0	−1·76383	0	1·78174
	−4	4	0·43644	0	−2·46885	0
	4	−4	−0·43644	0	−2·46885	0
	−4	−4	0	2·26779	0	−1·06904
	4	5	2·58199	0	−1·82574	0
	−4	5	0	0	0	−3·16227
	4	−5	0	2·98142	0	1·05409
	−4	−5	2·58199	0	1·82574	0
	5	4	2·58199	0	1·82574	0
	−5	4	0	−2·98142	0	−1·05409
	5	−4	0	0	0	3·16227
	−5	−4	2·58198	0	−1·82574	0
	5	5	0	−2·02260	0	1·90692
	−5	5	0·38925	0	−2·75241	0
	5	−5	−0·38925	0	−2·75241	0
	−5	−5	0	2·47206	0	−1·27128
2	1	2	0	−0·44721	0	2·19089
	−1	2	1·41421	0	−1·73205	0
	1	−2	−1·41421	0	−1·73205	0
	−1	−2	0	2·23607	0	0
	1	3	1·73205	0	−1·41421	0
	−1	3	0	0	0	−2·23606
	1	−3	0	2·19089	0	0·44721
	−1	−3	1·73205	0	1·41421	0
	2	1	0	−0·44721	0	2·19089
	−2	1	1·41421	0	−1·73205	0
	2	−1	−1·41421	0	−1·73205	0
	−2	−1	0	2·23607	0	0
	2	2	1·41421	0	0	0
	−2	2	0	−0·89443	0	−1·09544
	2	−2	0	0·89443	0	1·09544
	−2	−2	1·41421	0	0	0
	2	3	0	−0·87831	0	1·91236
	−2	3	0·92582	0	−1·88982	0
	2	−3	−0·92582	0	−1·88982	0
	−2	−3	0	2·04939	0	−0·47809
	2	4	2·26778	0	−1·85164	0
	−2	4	0	0	0	−2·92770
	2	−4	0	2·86855	0	0·58554
	−2	−4	2·26779	0	1·85164	0
	3	1	1·73205	0	1·41421	0
	−3	1	0	−2·19089	0	−0·44721
	3	−1	0	0	0	2·23606
	−3	−1	1·73205	0	−1·41421	0
	3	2	0	−0·87831	0	1·91236
	−3	2	0·92582	0	−1·88982	0
	3	−2	−0·92582	0	−1·88982	0
	−3	−2	0	2·04939	0	−0·47809

TABLE 8.2 (*Continued*)

K	κ_f	κ_i	$G_{KK0}(\kappa_f, \kappa_i)$	$G_{KK-11}(\kappa_f, \kappa_i)$	$G_{KK1}(\kappa_f, \kappa_i)$	$G_{KK+11}(\kappa_f, \kappa_i)$
2	3	3	1·85164	0	0	0
	−3	3	0	−1·17108	0	−1·43427
	3	−3	0	1·17108	0	1·43427
	−3	−3	1·85164	0	0	0
	3	4	0	−1·19523	0	1·95180
	−3	4	0·75593	0	−2·16024	0
	3	−4	−0·75593	0	−2·16024	0
	−3	−4	0	2·15141	0	−0·78072
	3	5	2·67261	0	−2·18218	0
	−3	5	0	0	0	−3·45032
	3	−5	0	3·38061	0	0·69006
	−3	−5	2·67261	0	2·18218	0
	4	2	2·26778	0	1·85164	0
	−4	2	0	−2·86855	0	−0·58554
	4	−2	0	0	0	2·92770
	−4	−2	2·26779	0	−1·85164	0
	4	3	0	−1·19523	0	1·95180
	−4	3	0·75593	0	−2·16024	0
	4	−3	−0·75593	0	−2·16024	0
	−4	−3	0	2·15141	0	−0·78072
	4	4	2·18218	0	0	0
	−4	4	0	−1·38013	0	−1·69031
	4	−4	0	1·38013	0	1·69031
	−4	−4	2·18218	0	0	0
	4	5	0	−1·45644	0	2·03859
	−4	5	0·65795	0	−2·41747	0
	4	−5	−0·65795	0	−2·41747	0
	−4	−5	0	2·28869	0	−1·01929
	5	3	2·67261	0	2·18218	0
	−5	3	0	−3·38061	0	−0·69006
	5	−3	0	0	0	3·45032
	−5	−3	2·67261	0	−2·18218	0
	5	4	0	−1·45644	0	2·03859
	−5	4	0·65795	0	−2·41747	0
	5	−4	−0·65795	0	−2·41747	0
	−5	−4	0	2·28868	0	−1·01929
	5	5	2·46183	0	0	0
	−5	5	0	−1·55700	0	−1·90692
	5	−5	0	1·55700	0	1·90692
	−5	−5	2·46183	0	0	0
3	1	3	0	−0·37796	0	2·61861
	−1	3	1·73205	0	−2·00000	0
	1	−3	−1·73205	0	−2·00000	0
	−1	−3	0	2·64575	0	0
	1	4	2·00000	0	−1·73205	0
	−1	4	0	0	0	−2·64575
	1	−4	0	2·61861	0	0·37796
	−1	−4	2·00000	0	1·73205	0
	2	2	0	−0·41404	0	2·86855
	−2	2	1·89736	0	−2·19089	0
	2	−2	−1·89736	0	−2·19089	0
	−2	−2	0	2·89827	0	0

TABLE 8.2 (*Continued*)

K	κ_f	κ_i	$G_{KK0}(\kappa_f, \kappa_i)$	$G_{KK-11}(\kappa_f, \kappa_i)$	$G_{KK1}(\kappa_f, \kappa_i)$	$G_{KK+11}(\kappa_f, \kappa_i)$
3	2	3	1·54919	0	−0·44721	0
	−2	3	0	−0·67612	0	−1·46385
	2	−3	0	1·35225	0	0·87831
	−2	−3	1·54919	0	0·44721	0
	2	4	0	−0·75593	0	2·18218
	−2	4	1·15470	0	−2·00000	0
	2	−4	−1·15470	0	−2·00000	0
	−2	−4	0	2·26779	0	−0·43644
	2	5	2·58199	0	−2·23606	0
	−2	5	0	0	0	−3·41565
	2	−5	0	3·38061	0	0·48795
	−2	−5	2·58199	0	2·23607	0
	3	1	0	−0·37796	0	2·61861
	−3	1	1·73205	0	−2·00000	0
	3	−1	−1·73205	0	−2·00000	0
	−3	−1	0	2·64575	0	0
	3	2	1·54919	0	0·44721	0
	−3	2	0	−1·35225	0	−0·87831
	3	−2	0	0·67612	0	1·46385
	−3	−2	1·54919	0	−0·44721	0
	3	3	0	−0·82808	0	2·39045
	−3	3	1·26491	0	−2·19089	0
	3	−3	−1·26491	0	−2·19089	0
	−3	−3	0	2·48423	0	−0·47809
	3	4	2·00000	0	−0·57735	0
	−3	4	0	−0·87287	0	−1·88982
	3	−4	0	1·74574	0	1·13389
	−3	−4	2·00000	0	0·57735	0
	3	5	0	−1·04031	0	2·16225
	−3	5	0·95346	0	−2·20193	0
	3	−5	−0·95346	0	−2·20193	0
	−3	−5	0	2·28869	0	−0·72075
	4	1	2·00000	0	1·73205	0
	−4	1	0	−2·61861	0	−0·37796
	4	−1	0	0	0	2·64575
	−4	−1	2·00000	0	−1·73205	0
	4	2	0	−0·75593	0	2·18218
	−4	2	1·15470	0	−2·00000	0
	4	−2	−1·15470	0	−2·00000	0
	−4	−2	0	2·26779	0	−0·43644
	4	3	2·00000	0	0·57735	0
	−4	3	0	−1·74574	0	−1·13389
	4	−3	0	0·87287	0	1·88982
	−4	−3	2·00000	0	−0·57735	0
	4	4	0	−1·13960	0	2·36863
	−4	4	1·04447	0	−2·41209	0
	4	−4	−1·04447	0	−2·41209	0
	−4	−4	0	2·50713	0	−0·78954
	4	5	2·33549	0	−0·67420	0
	−4	5	0	−1·01929	0	−2·20684
	4	−5	0	2·03859	0	1·32410
	−4	−5	2·33549	0	0·67420	0
	5	2	2·58199	0	2·23606	0

TABLE 8.2 (*Continued*)

K	κ_f	κ_i	$G_{KK0}(\kappa_f, \kappa_i)$	$G_{KK-11}(\kappa_f, \kappa_i)$	$G_{KK1}(\kappa_f, \kappa_i)$	$G_{KK+11}(\kappa_f, \kappa_i)$
3	−5	2	0	−3·38061	0	−0·48795
	5	−2	0	0	0	3·41565
	−5	−2	2·58199	0	−2·23607	0
	5	3	0	−1·04031	0	2·16225
	−5	3	0·95346	0	−2·20193	0
	5	−3	−0·95346	0	−2·20193	0
	−5	−3	0	2·28869	0	−0·72075
	5	4	2·33549	0	0	0
	−5	4	0	−2·03859	0	−1·32410
	5	−4	0	1·01929	0	2·20683
	−5	−4	2·33549	0	−0·67420	0
	5	5	0	−1·39930	0	2·42366
	−5	5	0·91606	0	−2·64443	0
	5	−5	−0·91606	0	−2·64443	0
	−5	−5	0	2·59870	0	−1·03871

and therefore†

$$f(r, \kappa) = \frac{1}{2M_N}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa+1}{r}\right) g(r, \kappa). \tag{8.26}$$

If we insert this relation in eqn (8.25*b*) we obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa(\kappa+1)}{r^2} + 2M_N\{T - V(r)\}\right] g(r, \kappa) = 0. \tag{8.27}$$

Since $-\kappa(\kappa+1) = -l(l+1)$ this radial equation is obviously that of the Schrödinger equation.

† A relativistic wave function is described by

$$\phi = \begin{pmatrix} v \\ u \end{pmatrix}$$

where v and u are the small and large components, respectively. For a free particle we have then (see eqn (2.42))

$$v = -\frac{\boldsymbol{\sigma}\mathbf{p}}{W + M_N} u \simeq -\frac{\boldsymbol{\sigma}\mathbf{p}}{2M_N} u.$$

As shown in eqns (6.99) to (6.110) we have also

$$\mathrm{sign}(\kappa) f(r, \kappa) \chi^{\mu}_{-\kappa} = -\frac{\boldsymbol{\sigma}\mathbf{p}}{2M_N} g_{\kappa}(r, \kappa) \chi^{\mu}_{k}$$

with

$$f(r, \kappa) = \frac{1}{2M_N}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa+1}{r}\right) g(r, \kappa).$$

Thus the above relation between $f(r, \kappa)$ and $g(r, \kappa)$ corresponds to assuming the free particle relation between small and large components for $W \to M_N$.

Thus $g(r, \kappa)\chi_\kappa^\mu$ is now understood as the solution of the single particle Schrödinger equation. In this case, the radial wave functions $g(r, \kappa)$ must be normalized according to

$$\int_0^\infty g^2(r, \kappa) r^2 \, dr = 1. \tag{8.28}$$

It should also be mentioned that the way of going from the relativistic to the non-relativistic limit, which has just been discussed, leads to exactly the same results as we would obtain by applying eqns (8.10*a*) and (8.10*b*).†

In the context of single particle matrix elements, we should mention especially the following publications, where this topic is extensively treated: Brysk (1952), Talmi (1953), Rose and Osborn (1954), Lipnik and Sunier (1966), Delabaye and Lipnik (1966), de Forest and Walecka (1966), and Strubbe and Callebaut (1970).

We now turn to the problem of the radial matrix element calculation and discuss this point for, at least, some simple models.

We are confronted with three types of radial matrix elements, i.e.‡

(a) $$\int_0^\infty g_f(r, \kappa_f)\phi(r) g_i(r, \kappa_i) r^2 \, dr \tag{8.29a}$$

(b) $$\int_0^\infty g_f(r, \kappa_f)\phi(r) f_i(r, \kappa_i) r^2 \, dr \tag{8.29b}$$

(c) $$\int_0^\infty f_f(r, \kappa_f)\phi(r) g_i(r, \kappa_i) r^2 \, dr \tag{8.29c}$$

$$\left(\text{we have} \quad \phi(r) = j_L(qr) \quad \text{or} \quad \phi(r) = \left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r)\right).$$

The first one of these belongs to the so-called non-relativistic nuclear matrix elements (since large components are connected with large components), the two last ones characterize the so-called relativistic nuclear matrix elements (because small components are connected with large components).

†It should be noted that

$$\frac{1}{2M_N}\{Y_L(\hat{r}_j) \otimes \mathbf{p}_j\}^K = \frac{\mathbf{p}_j \boldsymbol{\sigma}_j}{2M_N}\{Y_L(\hat{r}_j) \otimes \boldsymbol{\sigma}_j\}^K - \frac{1}{2M_N}\frac{\boldsymbol{\sigma}[\nabla \times j_L(qr)\mathbf{Y}_{KL}^M(\hat{r})]}{j_L(qr)}.$$

‡The terms with the radial matrix element

$$\int_0^\infty f_f(r, \kappa_f)\phi(r) f_i(r, \kappa_i) r^2 \, dr$$

are small relativistic corrections to the non-relativistic matrix elements and will be neglected in the following.

We have seen that a convenient way to go to the non-relativistic limit is to express the $f(r, \kappa)$ through the $g(r, \kappa)$ by the application of eqn (8.26). Doing so and making use of the Schrödinger equation, eqn (8.27), we can express the last two radial matrix elements, eqns (8.29*b*) and (8.29*c*), in terms of the first one, eqn (8.29*a*), for special choice of $\phi(r) = (r/R)^L$. After some transformations we then get

$$\int_0^\infty g_f(r, \kappa_f)\left(\frac{r}{R}\right)^L f_i(r, \kappa_i) r^2 \, dr = \frac{R}{2(L+1)} \int g_f(r, \kappa_f) \times [(T_i - T_f) - \{V_i(r) - V_f(r)\}]\left(\frac{r}{R}\right)^{L+1} g_i(r, \kappa_i) r^2 \, dr + \frac{(\kappa_f - \kappa_i + 1 + L)(\kappa_f + \kappa_i - L)}{4M_N R(L+1)} \int g_f(r, \kappa_f)\left(\frac{r}{R}\right)^{L-1} g_i(r, \kappa_i) r^2 \, dr \tag{8.30a}$$

and

$$\int_0^\infty f_f(r, \kappa_f)\left(\frac{r}{R}\right)^L g_i(r, \kappa_i) r^2 \, dr = -\frac{R}{2(L+1)} \int_0^\infty g_f(r, \kappa_f) \times [(T_i - T_f) - \{V_i(r) - V_f(r)\}]\left(\frac{r}{R}\right)^{L+1} g_i(r, \kappa_i) r^2 \, dr - \frac{(\kappa_f - \kappa_i - 1 - L)(\kappa_f + \kappa_i - L)}{4M_N R(L+1)} \int_0^\infty g_f(r, \kappa_f)\left(\frac{r}{R}\right)^{L-1} g_i(r, \kappa_i) r^2 \, dr. \tag{8.30b}$$

For further explicit calculations we have to use special models. The simplest one in this context is the Harmonic oscillator, i.e.

$$V(r) = \frac{M_N}{2} \omega^2 r^2 = \frac{1}{2M_N b^4} r^2. \tag{8.31}$$

ω is the oscillator frequence and b the oscillator parameter which are related by

$$\omega = \frac{1}{M_N b^2}. \tag{8.32}$$

The solutions for this type of potential can be found in many well-known textbooks of quantum mechanics and are given by

$$g_\nu(r, \kappa) = g_\nu(r, l) = \sqrt{\left\{\frac{2\Gamma(l+\nu+\tfrac{1}{2})}{b^3(\nu-1)!}\right\}} \frac{1}{\Gamma\left(l+\frac{3}{2}\right)} \left(\frac{r}{b}\right)^l \times e^{-\frac{1}{2}(r/b)^2} {}_1F_1\left(-\nu+1; l+\frac{3}{2}; \frac{r^2}{b^2}\right) \tag{8.33}$$

where

$$ {}_1F_1(a; b; x) = 1 + \frac{a}{b}x + \frac{a(a+1)x^2}{b(b+1)2!} \cdots \tag{8.34} $$

is the confluent hypergeometric function. In our case, where a is a non-positive integer the series terminate and ${}_1F_1$ is a polynomial of degree $-a$ in x.

The normalization is chosen to be

$$ \int_0^\infty g_\nu^2(r, l) r^2 \, dr = 1. \tag{8.35} $$

The energy eigen states are then given by

$$ E = \omega\left(N + \frac{3}{2}\right) = \omega[2(\nu - 1) + l + \tfrac{3}{2}]. \tag{8.36} $$

Before proceeding further we would, however, like to make some remarks about the correct choice of the length parameter b.

The uniformly charged sphere radius R is related to mean square radius $\langle r^2 \rangle$ by

$$ R^2 = \tfrac{5}{3}\langle r^2 \rangle. \tag{8.37} $$

The mean square radius in the oscillator model is, on the other hand, directly determined by calculating $\langle r^2 \rangle$ for each oscillator orbit and summing over all occupied states

$$ \langle r^2 \rangle_{sh} = \frac{1}{Z} \sum (2j+1)(2(\nu - 1) + l + \tfrac{3}{2}) b^2 \tag{8.38} $$

where $2j+1$ is the number of protons in each orbit. The experimentally deduced radius (for instance by electron scattering) $\langle r^2 \rangle_{exp}$ is now related to the shell model radius by

$$ \langle r^2 \rangle_{exp} = \langle r^2 \rangle_{sh} + \frac{3}{2}\left(a_p^2 - \frac{b^2}{A}\right). \tag{8.39} $$

The second term is a correction for the finite size of the proton ($a_p = 0{\cdot}65$ fm) and for the fact that the origin of co-ordinates is taken to be the centre of the shell model and not the centre of mass of the nucleus. The length parameter† b is, therefore, fixed by the choice of the nuclear radius R (Towner *et al.* 1976).

† A rough estimation of the length parameter b can be obtained in the following way. The number of nucleons in a shell is given by

$$ 2(N+1)(N+2) $$

(N = oscillator quantum number).

(*footnote continued on page 280*)

The introduction of the harmonic oscillator wave functions now has the great advantage that many of the radial matrix elements can be calculated analytically. In the following we will, therefore, list a number of useful formulae for those matrix elements, evaluated with harmonic oscillator wave functions.

Let us start with radial matrix elements of the type of eqn (8.29*a*).

The radial matrix element of the Bessel function is given by† (de Forest and Walecka 1966)

$$(\nu_f l_f \,|j_L(qr)|\, \nu_i l_i) = \frac{2^L}{(2L+1)!!}\, y^{L/2} e^{-y} \sqrt{\{(\nu_f - 1)!\ (\nu_i - 1)!\}}$$
$$\times \{\Gamma(\nu_f + l_f + \tfrac{1}{2})\Gamma(\nu_i + l_i + \tfrac{1}{2})\}^{1/2}$$
$$\times \sum_{m'=0}^{\nu_f - 1} \sum_{m=0}^{\nu_i - 1} \frac{(-1)^{m'+m}}{m'!\ m!\ (\nu_f - m' - 1)!\ (\nu_i - m - 1)!}$$
$$\times \frac{\Gamma(\frac{1}{2}[l_f + l_i + 2m' + 2m + L + 3])}{\Gamma(m' + l_f + \frac{3}{2})\Gamma(m + l_i + \frac{3}{2})}\, {}_1F_1(\tfrac{1}{2}[L - l_f - l_i - 2m' - 2m]; L + \tfrac{3}{2}; y) \tag{8.40}$$

† $(\nu_f l_f \,|j_L(qr)|\, \nu_i l_i) = \int_0^\infty g_{\nu_f}(r, l_f) j_L(qr) g_{\nu_i}(r, l_i) r^2\, dr.$

Thus the total number A of nucleons is given by

$$A = \sum_{N=0}^{N_{max}} 2(N+1)(N+2) = \tfrac{2}{3}(N_{max}+1)(N_{max}+2)(N_{max}+3) \simeq \tfrac{2}{3}(N_{max}+2)^3$$

On the other hand we have for the mean square radius within a shell N

$$\langle r^2 \rangle_N = b^2(N + \tfrac{3}{2})$$

and for the mean square radius of the nucleus

$$\langle r^2 \rangle = \frac{2b^2}{A} \sum_{N=0} (N + \tfrac{3}{2})(N+1)(N+2) \simeq \frac{b^2}{2A}(N_{max}+2)^4$$

if we assume neutron and proton radii being equal.

Therefore we obtain with $R = r_0 A^{1/3}$

$$r_0^2 A^{2/3} = \frac{5}{3}\frac{b^2}{2A}(N_{max}+2)^4 = \frac{5}{3}\frac{b^2}{2A}\{\tfrac{3}{2}A\}^{4/3}.$$

Finally, we then find ($r_0 = 1{\cdot}2$ fm)

$$b = 2^{7/6} 3^{-1/6} 5^{-1/2} r_0 A^{1/6}$$
$$= 2^{7/6} 3^{-1/6} 5^{-1/2} R A^{-1/6}$$
$$= 2{\cdot}60 \times 10^{-3} A^{1/6}$$

This corresponds to $\hbar\omega = 41 \times A^{-1/3}$ MeV.

where

$$y = \left(\frac{qb}{2}\right)^2 . \tag{8.41}$$

${}_1F_1$ is the confluent hypergeometric function discussed before.

For $\nu_i = \nu_f = 1$ this formula can be simplified to

$$(1l_f\,|j_L(qr)|\,1l_i) = \frac{2^{L/2}}{(2L+1)!!}\,y^{L/2}e^{-y} \times \frac{(l_f+l_i+L+1)!!}{\sqrt{\{(2l_f+1)!!\,(2l_i+1)!!\}}}\,{}_1F_1(\tfrac{1}{2}[L-l_f-l_i];\,L+\tfrac{3}{2};\,y). \tag{8.42}$$

In order to get an impression of the explicit behaviour of these matrix elements (and also of the form factors), as function of q^2 we will give also some special examples. For the s, p, d shells we have

$$1s \to 1s \quad (10\,|j_0(qr)|\,10) = e^{-(qb/2)^2} \tag{8.43a}$$

(only $L = 0$ possible)

$$1p \to 1p \quad (11\,|j_0(qr)|\,11) = \{1-\tfrac{1}{6}(qb)^2\}e^{-(qb/2)^2} \tag{8.43b}$$

$$(11\,|j_2(qr)|\,11) = \tfrac{1}{6}(qb)^2e^{-(qb/2)^2} \tag{8.43c}$$

$$1d \to 1d \quad (12\,|j_0(qr)|\,12) = \{1-\tfrac{1}{3}(qb)^2+\tfrac{1}{60}(qb)^4\}e^{-(qb/2)^2} \tag{8.43d}$$

$$(12\,|j_2(qr)|\,12) = \tfrac{7}{30}(qb)^2\{1-\tfrac{1}{14}(qb)^2\}e^{-(qb/2)^2} \tag{8.43e}$$

$$2s \to 2s \quad (20\,|j_0(qr)|\,20) = \{1-\tfrac{1}{3}(qb)^2+\tfrac{1}{24}(qb)^2\}e^{-(qb/2)^2} \tag{8.43f}$$

$$2s \to 1d \quad (20\,|j_2(qr)|\,12) = -\tfrac{1}{3}\sqrt{\tfrac{2}{5}}(qb)^2\{1-\tfrac{1}{8}(qb)^2\}e^{-(qb/2)^2}. \tag{8.43g}$$

From the geometrical coefficients $G_{KLs}(\kappa_f, \kappa_i)$ we know that $l_f + l_i + L$ must always be even (see eqn (6.141)). Radial matrix elements of the type listed above for $L = 1$ are therefore meaningless.

For the form factor coefficients we need radial matrix elements of the type† $(\nu_f l_f\,|r^L|\,\nu_i l_i)$. A very useful expression for the calculation of these radial matrix elements has been given by (Nilsson 1955)

$$(\nu_f l_f\,|r^L|\,\nu_i l_i) = (-1)^{\nu_i+\nu_f}b^L\left[\frac{\Gamma(\nu_i)\Gamma(\nu_f)}{\Gamma(\nu_i+t-\tau_i)\Gamma(\nu_f+t-\tau_f)}\right]^{1/2} \times \tau_i!\;\tau_f!\sum_\sigma \frac{\Gamma(t+\sigma+1)}{\sigma!\,(\nu_i-\sigma-1)!\,(\nu_f-\sigma-1)!\,(\sigma+\tau_i-\nu_i+1)!\,(\sigma+\tau_f-\nu_f+1)!} \tag{8.44}$$

† $(\nu_f l_f\,|r^L|\,\nu_i l_i) = \int_0^\infty g_{\nu_f}(r, l_f)r^L g_{\nu_i}(r, l_i)r^2\,dr.$

where

$$\tau_i = \tfrac{1}{2}(l_f - l_i + L) \tag{8.45a}$$

$$\tau_f = \tfrac{1}{2}(l_i - l_f + L) \tag{8.45b}$$

$$t = \tfrac{1}{2}(l_i + l_f + L + 1) \tag{8.45c}$$

and where the upper and lower limits of the summation variable σ are

$$\max(\nu_i - \tau_i - 1, \nu_f - \tau_f - 1) \leqslant \sigma \leqslant \min(\nu_i - 1, \nu_f - 1). \tag{8.46}$$

If no σ fulfils the above relation, the radial matrix element is zero. Some of the most important special cases of the above equation are the following (Faessler and Sheline 1966):

$$(\nu l - 1 \,|r|\, \nu l) = b(\nu + l - \tfrac{1}{2})^{1/2} \tag{8.47a}$$

$$(\nu - 1, l + 1 \,|r|\, \nu l) = -b(\nu - 1)^{1/2} \tag{8.47b}$$

$$(\nu + 1, l - 1 \,|r|\, \nu l) = -b\nu^{1/2} \tag{8.47c}$$

$$(\nu l + 1 \,|r|\, \nu l) = b(\nu + l + \tfrac{1}{2})^{1/2} \tag{8.47d}$$

and

$$(\nu l \,|r^2|\, \nu l) = b^2(2\nu + l - \tfrac{1}{2}) \tag{8.48a}$$

$$(\nu + 1, l - 2 \,|r^2|\, \nu l) = -2b^2[\nu(\nu + l - \tfrac{1}{2})]^{1/2} \tag{8.48b}$$

$$(\nu - 1, l + 2 \,|r^2|\, \nu l) = -2b^2[(\nu - 1)(\nu + l + \tfrac{1}{2})]^{1/2} \tag{8.48c}$$

$$(\nu - 1, l \,|r^2|\, \nu l) = -b^2[(\nu - 1)(\nu + l - \tfrac{1}{2})]^{1/2} \tag{8.48d}$$

$$(\nu l - 2 \,|r^2|\, \nu l) = b^2[\nu(\nu + l + \tfrac{1}{2})]^{1/2} \tag{8.48e}$$

$$(\nu - 1, l - 2 \,|r^2|\, \nu l) = -b^2[(\nu + l - \tfrac{1}{2})(\nu + l - \tfrac{3}{2})]^{1/2} \tag{8.48f}$$

$$(\nu l + 2 \,|r^2|\, \nu l) = b^2[(\nu + l + \tfrac{3}{2})(\nu + l + \tfrac{1}{2})]^{1/2} \tag{8.48g}$$

$$(\nu - 2, l + 2 \,|r^2|\, \nu l) = b^2[(\nu - 1)(\nu - 2)]^{1/2} \tag{8.48h}$$

$$(\nu + 2, l - 2 \,|r^2|\, \nu l) = b^2[\nu(\nu + 1)]^{1/2}. \tag{8.48i}$$

In many cases the radial matrix elements involving derivatives, eqns (8.29*b*) and (8.29*c*), can also be calculated analytically for harmonic oscillator wave functions.

If in eqns (8.29*b*) and (*c*) $\phi(r)$ is equal or proportional to r^L we can apply the eqns (8.30*a*) and (8.30*b*). For the harmonic oscillator we have†

$$\{(T_i - T_f) - [V_i(r) - V_f(r)]\} = \frac{1}{M_N b^2}\{2(\nu_i - \nu_f) + l_i - l_f\}.$$

Then these radial matrix elements involving derivatives can be expressed in terms of the radial matrix elements $(\nu_f l_f \,|r^L|\, \nu_i l_i)$.

† Note that

$$V_i(r) = V_f(r) = \frac{1}{2M_N b^4} r^2.$$

In the case $\phi(r) = j_L(qr)$ we have to make allowance for the possibility to differentiate explicitly the harmonic oscillator wave functions given in eqn (8.33). Doing so we are, however, able to derive some recurrence relations of which the most important ones are listed in the following (de Forest and Walecka 1966):

Taking into account

$$\frac{1}{2M_N}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa+1}{r}\right)=\begin{cases}\dfrac{1}{2M_N}\left(\dfrac{\mathrm{d}}{\mathrm{d}r}-\dfrac{l}{r}\right) & \text{for } \kappa<0 \quad (8.49a)\\[2ex] \dfrac{1}{2M_N}\left(\dfrac{\mathrm{d}}{\mathrm{d}r}+\dfrac{l+1}{r}\right) & \text{for } \kappa>0 \quad (8.49b)\end{cases}$$

we obtain for $\nu = 1$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l}{r}\right)g_1(r,l)=-\frac{1}{b}\sqrt{(l+\tfrac{3}{2})}g_1(r,l+1) \quad (8.50a)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{l+1}{r}\right)g_1(r,l)=\frac{1}{b}\sqrt{\{2(2l+1)\}}g_1(r,l-1)-\frac{1}{b}\sqrt{(l+\tfrac{3}{2})}g_1(r,l+1). \quad (8.50b)$$

For $\nu \leqslant 3$ the following recursion relations have to be used in addition:

$$g_2(r,l)=\sqrt{(l+\tfrac{3}{2})}g_1(r,l)-\sqrt{(l+\tfrac{5}{2})}g_1(r,l+2) \quad (8.51)$$

$$g_3(r,l)=\frac{1}{\sqrt{2}}[\sqrt{\{(l+\tfrac{5}{2})(l+\tfrac{3}{2})\}}g_1(r,l)-2(l+\tfrac{5}{2}) \times g_1(r,l+2)+\sqrt{\{(l+\tfrac{9}{2})(l+\tfrac{7}{2})\}}g_1(r,l+4)]. \quad (8.52)$$

It is interesting to evaluate the radial matrix elements involving derivatives for some simple cases. Choosing the same examples as given in eqns (8.43a–g) we obtain

$1s_{1/2} \rightarrow 1s_{1/2}$

$$\left.\begin{aligned}&\int f_i j_1(qr) g_f r^2\,\mathrm{d}r\\ &\int f_f j_1(qr) g_i r^2\,\mathrm{d}r\end{aligned}\right\} = -\frac{q}{4M_N}\mathrm{e}^{-(bq/2)^2} \quad (8.53a)$$

$1p_{1/2} \rightarrow 1p_{1/2}$

$$\left.\begin{aligned}&\int f_i j_1(qr) g_f r^2\,\mathrm{d}r\\ &\int f_f j_1(qr) g_i r^2\,\mathrm{d}r\end{aligned}\right\} = \frac{q}{12M_N}\{1+\tfrac{1}{2}(bq)^2\}\mathrm{e}^{-(bq/2)^2} \quad (8.53b)$$

$1p_{1/2} \rightarrow 1p_{3/2}$

$$\int f_i j_1(qr) g_f r^2\,\mathrm{d}r = \frac{q}{12M_N}\{1+\tfrac{1}{2}(bq)^2\}\mathrm{e}^{-(bq/2)^2} \quad (8.53c)$$

$$\int f_f j_1(qr) g_i r^2 \, dr = -\frac{5q}{12M_N}\{1-\tfrac{1}{10}(bq)^2\}e^{-(bq/2)^2} \tag{8.53d}$$

$1p_{3/2} \rightarrow 1p_{3/2}$

$$\left.\begin{aligned}\int f_i j_1(qr) g_f r^2 \, dr \\ \int f_f j_1(qr) g_i r^2 \, dr\end{aligned}\right\} = -\frac{5q}{12M_N}\{1-\tfrac{1}{10}(bq)^2\}e^{-(bq/2)^2} \tag{8.53e}$$

$1d_{3/2} \rightarrow 1d_{3/2}$

$$\left.\begin{aligned}\int f_i j_1(qr) g_f r^2 \, dr \\ \int f_f j_1(qr) g_i r^2 \, dr\end{aligned}\right\} = \frac{q}{4M_N}\{1+\tfrac{2}{15}(bq)^2-\tfrac{1}{60}(bq)^4\}e^{-(bq/2)^2} \tag{8.53f}$$

$1d_{3/2} \rightarrow 1d_{5/2}$

$$\int f_i j_1(qr) g_f r^2 \, dr = \frac{q}{4M_N}\{1+\tfrac{2}{15}(bq)^2-\tfrac{1}{60}(bq)^4\}e^{-(bq/2)^2} \tag{8.53g}$$

$$\int f_f j_1(qr) g_i r^2 \, dr = -\frac{7q}{12M_N}\{1-\tfrac{1}{5}(bq)^2+\tfrac{1}{140}(bq)^4\}e^{-(bq/2)^2} \tag{8.53h}$$

$1d_{5/2} \rightarrow 1d_{5/2}$

$$\left.\begin{aligned}\int f_i j_1(qr) g_f r^2 \, dr \\ \int f_f j_1(qr) g_i r^2 \, dr\end{aligned}\right\} = -\frac{7q}{12M_N}\{1-\tfrac{1}{5}(bq)^2+\tfrac{1}{140}(bq)^4\}e^{-(bq/2)^2} \tag{8.53i}$$

$2s_{1/2} \rightarrow 1d_{5/2}$

$2s_{1/2} \rightarrow 1d_{3/2}$

$$\int f_i j_1(qr) g_f r^2 \, dr = -\frac{7q}{24\sqrt{(10)}\,M_N}(bq)^2\{1-\tfrac{1}{14}(bq)^2\}e^{-(bq/2)^2}. \tag{8.53j}$$

If initial and final states do not have the same quantum numbers, of course, both types of radial integrals $\int f_f j_L(qr) g_i r^2 \, dr$ and $\int f_i j_L(qr) g_f r^2 \, dr$ have different values.

In eqns (8.43) we have always listed the matrix elements for the examples $1p_{1/2} \rightarrow 1p_{3/2}$, $d_{3/2} \rightarrow d_{5/2}$, $2s_{1/2} \rightarrow d_{5/2}$, $2s_{1/2} \rightarrow d_{3/2}$.

The corresponding results for the cases where initial and final states are interchanged can, on the other hand, simply be obtained replacing f_f by f_i and g_i by g_f and vice versa.

In the special examples, where we have calculated explicitly the radial matrix elements, those with $L = 0$ and $L = 2$ always belong to the non-

relativistic nuclear matrix elements and those with $L = 1$ to the relativistic ones. The reason is that for the non-relativistic matrix elements $L + l_f + l_i$ must always be even (this follows from the properties of the geometrical coefficients $G_{KLs}(\kappa_f, \kappa_i)$, see eqn (6.141)) and for the relativistic ones here $L + l_f + l_i$ must be odd.

The differentiation of the radial wave function has the effect of decreasing or increasing l by one unit. Thus for $L = 1$ the above relations can only be fulfilled for radial matrix involving derivatives, i.e. for those belonging to relativistic matrix elements.

By inspection of the above examples we see also that the relativistic matrix elements are smaller than the non-relativistic ones by a factor of about $1/MR \sim 1/20$.

As we have illustrated in the examples given above the q^2-dependence† of the form factors is essentially determined by the radial matrix elements of the type $\phi(r) = j_L(qr)$, i.e. exactly by these radial matrix elements divided by $(qR)^L/(2L+1)!!$. For oscillator wave functions the q^2-behaviour of the form factors for higher q^2-values ($bq/2 > 1$) is governed by the factor

$$e^{-(bq/2)^2}.$$

It should, however, be mentioned that especially the Gaussian shape of the oscillator wave functions and the exponential tail (for instance for a Saxon–Woods potential) of the true wave functions differ outside the nucleus. Therefore the q^2-dependence might also differ from a simple Gaussian behaviour.

On the other hand, for simplicity, oscillator wave functions are usually taken for explicit calculations. The advantages of this type of wave functions have been demonstrated before. Beyond that, it can be shown that the radial wave functions for other more refined potentials, as for example Saxon–Woods, can very suitably be expanded into oscillator wave functions (see Faessler and Sheline 1966). The first four to six terms of the expansion contain 99% of the true wave function in most cases. With such an expansion one has not only the advantages of representing the more refined wave functions with only few numbers, but also one retains the possibility to use the analytic formulae for the radial matrix elements discussed before.

† It should be mentioned that the form factors F_{KLs} for the limit $q^2 \to 0$ can also be calculated by applying the formulae for the radial matrix elements given in eqns (8.44) and (8.30*a* and *b*) since for $q \to 0$

$$j_L(qr) \to \frac{(qr)^L}{(2L+1)!!}.$$

8.1.1.3. Many-particle matrix elements in the j–j coupling scheme

In Section 8.1.1.2. a detailed treatment of the single-particle matrix elements was given. We now are going to discuss the more complicated case of more particles in initial and final nuclear states. For that purpose we will start with the simplest case of two nucleons in a central potential. The antisymmetric wave functions for nucleon 1 in shell j_a and nucleon 2 in shell j_b can then be written as (see, for example, de Shalit and Talmi 1963)

$$\phi(j_a j_b, JMTT_3) = \phi(j_a j_b, JM)\chi(\tfrac{1}{2}\tfrac{1}{2}, TT_3). \tag{8.54}$$

The spin of the composite system and its third component is denoted by J and M, and the isospin and its third component by T and T_3, respectively. $\phi(j_a j_b, JM)$ describes the spin angular part of the wave function and is given by

$$\phi(j_a j_b, JM) = \frac{1}{\sqrt{\{2(1+\delta_{j_a j_b})\}}} \sum_{m_a m_b} C(j_a j_b J; m_a m_b)$$
$$\{\phi_1(j_a m_a)\phi_2(j_b m_b) + (-1)^T \phi_1(j_b m_b)\phi_2(j_a m_a)\}. \tag{8.55}$$

$\chi(\frac{1}{2}\frac{1}{2}TT_3)$ is the isospin part of the wave function and reads as

$$\chi(\tfrac{1}{2}\tfrac{1}{2}TT_3) = \sum_{t_3^{(a)} t_3^{(b)}} C(\tfrac{1}{2}\tfrac{1}{2}T; t_3^{(a)} t_3^{(b)}) \chi_1^{t_3^{(a)}} \chi_2^{t_3^{(b)}}. \tag{8.56}$$

The antisymmetry of the composite state is guaranteed since the antisymmetric (symmetric) isospin part with $T=0$ ($T=1$) is always combined with the symmetric (antisymmetric) spin angular part.

For configurations of two nucleons in identical orbits $j_a = j_b$ the wave functions $(jj, JMTT_3)$ is different from zero only if

$$T + J = \text{odd}. \tag{8.57}$$

The beta-decay transition matrix element between these two particle states is now given by

$$(-1)^{J_f - M_f} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \mathfrak{M}_{KLs} = \sqrt{\left(\frac{4\pi}{2J_i+1}\right)} \langle \phi(j_a' j_b', J_f M_f T_f T_{3f}) |$$
$$\times \sum_{n=1,2} \{O_{KLs}^M t_-\}_n \, |\phi(j_a j_b, J_i M_i T_i T_{3i})\rangle \tag{8.58}$$

(for β^+-decay and electron capture t_- has to be replaced by t_+).

The operators O_{KLs}^M which should be applied to the spin angular part of the wave functions, have been discussed extensively before† (see eqns (8.6) and (8.14)). $n = 1, 2$ indicates the summation over nucleon 1 and 2.

† $O_{KLs}^M = j_L(qr)(1+\lambda\gamma_5)T_{KLs}^M(\hat{r})$ (in the case of the form factors)

or

$$O_{KLs}^M = \left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r)(1+\lambda\gamma_5)T_{KLs}^M(\hat{r}) \text{ (in the case of the form factor coefficients).}$$

Proceeding further, we will start with the treatment of the isospin part. After rewriting the isospin part of the single-particle matrix element as follows

$$\langle \chi_n^{t'_3}|\,\{t_-\}_n\,|\chi_n^{t_3}\rangle = (-1)^{\frac{1}{2}-t'_3}\begin{pmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ -t'_3 & -1 & t_3\end{pmatrix}\sqrt{2}\langle\tfrac{1}{2}|\,|\mathbf{t}|\,|\tfrac{1}{2}\rangle \tag{8.59}$$

we obtain

$$\begin{aligned}
&\langle\chi(\tfrac{1}{2}\tfrac{1}{2}T_fT_{3f})|\sum_{n=1,2}\{t_-\}_n\,|\chi(\tfrac{1}{2}\tfrac{1}{2}T_iT_{3i})\rangle\\
&\quad=\sum_{t^{(a)}t^{(b)}}\sum_{t'^{(a)}t'^{(b)}}\sqrt{(2T_i+1)}\,\sqrt{(2T_f+1)}\,\sqrt{2}\\
&\quad\times(-1)^{-T_{3i}-T_{3f}+\frac{1}{2}}\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & T_i\\ t_3^{(a)} & t_3^{(b)} & -T_{3i}\end{pmatrix}\\
&\quad\times\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & T_f\\ t_3'^{(a)} & t_3'^{(b)} & -T_{3f}\end{pmatrix}\Big\{(-1)^{-t'^{(a)}_3}\begin{pmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ -t_3'^{(a)} & -1 & t_3^{(a)}\end{pmatrix}\\
&\quad\times\delta_{t'^{(b)}_3 t^{(b)}_3}+(-1)^{-t'^{(b)}_3}\begin{pmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ -t_3'^{(b)} & -1 & t_3^{(b)}\end{pmatrix}\delta_{t'^{(a)}_3 t^{(a)}_3}\Big\}\\
&\quad\times\langle\tfrac{1}{2}|\,|\mathbf{t}|\,|\tfrac{1}{2}\rangle=(-1)^{T_f-T_{3f}}\begin{pmatrix}T_f & 1 & T_i\\ -T_{3f} & -1 & T_{3i}\end{pmatrix}\sqrt{[(2T_i+1)(2T_f+1)]}\\
&\quad\times\begin{Bmatrix}\frac{1}{2} & T_f & \frac{1}{2}\\ T_i & \frac{1}{2} & 1\end{Bmatrix}[(-1)^{T_i}+(-1)^{-T_f}]\langle\tfrac{1}{2}|\,|\mathbf{t}|\,|\tfrac{1}{2}\rangle.
\end{aligned}\tag{8.60}$$

Before proceeding further it should be mentioned that‡

$$\langle\tfrac{1}{2}|\,|\mathbf{t}|\,|\tfrac{1}{2}\rangle=\sqrt{\tfrac{3}{2}}\,. \tag{8.61}$$

Similarly, as shown above, we also introduce reduced single-particle matrix elements for the spin angular part by

$$\langle j'm'|\,O^M_{KLs}\,|jm\rangle=(-1)^{j'-m'}\begin{pmatrix}j' & K & j\\ -m' & M & m\end{pmatrix}\langle j'|\,|O_{KLs}|\,|j\rangle. \tag{8.62}$$

Then we have to combine the application of the spin angular operator O^M_{KLs} on particle n with the corresponding application of operator t_- on the same particle. The Racah recoupling for the spin angular part can be carried out analogously to that for the isospin part. Thus, we obtain the

‡ Note that for β^+-decay and electron capture we have

$$\begin{pmatrix}T_f & 1 & T_i\\ -T_{3f} & -1 & T_{3i}\end{pmatrix}\to\begin{pmatrix}T_f & 1 & T_i\\ -T_{3f} & 1 & T_{3i}\end{pmatrix}.$$

result (see also Rose and Osborn 1954):

$$\langle\phi(j'_a j'_b, J_f M_f T_f T_{3f})|\sum_{n=1,2}\{O^M_{KLs}t_-\}_n|\phi(j_a j_b, J_i M_i T_i T_{3i})\rangle$$

$$= \sqrt{\left\{\frac{(2J_i+1)(2J_f+1)(2T_i+1)(2T_f+1)2}{(1+\delta_{i_a i_b})(1+\delta_{i'_a i'_b})}\right\}}(-1)^{J_f-M_f+T_f-T_{3f}+K+1}$$

$$\times\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix} T_f & 1 & T_i \\ -T_{3f} & -1 & T_{3i}\end{pmatrix}\begin{Bmatrix} \frac{1}{2} & T_f & \frac{1}{2} \\ T_i & \frac{1}{2} & 1\end{Bmatrix}$$

$$\times\left\{(-1)^{T_i+J_i+j'_a+j_b-1}\begin{Bmatrix} j'_a & J_f & j_b \\ J_i & j_a & K\end{Bmatrix}\langle j'_a||O_{KLs}||j_a\rangle\,\delta_{j'_b j_b}\right.$$

$$+(-1)^{T_f+J_f+j_a+j_b-1}\begin{Bmatrix} j'_b & J_f & j_a \\ J_i & j_b & K\end{Bmatrix}\langle j'_b||O_{KLs}||j_b\rangle\,\delta_{j'_a j_a}$$

$$+(-1)^{j'_a+j_b}\begin{Bmatrix} j'_a & J_f & j_a \\ J_i & j_b & K\end{Bmatrix}\langle j'_a||O_{KLs}||j_b\rangle\delta_{j_a j'_b}+(-1)^{T_i+T_f+J_i-J_f+1}\begin{Bmatrix} j'_b & J_f & j_b \\ J_i & j_a & K\end{Bmatrix}$$

$$\left.\times\langle j'_b||O_{KLs}||j_a\rangle\delta_{j_b j'_a}\right\}\langle\tfrac{1}{2}||\mathbf{t}||\tfrac{1}{2}\rangle. \tag{8.63}$$

We see, that the transition matrix element is zero if all four spins j_a, j_b, j'_a, j'_b are different. A non-vanishing matrix element is obtained if at least one of the spins, j_a or j_b in the initial state is equal to one of the spins j'_a or j'_b in the final state. In that case only one of the four terms survives. In the case where $j_a = j'_a$ and $j'_b = j_b$ the first two terms contribute, and for $j_a = j'_b$ and $j'_a = j_b$ the last two. If, on the other hand, either two shells in the initial state or two shells in the final state are also equal, two terms contribute.

For a configuration where all the four nucleons in initial and final state are in the same shell, eqn (8.63) simplifies to

$$\langle\phi(j^2, J_f T_f T_3)\|\sum_{n=1,2}\{O_{KLs}t_-\}_n\|\phi(j^2, J_i T_i T_{3i})\rangle$$

$$=(-1)^{T_f-T_{3f}}\begin{pmatrix} T_f & 1 & T_i \\ -T_{3f} & -1 & T_{3i}\end{pmatrix}$$

$$\times\frac{1}{\sqrt{2}}\langle j||O_{KLs}||j\rangle\langle\tfrac{1}{2}||\mathbf{t}||\tfrac{1}{2}\rangle\sqrt{\{(2J_i+1)(2J_f+1)(2T_i+1)(2T_f+1)\}}$$

$$\times(-1)^K\begin{Bmatrix} j & J_f & j \\ J_i & j & K\end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & T_f & \frac{1}{2} \\ T_i & \frac{1}{2} & 1\end{Bmatrix}$$

$$\times\{1+(-1)^{T_i+J_i+1}+(-1)^{T_f+J_f+1}+(-1)^{T_i+J_i+T_f+J_f}\}. \tag{8.64}$$

Here we have written this matrix element in a form where it is reduced in the spin space but not in the isospin space.

With the exception of the reduced single particle matrix element $\langle j||O_{KLs}||j\rangle$ all terms are independent of the multipolarity L and the spin s of the tensor operators O_{KLs}. The above formula can, therefore, also be written as

$$\langle\phi(j^2, J_fT_fT_{3f})\| \sum_{n=1,2}\{O_{KLs}t_-\}_n \|\phi(j^2, J_iT_iT_{3i})\rangle = C(K)\langle j||O_{KLs}||j\rangle \quad (8.65)$$

where the factor $C(K)$ contains the two particle structure. That means we have also† (see eqn (8.58))

$$\mathfrak{M}_{KLs}(q^2) = C(K)\mathfrak{M}_{KLs}(q^2)_{\text{single particle}}. \quad (8.66)$$

Let us now consider the more general case of N nucleons in initial and final state within one shell with spin j. Since in the impulse approximation treatment we are always dealing with one-particle operators we have to build up a normalized and totally antisymmetric (with respect to permutation of all N nucleons) wave function of the form:

$$\phi(j^N v\alpha JM) = \sum_{v_1\alpha_1J_1T_1}[j^{N-1}v_1T_1J_1\alpha_1\,|\}\,j^NvTJ\alpha] \times\{\phi(j^{N-1}v_1T_1J_1\alpha_1)\otimes\phi(j)\}^J_M. \quad (8.67)$$

The bracket and the $\otimes$-sign on the right side symbolize, as before, vector coupling of the angular momentum J_1 and j to a total angular momentum J with third component M. v is the seniority and α indicates additional quantum numbers needed. The coefficients in eqn (8.67) are called coefficients of fractional parentage (CFP). Because an exhaustive discussion of their properties can be found in the book by de Shalit and Talmi (1963) we will not go into further details here.

The above formula, eqn (8.67), represents a coupling of one spectator particle to the other $N-1$ particles in a completely antisymmetric way (in all N particles). Since the Racah recoupling can be carried out analogously to the recoupling in the two particle case, the many particle factor $C(K)$ for a transition within a shell filled with N nucleons is then easily found to be

$$C(K) = (-1)^{T_f-T_{3f}}\begin{pmatrix} T_f & 1 & T_i \\ -T_{3f} & -1 & T_{3i}\end{pmatrix}N\sqrt{\{3(2J_i+1)(2J_f+1)(2T_i+1)\times(2T_f+1)\}}$$
$$\times\sum_{v_1\alpha_1T_1J_1}[j^Nv_fT_fJ_f\alpha_f\,\{|\,j^{N-1}v_1T_1J_1\alpha_1][j^{N-1}v_1T_1J_1\alpha_1\,|\}\,j^Nv_iT_iJ_i\alpha_i]$$
$$\times(-1)^{T_1+\frac{1}{2}+T_f+1+J_1+j+J_f+K}\begin{Bmatrix}\frac{1}{2} & T_f & T_1\\ T_i & \frac{1}{2} & 1\end{Bmatrix}\begin{Bmatrix} j & J_f & J_1\\ J_i & j & K\end{Bmatrix} \quad (8.68)$$

(see de Shalit and Talmi 1963).

† Note that

$$\mathfrak{M}_{KLs_{\text{single particle}}} = \sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\langle j'||O_{KLs}||j\rangle.$$

For $N=2$, where $J_1=j$ and $T_1=\frac{1}{2}$, this formula should be identical with eqn (8.64). Because for $N=2$ the fractional parentage coefficients are equal to 1 and J_f+T_f must be odd, we conclude immediately that this requirement is fulfilled.

For practical applications of the formula given above extensive tables of fractional parentage coefficients can, for example, be found in the publications by Bayman and Lande (1966), Shlomo (1972), and also in the book by de Shalit and Talmi (1963).

Some special values for $C(K)$ are also listed in Table 8.3.

Up to now we have only considered a transition within one shell filled with N particles. It remains, however, to discuss more complicated situations, for example, one where we have two groups of equivalent nucleons in different shells. For that purpose we consider a final state configuration consisting of N_a nucleons in a shell with spin j_a and N_b nucleons in a shell with spin j_b. For the further treatment of that problem it is more suitable to go back to eqn (8.2) and to write the single particle operator in terms of annihilation and creation operators, i.e.

$$O=\sum_{n=1}^{N}\{O^M_{KLs}t_-\}_n \rightarrow \sum_{m_a m_b}\sum_{t_{3a}t_{3b}}\langle j_b\tfrac{1}{2}|\,O^M_{KLs}t_-\,|j_a\tfrac{1}{2}\rangle\, a_{j_a m_a;\frac{1}{2}t_{3a}}\, a^+_{j_b m_b;\frac{1}{2}t_{3b}}. \tag{8.69}$$

By introducing reduced (in spin and isospin space) matrix elements this formula can be rewritten as

$$\begin{aligned} O=&\sum_{m_a m_b}\sum_{t_{3a}t_{3b}}(-1)^{j_a-m_a+\frac{1}{2}-t_{3b}} \\ &\times\sqrt{2}\begin{pmatrix} j_b & K & j_a \\ -m_b & M & m_a\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ -t_{3b} & -1 & t_{3a}\end{pmatrix} \\ &\times\langle j_b\|O_{KLs}\|j_a\rangle\,\langle\tfrac{1}{2}\|\mathbf{t}\|\tfrac{1}{2}\rangle\, a_{j_a}a^+_{j_b} \\ =&\sum_{m_a m_b}\sum_{t_{3a}t_{3b}}\frac{\sqrt{2}}{\sqrt{\{3(2K+1)\}}}C(j_b j_a K;\, m_b-m_a)C(\tfrac{1}{2}\tfrac{1}{2}1;\, t_{3b}-t_{3a}) \\ &\times\langle j_b\|O_{KLs}\|j_a\rangle\,\langle\tfrac{1}{2}\|\mathbf{t}\|\tfrac{1}{2}\rangle\,(-1)^{j_a+m_a}a_{j_a}a^+_{j_b} \end{aligned} \tag{8.70}$$

(in order to avoid too many indices we have written a_{j_a} and $a^+_{j_b}$ instead of $a_{j_a m_a;\frac{1}{2}t_{3a}}$ and $a_{j_b m_b;\frac{1}{2}t_{3b}}$).

This then simply reads† as

$$O=\frac{\sqrt{2}}{\sqrt{\{3(2K+1)\}}}\langle j_b\|O_{KLs}\|j_a\rangle\,\langle\tfrac{1}{2}\|\mathbf{t}\|\tfrac{1}{2}\rangle\,\{a^+_{j_b}\otimes\tilde{a}_{j_a}\}^{K;1}_{M;-1} \tag{8.71}$$

† $\{a^+_{j_b}\otimes\tilde{a}_{j_a}\}^{K,r}_{M,r_3}$ means vector coupling of j_b and j_a in spin and isospin space where K and M is the tensor rank and its third component in spin space, and r and r_3 in the isospin space, respectively.

TABLE 8.3 *Examples of the many-particle factor $C(K)$ for transitions within the $d_{5/2}$-shell*

Number of particles N	Transition	K	$C(K)$
1	J: $5/2^+ \rightarrow 5/2^+$ T: $1/2 \rightarrow 1/2$ T_3: $1/2 \rightarrow -1/2$	0 1 2 3 4	1 1 1 1 1
2	J: $2^+ \rightarrow 2^+$ T: $1 \rightarrow 1$ T_3: $1 \rightarrow 0$ v: $2 \rightarrow 2$	0 1 2 3 4	1·2910 0·5345 −0·4124 −0·6429 0·4792
2	J: $2^+ \rightarrow 2^+$ T: $1 \rightarrow 0$ T_3: $1 \rightarrow 0$ v: $2 \rightarrow 2$	0 1 2 3 4	0 0 0 0 0
3	J: $5/2^+ \rightarrow 5/2^+$ T: $3/2 \rightarrow 3/2$ T_3: $3/2 \rightarrow 1/2$ v: $1 \rightarrow 1$	0 1 2 3 4	1·7320 0·5773 0 0·5773 0
3	J: $5/2^+ \rightarrow 5/2^+$ T: $3/2 \rightarrow 1/2$ T_3: $3/2 \rightarrow 1/2$ v: $1 \rightarrow 1$	0 1 2 3 4	0 0·6172 0·9258 0·6172 0·9258
4	J: $2^+ \rightarrow 2^+$ T: $1 \rightarrow 1$ T_3: $1 \rightarrow 0$ v: $2 \rightarrow 2$	0 1 2 3 4	1·2910 0·0891 −0·2062 −0·1071 0·2396
4	J: $2^+ \rightarrow 2^+$ T: $1 \rightarrow 0$ T_3: $1 \rightarrow 0$ v: $2 \rightarrow 2$	0 1 2 3 4	0 0·5040 −0·2916 −0·6061 0·3388

where

$$\bar{a}_j = (-1)^{j+m} a_{j-m}. \qquad (8$$

Here we now have the situation of a tensor product operator

$$\{a^+_{j_b} \otimes \bar{a}_{j_a}\}^{K;1}_{M;-1}$$

where the two tensor operators $\bar{a}_{j_a}$ and $a^+_{j_b}$ act in different spaces.

The reduced matrix element of such a tensor product operator is now easily obtained by applying a well known formula (see Edmonds 1964; de Shalit and Talmi 1963) which we have already used earlier. The result then reads as†

$$\langle (J_{b_f}T_{b_f})(J_{a_f}T_{a_f})(J_fT_f)\| \{a^+_{j_b}\otimes \bar{a}_{j_a}\}^{K;1}_{M;-1}\| (J_{b_i}T_{b_i})(J_{a_i}T_{a_i})(J_iT_i)\rangle$$
$$= (-1)^{N_a}\sqrt{\{3(2J_i+1)(2J_f+1)(2K+1)(2T_i+1)(2T_f+1)\}}$$
$$\times \begin{Bmatrix} J_{b_f} & J_{b_i} & j_b \\ J_{a_f} & J_{a_i} & j_a \\ J_f & J_i & K \end{Bmatrix} \begin{Bmatrix} T_{b_f} & T_{b_i} & \frac{1}{2} \\ T_{a_f} & T_{a_i} & \frac{1}{2} \\ T_f & T_i & 1 \end{Bmatrix}$$
$$\times \langle J_{b_f}T_{b_f}\| a^+_{j_b}\| J_{b_i}T_{b_i}\rangle \langle J_{a_f}T_{a_f}\| \bar{a}_{j_a}\| J_{a_i}T_{a_i}\rangle \quad (8.73)$$

The last two reduced matrix elements, however, are related to the fractional parentage coefficients introduced before (see, for example, Brussaard and Glaudemans 1977).

We have

$$\langle J_{a_f}T_{a_f}\| \bar{a}_{j_a}\| J_{a_i}T_{a_i}\rangle = (-1)^{J_{a_f}+j_a-J_{a_i}+T_{a_f}+\frac{1}{2}-T_{a_i}}$$
$$\times \sqrt{\{(N_a+1)(2J_{a_i}+1)(2T_{a_i}+1)\}}$$
$$\times [j_a^{N_a}v_{a_f}T_{a_f}J_{a_f}\alpha_{a_f}\,|\}\, j_a^{N+1}v_{a_i}T_{a_i}J_{a_i}\alpha_{a_i}] \quad (8.74)$$

$$\langle J_{b_f}T_{b_f}\| a^+_{j_b}\| J_{b_i}T_{b_i}\rangle = \sqrt{\{N_b(2J_{b_f}+1)(2T_{b_f}+1)\}}$$
$$\times [j^{N_b}v_{b_f}T_{b_f}J_{b_f}\alpha_{b_f}\,\{|\, j^{N_b-1}v_{b_i}T_{b_i}J_{b_i}\alpha_{a_i}]. \quad (8.75)$$

Combining the above eqns (8.71–8.75) we get finally

$$\langle \phi(j_b^{N_b}j_a^{N_a}, J_fT_fT_{3f})\| \sum_n \{O_{KLs}t_-\}_n \|\phi(j_b^{N_b-1}j_a^{n_a+1}, J_iT_iT_{3i})\rangle$$
$$= C(K)\,\langle j_b\|\, |O_{KLs}|\, |j_a\rangle \quad (8.76)$$

where the many particle factor $C(K)$ is now given by

$$C(K) = (-1)^{T_f-T_{3f}}\begin{pmatrix} T & 1 & T_i \\ -T_{3f} & -1 & T_{3i} \end{pmatrix}(-1)^{N_a}\sqrt{\{(N_a+1)N_b\}}$$
$$\times [j_a^{N_a}v_{a_f}T_{a_f}J_{a_f}\alpha_{a_f}\,|\}\, j_a^{N_a+1}v_{a_i}T_{a_i}J_{a_i}][j_b^{N_b}v_{b_f}T_{b_f}J_{b_f}\alpha_{a_f}\,\{|\, j_b^{N_b-1}v_{b_i}T_{b_i}J_{b_i}\alpha_{b_i}]$$
$$\times (-1)^{J_{a_f}+j_a-J_{a_i}+T_{a_f}+\frac{1}{2}-T_{a_i}}\sqrt{\{3(2J_i+1)(2J_f+1)(2J_{a_i}+1)}$$
$$\times (2J_{b_f}+1)(2T_f+1)(2T_i+1)(2T_{a_i}+1)(2T_{b_f}+1)\}$$
$$\times \begin{Bmatrix} J_{b_f} & J_{a_f} & J \\ J_{b_i} & J_{a_i} & J_i \\ j_a & j_b & K \end{Bmatrix} \begin{Bmatrix} T_{b_f} & T_{a_f} & T_f \\ T_{b_i} & T_{a_i} & T_i \\ \frac{1}{2} & \frac{1}{2} & 1 \end{Bmatrix} \quad (8.77)$$

(see de Shalit and Talmi 1963).

† The factor $(-1)^{N_a}$ takes into account the corresponding reordering of the anticommuting annihilation and creation operators (see Brussaard and Glaudemans 1977).

Recently a graphical method for the evaluation of matrix elements between states of many particle configurations has also been derived which is very useful for an explicit calculation (see Keh-Ning Huang 1979).

In the preceding we have derived formulae for the many-particle matrix elements by assuming that nucleon configurations can be described by the simple independent particle shell model in j–j coupling. However, the assumptions made above that the valence nucleons belong to the same shell or subshell is unrealistic. Because of the residual interaction the different nucleons are spread out over all shells or subshells. The distribution of the nucleons over the different shells has then to be calculated theoretically by introducing explicit forms for the residual interaction.

For practical reasons (otherwise the dimension of the matrices to be handled is too large) the full space is, however, usually truncated. Calculations are then restricted to a very few shells or subshells, respectively. The most famous examples in this context are the calculations within the configuration space of the p-shell and of the s–d-shell (see, for instance, Cohen and Kurath 1965; Halbert *et al.* 1971; Brussaard and Glaudemans 1977).

The usual way to treat the problem for a nucleus consisting of A nucleons is to assume an inert core of N_C nucleons and $N = A - N_C$ valence nucleons. Then the model wave functions ψ^{NJT}, which describe the states of N active particles with total angular momentum J and isospin T, are given by the expansion

$$\psi(NJT) = \sum_{k}^{\mathrm{Dim}} a_k \phi_k(NJT). \tag{8.78}$$

The $\phi_k(NJT)$ are the antisymmetric basis many-particle wave functions (independent particle wave functions). The sum runs over all (Dim) basis vectors $\phi_k(NJT)$ of a basis which spans the truncated space chosen. In the s–d shell ($17 \leqslant A \leqslant 39$) the basis wave functions are, for example, given by

$$\phi_k(NJT) = (1s_{1/2})^4_{0,0}(1p_{3/2})^8_{0,0}(1p_{1/2})^4_{0,0} \times \{\{(1d_{5/2})^{n_1}_{J_1,T_1} \otimes (2s_{1/2})^{n_2}_{J_2,T_2}\}_{J_{12}T_{12}} \otimes (1d_{3/2})^{n_3}_{J_3,T_3}\}_{JT} \tag{8.79}$$

where $N = n_1 + n_2 + n_3 = A - 16$ (^{16}O is assumed to be the inert core).

The Hamiltonians are built up by a one body part

$$H_0 = \sum_n \frac{1}{2M_N}\mathbf{p}^2 + \tfrac{1}{2}M_n\omega^2\mathbf{r}^2 - \xi(\mathbf{l}_n\mathbf{s}_n) + \tfrac{1}{6}\eta(\mathbf{l}^2 - 6) \tag{8.80}$$

and a two body part

$$H_1 = \sum_{m<n} V_{mn}. \tag{8.81}$$

The parameters ξ and η are usually chosen in such a way that the single particle levels (in s–d shell for ^{17}O) are correctly reproduced.

For the residual interactions either matrix elements V_{mn} derived from first principles (see, for example, Kuo 1967) or effective interactions obtained by different least squares fits to energy levels (see, for example, Halbert *et al.* 1971; Preedom and Wildenthal 1972; Brussaard and Glaudemans 1977) are used. Then the expansion amplitudes a_t of the eigen functions of the Hamiltonian given above are obtained via matrix diagonalization.

Finally the many-particle beta-transition matrix elements are determined by

$$\langle \psi_f(NJ_fT_f)| |O_{KLs}| |\psi_i(NJ_iT_i)\rangle$$

$$= \sum_{k,l} a_k a_l \langle \phi_l(NJ_fT_f)| |O_{KLs}| |\phi_k(NJ_iT_i)\rangle$$

$$= \sum_{k,l} a_k a_i C_{kl}(K) \langle j_l| |O_{KLs}| |j_k\rangle. \quad (8.82)$$

The factors $C_{kl}(K)$ are the many-particle factors introduced before. Usually the quantities

$$\rho_{kl} = a_k a_i C_{kl}(K) = \langle \psi(NJ_fT_f)| |\{a_{j_l}^{+} \otimes a_{j_k}\}^K| |\psi(NJ_iT_i)\rangle \quad (8.83)$$

are referred to as the one body transition densities. These transition densities contain all the necessary information about the nuclear states in question which are needed for a specific shell-model calculation of beta-transition matrix elements.

Examples for explicit calculations of these transition densities in the s–d shell are, for example, given in the papers by Calaprice *et al.* (1977) and by Wildenthal and Chung (1979). Apart from these some other publications which contain extensive theoretical calculations of beta-decay observables should also be mentioned, i.e. those by Brown *et al.* (1978), and Lanford and Wildenthal (1973).

The preceding discussion gives an illustration of the fact that, however complicated the nuclear states may be, the exact nuclear matrix elements between many-body states can be expanded in a linear combination of single-particle matrix elements.

8.1.1.4. Many-particle matrix elements in the SU(s)–SU(3) coupling scheme

In the foregoing section the application of the j–j coupling scheme to the calculation of beta-decay transition matrix elements has been outlined in detail. Because of the complexity of the many body problem and the lack of knowledge of the basic nucleon–nucleon interaction inside the nucleus,

other simple models have also been constructed which for certain mass regions are more suitable for a description of the nuclear structure than the j–j coupling. One of these models which has been used for the calculation of beta-decay (and also muon capture) observables in the last years is the SU(s)–SU(3) model. We will, therefore, discuss this model in the following in more detail.

For that purpose let us start again with the two nucleon problem. Now, however, we will construct the two particle wave function in L–S coupling, i.e. we couple first $\mathbf{l}_a$ and $\mathbf{l}_b$ to $\mathbf{L}$, $\mathbf{s}_a$ and $\mathbf{s}_b$ to $\mathbf{S}$ and then $\mathbf{L}$ and $\mathbf{S}$ to $\mathbf{J}$. Of course, we have at the same time also to couple the isospins $\mathbf{t}_a$ and $\mathbf{t}_b$ to $\mathbf{T}$. The completely antisymmetric wave function can then be written as (see, for example, de-Shalit and Talmi 1963)

$$\phi(l_a l_b, JMTT_3)\frac{1}{\sqrt{\{2(1+\delta_{l_a l_b})\}}}\sum_{M_L M_S} C(LSJ; M_L M_S)$$
$$\times \sum_{m_a m_b} C(l_a l_b L; m_a m_b)[\phi_1(l_a m_a)\phi_2(l_b m_b)+(-1)^{S+T+1}$$
$$\times \phi_2(l_a m_a)\phi_1(l_b m_b)]\chi(\tfrac{1}{2}\tfrac{1}{2}, SM_S)\chi(\tfrac{1}{2}\tfrac{1}{2}, TT_3) \quad (8.84)$$

where the spin and isospin parts read as (see also eqn 8.56)

$$\chi(\tfrac{1}{2}\tfrac{1}{2}, SM_S) = \sum_{m_s^{(a)} m_s^{(b)}} C(\tfrac{1}{2}\tfrac{1}{2}S; m_s^{(a)} m_s^{(b)})\chi_1^{m^{(a)}}\chi_2^{m^{(b)}} \quad (8.85)$$

$$\chi(\tfrac{1}{2}\tfrac{1}{2}TT_3) = \sum_{t_3^{(a)} t_3^{(b)}} C(\tfrac{1}{2}\tfrac{1}{2}T; t_3^{(a)} t_3^{(b)})\chi_1^{t_3^{(a)}}\chi_2^{t_3^{(b)}}. \quad (8.86)$$

Before proceeding further we will first discuss the symmetry properties of this wave function in a more general way so that the treatment of more than two particles can later on be handled more easily.† The mathematical tool by which symmetry is described is the theory of groups. The next section is, therefore, devoted to a description of some results of group theory for our special case (see, for example, Hamermesh 1962; Harvey 1968, Kramer and Moshinsky 1968; Elliot 1969; Parikh 1978).

In this context we now start with the spin–isospin part of the above wave functions.

At the beginning we consider the 2×2 unitary transformation in the complex two dimensional spin space and in the complex two dimensional isospin space. If U is a unitary 2×2 matrix ($U^+ = U^{-1}$) we have in the spin space

$$\phi' = U_s\phi \quad (8.87)$$

† It should be noted that for $l_a = l_b$ we have $L+S+T=$ odd.

and the isospin space

$$\phi' = U_t \phi. \tag{8.88}$$

Explicitly U_s and U_t can be written in the form

$$U_s = e^{i\frac{1}{2}\sum_{n=1,2} \mathbf{a}\{\sigma\}_n} = e^{i\mathbf{a}\mathbf{S}} \tag{8.89}$$

$$U_t = e^{i\frac{1}{2}\sum_{n=1,2} \mathbf{b}\{\tau\}_n} = e^{i\mathbf{b}\mathbf{T}} \tag{8.90}$$

where† the $\boldsymbol{\sigma}$'s and $\boldsymbol{\tau}$'s are the Pauli spin matrices in spin and isospin space. The $\mathbf{a} = \{a_1, a_2, a_3\}$ and $\mathbf{b} = \{b_1, b_2, b_3\}$ are six arbitrary real numbers. The application of the transformation U_s and U_t, respectively, on our wave functions corresponds to the Euler angle rotations about the angles $a_3, a_2; a_1$ in the spin space and b_3, b_2, b_1 in the isospin space, respectively.‡

Since now

$$\det(e^A) = e^{\mathrm{tr}(A)} \tag{8.91}$$

and since the trace of each component of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ is zero, the U_s and U_t have unit determinant. Thus the covering group is the special unitary group (or unitary unimodular group) in two dimensions SU(2).

If we now assume that the Hamiltonian H_0 of the two particles is spin as well as isospin independent, then it would be invariant not only under the spin SU(2) group and the isospin SU(2) group, but also under the wider group $SU(4) = SU(2) \times SU(2)$ in the four dimensional space of spin–isospin.

Then we have§

$$H_0 = e^{i\mathbf{a}\mathbf{S}} H_0 e^{-i\mathbf{a}\mathbf{S}} \tag{8.92}$$

$$H_0 = e^{i\mathbf{b}\mathbf{T}} H_0 e^{-i\mathbf{b}\mathbf{T}} \tag{8.93}$$

† $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are hermitian, i.e. $\boldsymbol{\sigma}^+ = \boldsymbol{\sigma}$ and $\boldsymbol{\tau}^+ = \boldsymbol{\tau}$.

‡ By expanding

$$e^{i\frac{1}{2}a_1\sigma_1} = 1\left\{1 - \left(\frac{a_1}{2}\right)^2 \Big/ 2! + \left(\frac{a_1}{2}\right)^4 \Big/ 4! - \ldots\right\} + i\sigma_1\left\{\frac{a_1}{2} - \left(\frac{a_1}{2}\right)^3 \Big/ 3! + \left(\frac{a_1}{2}\right)^5 \Big/ 5! - \ldots\right\} = 1\cos\left(\frac{a_1}{2}\right) + i\sigma_1 \sin\left(\frac{a_1}{2}\right)$$

and also

$$e^{i\frac{1}{2}a_2\sigma_2} = 1\cos\left(\frac{a_2}{2}\right) + i\sigma_2 \sin\left(\frac{a_2}{2}\right)$$

$$e^{i\frac{1}{2}a_3\sigma_3} = 1\cos\left(\frac{a_3}{2}\right) + i\sigma_3 \sin\left(\frac{a_3}{2}\right)$$

this fact is immediately verified.

§ Note that

$$e^{icO} H_0 e^{-icO} = H_0 + ic[O, H_0] - \frac{c^2}{2!}[O, [O, H_0]] +$$

where O denotes any hermitian operator.

and therefore

$$[H_0, \mathbf{S}] = 0 \tag{8.94}$$

$$[H_0, \mathbf{T}] = 0. \tag{8.95}$$

But in addition we get (see, for example, Elliot 1969; Parikh 1978)

$$H_0 = e^{i\Sigma_{qq'} c_{qq'} Y_{qq'}} H_0 e^{-i\Sigma_{qq'} c_{qq'} Y_{qq'}} \tag{8.96}$$

and, therefore

$$[H_0, Y_{qq'}] = 0. \tag{8.97}$$

Y is a double tensor operator of degree 1 in both spin and isospin spaces. Its components are given by

$$Y_{qq'} = \tfrac{1}{2} \sum_{n=1,2} \{\sigma_q \tau_{q'}\}_n. \tag{8.98}$$

Usually this scheme is denoted by the supermultiplet scheme.

The symmetry properties of the above wave function belonging to an SU(4) invariant Hamiltonian can be categorized by the so-called Young tableau (see, for example, Hamermesh 1962; de-Shalit and Talmi 1963; Elliot 1969; Eisenberg and Greiner 1972).

For our two particle system this corresponds to

$$\boxed{1}\boxed{2} \quad \text{and} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \tag{8.99}$$

where the former is the symmetric case and the latter the antisymmetric case of the spin–isospin function. The shapes of the Young tableaus are described by the so-called Young patterns $[f]$, i.e. by

$$[f] = [2] \quad \text{and} \quad [f] = [11]. \tag{8.100}$$

As before, the former denotes the symmetric case and the latter the antisymmetric case. Since S and T commute with H_0 they are good quantum numbers.

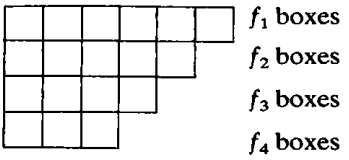
By inspection of eqn (8.55) we see immediately that for the symmetric case

$$[f] = [2] \qquad \begin{array}{l} S = 0, \quad T = 0 \\ S = 1, \quad T = 1 \end{array} \tag{8.101}$$

and for the antisymmetric case

$$[f] = [11] \qquad \begin{array}{l} S = 1, \quad T = 0 \\ S = 0, \quad T = 1. \end{array} \tag{8.102}$$

In general, for N particles we would have



(8.103)

where $f_1+f_2+f_3+f_4=N$ and $f_1 \geqslant f_2 \geqslant f_3 \geqslant f_4$. Then the Young patterns are designated as $[f_1 f_2 f_3 f_4]$.

For $N=3$, for example, the totally symmetric case is denoted by

$$[f]=[3] \tag{8.104}$$

and the totally antisymmetric by

$$[f]=[111]. \tag{8.105}$$

In this case we have also states of mixed symmetry which are characterized by

$$[f]=[21]. \tag{8.106}$$

The Young diagrams indicate symmetrization with respect to nucleon numbers within rows and antisymmetrization within columns. Because we have only four different single particle functions of the spin and isospin co-ordinates the columns cannot contain more than four nucleon numbers. The possible diagrams are thus restricted to†

$$f_1+f_2+f_3+f_4=N. \tag{8.107}$$

As a consequence of the assumption that the Hamiltonian is invariant under the transformation U of SU(4), the Young patterns $[f]$ can be used as a quantum number to characterize many particle states.

After the treatment of the spin–isospin symmetry we have to consider the orbital symmetry. The wave functions $\phi(lm)$ of a single nucleon span a vector space of dimension s where (see Elliot 1958; Harvey 1968; Elliot 1969)

$$\begin{aligned} s &= (2l_a+1)+(2l_b+1) \quad \text{if} \quad l_a \neq l_b \\ s &= 2l+1 \qquad\qquad\quad \text{if} \quad l_a = l_b = l. \end{aligned} \tag{8.108}$$

† The three differences in length of the rows are needed in this case only. Usually one uses the notation

$$\begin{aligned} P &= \tfrac{1}{2}(f_1+f_2-f_3-f_4), \\ P' &= \tfrac{1}{2}(f_1+f_3-f_2-f_4), \\ P'' &= \tfrac{1}{2}(f_1+f_4-f_2-f_3). \end{aligned}$$

The irreducible representations of SU(4) are therefore characterized by (P, P', P'').

The most general group of transformations which preserve the orthonormality of these functions is then the group $U(s)$, and if we ignore transformations with a simple overall change of a phase the group SU(s). The Pauli principle requires that the whole two (or many) particle wave function is antisymmetric. This is, however, only the case if the orbital part is symmetric (antisymmetric) and the spin–isospin part antisymmetric (symmetric). That means the respective Young patterns of the orbital part are adjoint to those of the spin–isospin part (see Elliot 1969), i.e.

orbital symmetry	spin–isospin symmetry	
$[f]=[2]$	$[\tilde{f}]=[11]$	(8.109a)
$[f]=[11]$	$[\tilde{f}]=[2]$.	(8.109b)

Let us now assume H_0 to be the oscillator Hamiltonian in three dimensions,† i.e.

$$H_0=\sum_{n=1,2}\frac{1}{2M_N}\{p^2+r^2/b^4\}_n. \tag{8.110}$$

Then we can define a general group of unitary transformations

$$U_l=e^{i\Sigma_{j,k}C_{jk}A_{jk}} \tag{8.111}$$

where the C_{jk}'s are real numbers (see Elliot 1958, 1969; Eisenberg and Greiner 1972; Parikh 1978) and the generators of the group are given by

$$H_0=\frac{1}{2Mb^2}(A_{xx}+A_{yy}+A_{zz}) \tag{8.112a}$$

$$L_{+1}=\tfrac{1}{2}\sqrt{2}\,(A_{xz}-A_{zx})+\frac{i}{2}\sqrt{2}\,(A_{yz}-A_{zy}) \tag{8.112b}$$

$$L_0=i(A_{yx}-A_{xy}) \tag{8.112c}$$

$$L_{-1}=\tfrac{1}{2}\sqrt{2}\,(A_{xz}-A_{zx})-\frac{i}{2}\sqrt{2}\,(A_{yz}-A_{zy}) \tag{8.112d}$$

$$Q_0=-A_{xx}-A_{yy}+2A_{zz} \tag{8.112e}$$

$$Q_1+Q_{-1}=-i\sqrt{6}\,(A_{yz}+A_{zy}) \tag{8.112f}$$

$$Q_1-Q_{-1}=-\sqrt{6}\,(A_{zx}+A_{xz}) \tag{8.112g}$$

$$Q_2+Q_{-2}=\sqrt{6}\,(A_{xx}-A_{yy}) \tag{8.112h}$$

$$Q_2-Q_{-2}=i\sqrt{6}\,(A_{xy}+A_{yx}). \tag{8.112i}$$

† Then our wave functions $\phi(lm)$ are given by

$$\phi(lm)=\phi_r(r,l)Y_l^m(\hat{r})$$

with the radial wave functions defined by eqn (8.33).

Here the L_q are the three rotation operators

$$L_q = \sum_{n=1,2} \{L'_q\}_n \tag{8.113}$$

with

$$L'_q = (\mathbf{r} \times \mathbf{p})_q \tag{8.114}$$

and the Q_q the five components of a second degree tensor operaor

$$Q_q = \sum_{n=1,2} \{Q'_q\}_n \tag{8.115}$$

with

$$Q'_q = \sqrt{\left(\frac{4\pi}{5}\right)} \frac{1}{b^2} [r^2 Y_2^q(\hat{r}) + b^2 p^2 Y_2^q(\hat{p})]. \tag{8.116}$$

Since

$$[H_0, A_{ij}] = 0 \tag{8.117}$$

the U's comprise a symmetry group for H_0. The group built up by the nine operators is the group U_3. Within a given shell the transformation

$$U_1^{(0)} = e^{iC_0 H_0} \tag{8.118}$$

changes the basis functions by a phase only. Therefore we can omit the operator connected with H_0 and thus end up with the subgroup SU(3).

We should also mention that L_q, $q = -1, 0, +1$, generates an additional subgroup R_3, the group of rotations in three dimensions, i.e. L is also a good quantum number. That means our states of two nucleons are classified by means of the following two chains of groups

$$\mathrm{SU(s)} \supset \mathrm{SU(3)} \supset \mathrm{R(3)} \tag{8.119}$$

for the orbital part.

From the fact that in SU(3) there is only one way of constructing a totally antisymmetric tensor of rank 3, i.e. the Slater determinant, the columns in the Young diagram cannot contain more than three numbers (three boxes). Thus we can denote these states by the Young patterns $[h_1 h_2 h_3]$. In practice, however, two numbers $(\lambda\mu)$ are only needed which are given by

$$\lambda = h_1 - h_2, \qquad \mu = h_2 - h_3. \tag{8.120}$$

Usually they are called Elliot numbers (Elliot 1958). From the discussion above it then follows that our orbital states or wave functions can be classified according to $[f]\ \lambda\mu L$. This classification can be applied not only for $N = 2$ particles but also for every number N of nucleons within an oscillator shell.

In order to elucidate the whole formalism in a little more detail we will now consider explicitly the p-shell and the s–d-shell. Let us begin with the

p-shell. In the shell with the oscillator quantum number $N=1$ there is only one single orbital shell, the 1p-shell. Therefore the full group $(s=(2l+1)=3)$ is here $SU(s)=SU(3)$ and for this reason identical with the harmonic oscillator SU(3) classification. Thus, this last classification tells us nothing new.

In the s–d-shell we have, on the other hand, $s=2l_a+1+2l_b+1=6$ and therefore a chain

$$SU(6)\supset SU(3)\supset R(3). \tag{8.121}$$

Now one nucleon in the s–d-shell transforms, according to SU(3) representation

$$(\lambda\mu)=(20) \tag{8.122}$$

and therefore two nucleons as

$$(20)\times(20)=(40)+(02)+(21). \tag{8.123}$$

Since (Elliot 1958)

$$L=K, K+1, K+2, \ldots, (K+\max\{\lambda,\mu\}) \tag{8.124}$$

where the integer $K=\min\{\lambda,\mu\}, \min\{\lambda,\mu\}-2, \ldots, 1$ or 0 with the exception that if $K=0$

$$L=\max\{\lambda,\mu\}, \max\{\lambda,\mu\}-2, \ldots, 1 \text{ or } 0. \tag{8.125}$$

We have

representation (λ,μ)	L
(40)	0, 2, 4
(02)	0, 2
(21)	1, 2, 3.

(8.126)

In the three dimensional Harmonic oscillator the s and d energy eigenvalues are degenerate. The wave function of our $N=2$ nucleons in the s–d-shell is therefore given by (see eqn (8.84))

$$\phi_{s-d}(JMTT_3)=\sum_{l_a l_b} a(l_a l_b)\phi(l_a l_b, JMTT_3) \tag{8.127}$$

where $l_a=0, 2$ and $l_b=0, 2$.

The coefficients $a(l_a l_b)$ have to be determined by the requirement that the SU(3) operators L_q and Q_q cannot couple two states belonging to different irreducible representations of SU(3). One starts with the (40) $L=4$ state, for which we have simply $a(22)=1$ and $a(00)$, $a(02)$, $=0$. Then one calculates the matrix element Q_q between this (40) $L=4$ state and the (02) $L=2$ state and determines the coefficients by the

TABLE 8.4 *Two-body wave functions in SU(3)–SU(4) coupling for the s–d-shell (Feldmeier et al. 1973)*

$\|f\|$	(λ, μ)	L	S	T	Orbital part
[2]	(4, 0)	0	0	1	$\sqrt{\frac{5}{9}}[ss]^0+\frac{2}{3}[dd]^0$
[2]	(4, 0)	2	0	1	$\sqrt{\frac{7}{18}}([sd]^2+[ds]^2)+\sqrt{\frac{2}{9}}[dd]^2$
[2]	(4, 0)	4	0	1	$[dd]^4$
[2]	(0, 2)	0	0	1	$\sqrt{\frac{2}{9}}[dd]^0-\frac{2}{3}[ss]^0$
[2]	(0, 2)	2	0	1	$\frac{1}{3}([sd]^2+[ds]^2)-\sqrt{\frac{7}{9}}[dd]^2$
[11]	(2, 1)	1	0	0	$-[dd]^1$
[11]	(2, 1)	2	0	0	$\sqrt{\frac{1}{2}}([sd]^2-[ds]^2)$
[11]	(2, 1)	3	0	0	$[dd]^3$
[2]	(4, 0)	0	1	0	
[2]	(4, 0)	2	1	0	
[2]	(4, 0)	4	1	0	
[2]	(0, 2)	0	1	0	same as above
[2]	(0, 2)	2	1	0	
[11]	(2, 1)	1	1	1	
[11]	(2, 1)	2	1	1	
[11]	(2, 1)	3	1	1	

requirement that this matrix element has to be zero. The (40) $L=2$ state then follows from orthogonality. The further proceeding for the other states is then similar and will not be described in further detail. The corresponding two body orbital wave functions in SU(3) coupling are listed in Table 8.4.

For the convenience of the reader we have also listed the SU(3)–SU(4) structure for $N \leqslant 4$ nucleons in the p- and s–d-shell (see Table 8.5).

TABLE 8.5 *TS structure of the supermultiplets with $N \leqslant 4$ (Elliot 1969)*

N	Orbital symmetry	Spin-isospin symmetry	$(PP'P'')$	$(2T+1, 2S+1)$	p-shell L	L	s–d-shell $(\lambda\mu)$
0	[0]	[0]	(0 0 0)	(11)	S	S	(00)
1	[1]	[1]	$(\frac{1}{2}\frac{1}{2}\frac{1}{2})$	(22)	P	SD	(20)
2	[2]	[11]	(1 0 0)	(31) (13)	SD	SDG	(40) (02)
	[11]	[2]	(1 1 1)	(11) (33)	P	PDF	(21)
3	[3]	[111]	$(\frac{1}{2}\frac{1}{2}-\frac{1}{2})$	(22)	PF	SDFGI	(60) (22) (00)
	[21]	[21]	$(\frac{3}{2}\frac{1}{2}\frac{1}{2})$	(22) (24) (42)	PD	SPDFGH	(41) (22) (11)
	[111]	[3]	$(\frac{3}{2}\frac{3}{2}\frac{3}{2})$	(22) (44)	S	PF	(30) (03)
4	[4]	[1111]	(0 0 0)	(11)	SDG	SDFGHIL	(80) (42) (04) (20)
	[31]	[211]	(1 1 0)	(13) (31) (33)	PDF	SPDFGHIK	(61) (42) (23) (31) (12) (20)
	[22]	[22]	(2 0 0)	(11) (33) (15) (51)	SD	SPDFGHI	(42) (31) (04) (20)
	[211]	[31]	(2 1 1)	(13) (31) (33) (35) (53)	P	PDFGH	(50) (23) (31) (12) (01)
	[1111]	[4]	(2 2 2)	(11) (33) (55)		PDF	(21)

After this short look into the L–S and SU(3)–SU(4) coupling scheme let us now come back to the calculation of the beta-transition matrix elements. In this context we have, however, to consider the single particle matrix elements given in eqns (8.21a–d) once again. These single particle matrix elements which are given in j–j coupling have now to be rewritten in a way suitable for L–S coupling, i.e.

$$\langle l'||A_{L'}||l\rangle\langle\tfrac{1}{2}||\Sigma^{(s')}||\tfrac{1}{2}\rangle = \frac{(2L'+1)(2s'+1)}{\sqrt{(2K+1)}} \times \sum_{jj'}\sqrt{\{(2j'+1)(2j+1)\}}\begin{Bmatrix} l' & l & L' \\ \frac{1}{2} & \frac{1}{2} & s' \\ j' & j & K \end{Bmatrix}\langle j'||O_{KLs}||j\rangle \quad (8.128)$$

where $\langle l'||A_{L'}||l\rangle$ is now the reduced single particle matrix element of the beta-decay operator in the orbital space. As shown before (see eqn 6.139) we have

$$\langle\tfrac{1}{2}||\Sigma^{(s')}||\tfrac{1}{2}\rangle = \sqrt{\{(2(2s+1)\}} \quad (8.129)$$

with $\Sigma^{(0)}=1$ and $\Sigma^{(1)}=\boldsymbol{\sigma}$.

The single particle matrix element $\langle l'||A_{L'}||l\rangle$ is then obtained by applying the above equation where the single particle matrix elements $\langle j'||O_{KLs}||j\rangle$ have to be taken from the eqns (8.21a–d). Thus for the non-relativistic matrix elements we get

$$\begin{aligned}\langle l'||A_L||l\rangle &= \langle l(\kappa_f)||A_L||l(\kappa_i)\rangle \\ &= \sqrt{\left\{\frac{(2L+1)(2l(\kappa_f)+1)(2l(\kappa_i)+1)}{2J_i+1}\right\}}\frac{(2L+1)!!}{(qR)^L} \\ &\quad\times i^{l(\kappa_f)+l(\kappa_i)+L}(-1)^{K-s+L}\begin{pmatrix} l(\kappa_f) & l(\kappa_i) & L \\ 0 & 0 & 0\end{pmatrix} \\ &\quad\times\int_0^\infty g_f(r,\kappa_f)j_L(qr)g_i(r,\kappa_i)r^2\,dr. \end{aligned} \quad (8.130)$$

Of course, this result follows also directly from a consideration† of the

† Let us, for example, consider the matrix element ${}^{V}\mathfrak{M}_{KL1}(\mathbf{q}^2)$. The corresponding operaor is (see eqn 4.10b)

$$T^{M}_{KL1}(\hat{\mathbf{r}}) \to i^L(-1)^{K-L}\frac{1}{2M_N}\{Y(\hat{\mathbf{r}})\otimes\mathbf{p}\}^K$$

which acts in the orbital space only. That means we have

$$\begin{aligned}\langle l'||A_K||l\rangle &= \frac{(2K+1)!!}{(qR)^K 2M_N}\,i^{L-l'+l}(-1)^{K-L}\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\langle l'||j_K(qr)\{Y_L\otimes\mathbf{p}\}^K||l\rangle \\ &= \frac{(2K+1)!!}{(qR)^K 2M_N}\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\,i^{L-l'+l}\{(-1)^{K-L+1} \\ &\quad\times\langle l'||j_K(qr)\{Y_L(\hat{\mathbf{r}})\otimes\boldsymbol{\nabla}_l\}^K||l\rangle - \langle l'||\{\boldsymbol{\nabla}_{l'}\otimes Y_L(\hat{\mathbf{r}})\}^K j_K(qr)||l\rangle\}.\end{aligned}$$

(footnote continued on page 304)

explicit operators given in eqns (8.10a and b). Beyond that we see that the index L of the matrix elements is the rank of the corresponding tensor operator in the orbital space and the index s the rank in the spin space. In the case of the relativistic matrix elements the situation is, however, a little more complicated.

By inspection of eqns (8.10a–b) we find that here the indices L and s are not identical with the corresponding ranks of the tensor operators in orbital and spin space. In this case we find for ${}^{A}\mathfrak{M}_{KK0}(\mathbf{q})$

$$\langle l(\kappa_f)|\,|A_{L'}|\,|l(\kappa_i)\rangle = \frac{(2L'+1)\sqrt{\{(2l(\kappa_f)+1)(2l(\kappa_i)+1)\}}}{\sqrt{(2J_i+1)}}$$

$$\times \frac{(2K+1)!!}{(qR)^K}(-1)^{l(\kappa_i)+1}\Bigg[\sum_{\substack{\kappa_i=l\\ \kappa_i=-l-1}} \mathrm{i}^{l(\kappa_f)+l(-\kappa_i)+K}\frac{\kappa_i}{2l(\kappa_i)+1}$$

$$\times \begin{Bmatrix} L' & K & 1 \\ l(-\kappa_i) & l(\kappa_i) & l(\kappa_f)\end{Bmatrix}\frac{\begin{pmatrix} l(\kappa_f) & l(-\kappa_i) & K \\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix} l(\kappa_i) & l(-\kappa_i) & 1 \\ 0 & 0 & 0\end{pmatrix}}$$

$$\times \int f_i(r,\kappa_i)j_K(qr)g_f(r,\kappa_f)r^2\,\mathrm{d}r + \sum_{\substack{\kappa_f=l\\ \kappa_f=-l-1}}(-1)^{L'-1}\mathrm{i}^{l(-\kappa_f)+l(\kappa_i)+K}$$

Here ∇_l acts on the initial state l only, and $\nabla_{l'}$ on the final state l' only. Further on we obtain (see Edmonds 1964)

$$\langle l'|\,|j_K(qr)\{Y_L(\hat{r})\otimes\nabla_l\}^K|\,|l\rangle$$

$$= (-1)^{K+l'+l}\sqrt{(2K+1)}\sum_{l''}\begin{Bmatrix} L & 1 & K \\ l & l' & l''\end{Bmatrix}\langle l'|\,|Y_L(\hat{r})|\,|l''\rangle\,\langle l''|\,|j_K(qr)\nabla|\,|l\rangle$$

and

$$\langle l'|\,|\{\nabla_{l'}\otimes Y_L(\hat{r})\}^K j_K(qr)|\,|l\rangle$$

$$= (-1)^{K+l'+l}\sqrt{(2K+1)}\sum_{l''}\begin{Bmatrix} 1 & L & K \\ l & l' & l''\end{Bmatrix}\langle l''|\,|Y_L(\hat{r})|\,|l'\rangle\,\langle l''|\,|j_K(qr)\nabla|\,|l'\rangle$$

The sum over l'' runs over the two terms $l''=l-1$, $l''=l+1$ or $l''=l'-1$, $l''=l'+1$, respectively. Because

$$\langle l_a|\,|Y_L(\hat{r})|\,|l_b\rangle = \frac{1}{\sqrt{(4\pi)}}(-1)^{l_a}\sqrt{\{(2L+1)(2l_a+1)(2l_b+1)\}}\begin{pmatrix} l_a & L & l_b \\ 0 & 0 & 0\end{pmatrix}$$

and

$$\langle l''|\,|j_K(qr)\nabla|\,|l\rangle = -\mathrm{sign}(\kappa_i)\sqrt{(k_i)}\,2M_N\int f_i(r,\kappa_i)$$

$$\langle l''|\,|j_K(qr)\nabla|\,|l'\rangle = -\mathrm{sign}(\kappa_f)\sqrt{(k_f)}2M_N\int f_f(r,\kappa_f)j_K(qr)g_i(r,\kappa_i)r^2\,\mathrm{d}r$$

we obtain finally the same formula for $\langle l'|\,|A_K|\,|l\rangle$ as given in eqn (8.133).

$$\times\frac{\kappa_f}{2l(\kappa_f)+1}\begin{Bmatrix} L' & K & 1 \\ l(-\kappa_f) & l(\kappa_f) & l(\kappa_i)\end{Bmatrix}\frac{\begin{pmatrix} l(-\kappa_f) & l(\kappa_i) & K \\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix} l(\kappa_f) & l(-\kappa_f) & 1 \\ 0 & 0 & 0\end{pmatrix}}$$

$$\times\int f_f(r,\kappa_f)j_K(qr)g_i(r,\kappa_i)r^2\,\mathrm{d}r\Big], \tag{8.131}$$

$$\langle\tfrac{1}{2}||\Sigma^{(s')}||\tfrac{1}{2}\rangle=\sqrt{6},\quad \text{i.e.}\quad s'=1 \tag{8.132}$$

and for ${}^{V}\mathfrak{M}_{KL1}(\mathbf{q}^2)$

$$\langle l'||A_K||l\rangle=\langle l(\kappa_f)||A_K||l(\kappa_i)\rangle$$

$$=\sqrt{\left\{\frac{(2L+1)(2K+1)(2l(\kappa_f)+1)(2l(\kappa_i)+1)}{(2J_i+1)}\right\}}\frac{(2L+1)!!}{(qR)^L}$$

$$\times(-1)^{l(\kappa_i)+1}\Bigg[\sum_{\substack{\kappa_i=l\\ \kappa_i=-l-1}} i^{l(\kappa_f)+l(-\kappa_i)+L}\frac{\kappa_i}{2l(\kappa_i)+1}$$

$$\times\begin{Bmatrix} K & L & 1 \\ l(-\kappa_i) & l(\kappa_i) & l(\kappa_f)\end{Bmatrix}\frac{\begin{pmatrix} l(\kappa_f) & l(-\kappa_i) & L \\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix} l(\kappa_i) & l(-\kappa_i) & 1 \\ 0 & 0 & 0\end{pmatrix}}$$

$$\times\int f_i(r,\kappa_i)j_L(qr)g_f(r,\kappa_f)r^2\,\mathrm{d}r$$

$$+\sum_{\substack{\kappa_f=l\\ \kappa_f=-l-1}}(-1)^{K-1}i^{l(-\kappa_f)+l(\kappa_i)+L}\frac{\kappa_f}{2l(\kappa_f)+1}\begin{Bmatrix} K & L & 1 \\ l(-\kappa_f) & l(\kappa_f) & l(\kappa_i)\end{Bmatrix}$$

$$\times\frac{\begin{pmatrix} l(-\kappa_f) & l(\kappa_i) & L \\ 0 & 0 & 0\end{pmatrix}}{\begin{pmatrix} l(\kappa_f) & l(-\kappa_f) & 1 \\ 0 & 0 & 0\end{pmatrix}}\int f_f(r,\kappa_f)j_L(qr)g_i(r,\kappa_i)r^2\,\mathrm{d}r\Big], \tag{8.133}$$

$$\langle\tfrac{1}{2}||\Sigma^{(s')}||\tfrac{1}{2}\rangle=\sqrt{2},\qquad \text{i.e. } s'=0. \tag{8.134}$$

Finally, our beta-decay transition operator is now written as

$$O^{M}_{KL's'}=\sum_{M_{s'}M_{L'}}C(s'L'K,M_{s'}M_{L'})\,\Sigma^{(s')}_{M_{s'}}A^{M_{L'}}_{L'} \tag{8.135}$$

where $s'=s$ and $L'=L$ for the non-relativistic matrix elements, and $s'=0$, $L'=K$ if $s=1$ and $s'=1$, $L'=K+1$, K, $K-1$ if $s=0$ for the relativistic matrix elements. In the latter case $A_{L'}$ has to be summed over the three possibilities of L'.

By using the analogous Racah recoupling technique as in the foregoing

Section the two particle matrix element is now obtained

$$\langle\phi(J_f, T_f T_{3f})\Big|\Big|\sum_{n=1,2}\{O^M_{KLs}t_-\}_n\Big|\Big|\phi(J_i, T_i T_{3i})\rangle = (-1)^{T_f-T_{3f}}\begin{pmatrix} T_f & 1 & T_i \\ -T_{3f} & -1 & T_{3i}\end{pmatrix}$$
$$\times\sqrt{\{2(2K+1)(2J_i+1)(2J_f+1)(2L_i+1)(2L_f+1)(2S_i+1)(2S_f+1)}$$
$$\times(2T_i+1)(2T_f+1)\}$$
$$\times\begin{Bmatrix} L_f & S_f & J_f \\ L_i & S_i & J_i \\ L' & s' & K\end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & S_f & \frac{1}{2} \\ S_i & \frac{1}{2} & s'\end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & T_f & \frac{1}{2} \\ T_i & \frac{1}{2} & 1\end{Bmatrix}$$
$$\times\Bigg\{\sum_{\substack{l_i\leqslant l_i' \\ l_f\leqslant l_f'}} a_i(l_i, l_i')a_f(l_f, l_f')(-1)^{L'+1+s'}\Bigg[(-1)^{L_i+S_i+T_i}\sqrt{(2-\delta_{l_i l_i'})}$$
$$\times\begin{Bmatrix} l_f & L_f & l_i' \\ L_i & l_i & L'\end{Bmatrix}\langle l_f||A_{L'}||l_i\rangle\,\delta_{l_i' l_i'}$$
$$+(-1)^{L_f+S_f+T_f}\sqrt{(2-\delta_{l_i' l_i'})}\begin{Bmatrix} l_f' & L_f & l_i \\ L_i & l_i' & L'\end{Bmatrix}\langle l_f'||A_{L'}||l_i'\rangle\,\delta_{l_f l_i}\Bigg]\Bigg\}$$
$$\times\langle\tfrac{1}{2}||\Sigma^{(s')}||\tfrac{1}{2}\rangle\langle\tfrac{1}{2}||\mathbf{t}||\tfrac{1}{2}\rangle. \tag{8.136}$$

The sum runs over $l = l_a, l_b$ and $l' = l_a, l_b$. If $l = l' = l_a$ or $l = l' = l_b$ we have $L_i + S_i + T_i =$ odd and $L_f + S_f + T_f =$ odd. The above formula can, for example, be applied to transitions within the 1s or 1p shell (then $l_a = l_b$), and to transitions within the s–d or f–p shell (then $l_a \neq l_b$). Note that $l + l'$ is even in the cases under consideration.

It is interesting to discuss the special case $L' = 0$ and $A_0 = 1$ which belongs to the matrix elements† ${}^{V}\mathfrak{M}^{(0)}_{000}$ (then $s' = 0$) and ${}^{A}\mathfrak{M}^{(0)}_{101}$ (then $s' = 1$).

The last term in eqn (8.136) comes out to be

$$\sum_{l\leqslant l'} a_i(l, l')a_f(l, l')(-1)^{1+s'}$$
$$\times\Bigg[(-1)^{L_i+S_i+T_i}\begin{Bmatrix} l & L_f & l' \\ L_i & l & 0\end{Bmatrix}\langle l||1||l\rangle + (-1)^{L_f+S_f+T_f}\begin{Bmatrix} l' & L_f & l \\ L_i & l' & 0\end{Bmatrix}\langle l'||1||l'\rangle\Bigg]$$
$$= \frac{(-1)^{s'+1}}{\sqrt{\{(2L_f+1)(2J_i+1)\}}}\{(-1)^{S_i+T_i}+(-1)^{S_f+T_f}\}\,\delta_{L_f L_i}. \tag{8.137}$$

Then the whole transition matrix element for this case differs from zero only if

$$(-1)^{S_i+T_i} = (-1)^{S_f+T_f}.$$

By inspection of eqns (8.101) and (8.102) we see, however, that this corresponds to the fact that the spin–isospin symmetry must be the same in initial and final state, i.e.

$$\left.\begin{matrix} {}^{V}\mathfrak{M}^{(0)}_{000} \\ {}^{A}\mathfrak{M}^{(0)}_{101}\end{matrix}\right\} \neq 0 \quad \text{only for} \quad \begin{matrix} [2]_i \to [2]_f \\ [11]_i \to [11]_f\end{matrix} \tag{8.138}$$

† These matrix elements are the Fermi and Gamow–Teller matrix elements.

where the Young patterns $[f]$ characterize the spin–isospin symmetry (as mentioned before the orbital symmetry is adjoint to the spin–isospin symmetry).

The above selection rule can also be derived from the fact that the operators belonging to the matrix elements ${}^{V}\mathfrak{M}^{(0)}_{000}$ and ${}^{A}\mathfrak{M}^{(0)}_{101}$ given as

$$ {}^{V}\mathfrak{M}^{(0)}_{000} \triangleq \sum_n \{t_-\}_n = T_- \tag{8.139a} $$

$$ {}^{A}\mathfrak{M}^{(0)}_{101} \triangleq \frac{1}{\sqrt{(2J_i+1)}} \sum_n \{\boldsymbol{\sigma} t_-\}_n \tag{8.139b} $$

are members of the group operators for SU(4). They can therefore not couple one symmetry representation $[f]$ to another. This last statement is, however, not restricted to the case of $N=2$ particles but it is valid for every number N of nucleons.

For the Fermi matrix element ${}^{V}\mathfrak{M}^{(0)}_{000}$ we can, on the other hand, easily calculate the corresponding value for every possible configuration even for those consisting of any number of nucleons.

By application of the ladder operator T_- on a wave function with the isospin quantum numbers T_i, T_{3i} (see, for example, Edmonds 1964) we obtain simply†

$$ {}^{V}\mathfrak{M}^{(0)}_{000} = \langle T_f T_{3f} | \, T_- \, | T_i T_{3i} \rangle = \sqrt{\{(T_i + T_{3i})(T_i - T_{3i}+1)\}}\, \delta_{T_f T_i} $$

$$ = (-1)^{T-T_{3f}} \begin{pmatrix} T & 1 & T \\ -T_{3f} & -1 & T_{3i} \end{pmatrix} \sqrt{\{2T(T+1)(2T+1)\}} \tag{8.140} $$

with $T = T_i = T_f$ and for $[\tilde{f}]_f = [\tilde{f}]_i$.

As mentioned earlier the irreducible representation of SU(4) can be characterized by the quantum numbers P, P' and P'' (see, for example, Hamermesh 1962; Elliot 1969). The significance of these quantum numbers is the following: P is the largest possible value of T_3 contained in the supermultiplet, P' is the largest value of S_3 for a state with $T_3 = P$ (P is also the largest value of S_3 and P' the largest value of T_3 for a state with $S_3 = P$ at the same time). Then P'' is the largest value of Y_{33} for a state with $T_3 = P$ and $S_3 = P'$ (or $S_3 = P$ and $T_3 = P'$).

These facts give us the possibility to calculate the Gamow–Teller matrix element ${}^{A}\mathfrak{M}^{(0)}_{101}$ for some special cases (states with the highest weight), i.e. if $T_f = T_i = T = T_{3i} = P$ and $S_f = S_i = S = S_{3i} = P'$.

† For β^+-decay we would have

$$ \langle T_f T_{3f} | \, T_+ \, | T_i T_{3i} \rangle = \sqrt{\{(T - T_{3i})(T + T_{3i}+1)\}}. $$

Then we have

$$
{}^{A}\mathfrak{M}^{(0)}_{101} = \sqrt{\{6(2J_f+1)(2L+1)\}} \begin{Bmatrix} L & S & J_f \\ L & S & J_i \\ 0 & 1 & 1 \end{Bmatrix}
\times \frac{(-1)^{T-T_{3f}} \begin{pmatrix} T & 1 & T \\ -T_{3f} & -1 & T_{3i} \end{pmatrix}}{\begin{pmatrix} S & 1 & S \\ -S & 0 & S \end{pmatrix} \begin{pmatrix} T & 1 & T \\ -T & 0 & T \end{pmatrix}}
\times \underbrace{\langle [\tilde{f}_f] \| Y_{33} \| [\tilde{f}_i] \rangle}_{P''} \delta_{[\tilde{f}_f][\tilde{f}_i]}. \tag{8.141}
$$

The $3j$-symbols can explicitly be evaluated with the result

$$
{}^{A}\mathfrak{M}^{(0)}_{101} = (-1)^{T-T_{3f}} \begin{pmatrix} T & 1 & T \\ -T_{3f} & -1 & T_{3i} \end{pmatrix}
\times \sqrt{\{6(2J_f+1)(2L+1)\}} \begin{Bmatrix} L & S & J_f \\ L & S & J_i \\ 0 & 1 & 1 \end{Bmatrix}
\times \sqrt{\{S(S+1)(2S+1)T(T+1)(2T+1)\}} \frac{P''}{PP'} \delta_{[\tilde{f}_f][\tilde{f}_i]} \tag{8.142}
$$

Beta-transitions where $J_i^\pi = J_f^\pi$, $T_i = T_f$, and $[\tilde{f}_i] = [\tilde{f}_f]$ are usually called superallowed transitions.

The residual interactions between the nucleons have now the consequence that the states with the highest orbital symmetry (lowest spin–isospin symmetry) are lying lowest in energy.

Consequently, for a given T value the supermultiplets with the smallest values of P, P' and P'' compatible with that isospin values correspond to lowest states in energy. Since as far as T is considered the lowest states in energy are also those with lowest possible T, i.e. those for $T = |T_3|$ we have $P = T = |T_3|$.

The quantum number P' is then associated with S where we assume the values $S = 0$ for an e–e nucleus (e = even, o = odd), $S = 1$ for an o–o nucleus and $S = \frac{1}{2}$ for an e–o or an o–e nucleus.

This situation is surveyed in Table 8.6. It is, however, important to note that because of the situation outlined above we often have for the states of interest for beta-decay $P = T$ and $P' = S$. Thus the above formula can be applied. It should, however, be noted that L remains undefined in a certain sense. For a discussion of selection rules see also Petrauskas and Vanagas (1968), Petrauskas and Sabaliauskas (1976), and Mukhopadhyay and Cannata (1974). Tests on the validity of the supermultiplet model in

TABLE 8.6 *Quantum numbers P, P′ and P″ for the ground and lowest nuclear states*

	A even		A odd	
	$Z=2k, N=2k$	$Z=2k+1, N=2k'+1$	$Z=2k, N=2k'+1$	$Z=2k+1, N=2k'$
(P, P', P'')	$(T, 0, 0)$	$(T, 1, 0)$	$(T\ \frac{1}{2}\ \frac{1}{2})$	$(T\ \frac{1}{2}\ -\frac{1}{2})$

nature have, by the way, been carried out by Franzini and Radicati (1963), Burdet *et al.* (1968, 1969). Explicit theoretical calculations of the level structure for a number of nuclei in the s–d-shell have also been performed (see Feldmeier *et al.* 1973; Conze *et al.* 1973).

As has been done before in the case of j–j coupling (see Section 8.1.1.3) the formula (8.136) specialized for two nucleons can also be generalized to nuclear states where we have N nucleons. Because the Racah recoupling technique does not differ from the two nucleon case we obtain similarly (see Kramer and Moshinsky 1968)

$$\langle\phi([f_f]J_fL_fS_fT_f)\Big|\Big|\sum_{n=1}^{A}\{O_{KLs}t_-\}_n\Big|\Big|\phi([f_i]J_iL_iS_iT_i)\rangle$$

$$=(-1)^{T_f-T_{3f}}\begin{pmatrix}T_f & 1 & T_i\\ -T_{3f} & 1 & T_{3i}\end{pmatrix}\sum_{[f_1]}N\sqrt{\binom{n_f}{n_{f_1}}}\sqrt{\binom{n_{f_1}}{n_{f_i}}}$$

$$\times\sqrt{\{(2K+1)(2J_i+1)(2J_f+1)(2L_i+1)(2L_f+1)(2S_i+1)(2S_f+1)(2T_i+1)\times(2T_f+1)\}}$$

$$\times\begin{Bmatrix}L_f & S_f & J_f\\ L_i & S_i & J_i\\ L' & s' & K\end{Bmatrix}\sum_{\substack{\lambda_1\mu_1\nu_1\\ ll'L_1}}(-1)^{L_1+l+L_f+L'}$$

$$\times\begin{Bmatrix}l' & L_f & L_1\\ L_i & l & L'\end{Bmatrix}\langle l'||A_{L'}||l\rangle$$

$$\times\langle[f_f]L_{f'},(\lambda_f\mu_f),\nu_f\{|l'[f_1]L_1,(\lambda_1\mu_1),\nu_1\rangle$$

$$\times\langle\nu_1,(\lambda_1\mu_1)L_1[f_1]l|\}[f_i]L_i,(\lambda_i\mu_i)\nu_i\rangle$$

$$\times\sum_{\alpha_1S_1T_1}(-1)^{S_i+\frac{1}{2}+S_f+s'}\begin{Bmatrix}\frac{1}{2} & S_f & S_1\\ S_i & \frac{1}{2} & s'\end{Bmatrix}\sqrt{2}$$

$$\times(-1)^{T_i+\frac{1}{2}+T_f+1}\begin{Bmatrix}\frac{1}{2} & T_f & T_1\\ T_i & \frac{1}{2} & 1\end{Bmatrix}$$

$$\times\langle\tfrac{1}{2}||\Sigma^{(s')}||\tfrac{1}{2}\rangle\langle\tfrac{1}{2}||\mathbf{t}||\tfrac{1}{2}\rangle\langle\gamma^n[\tilde{f}_f]S_fT_f\{|\tfrac{1}{2}\tfrac{1}{2}\gamma^{n-1}[\tilde{f}_1]\alpha_1S_1T_1\rangle$$

$$\times\langle\tfrac{1}{2}\tfrac{1}{2}\gamma^{n-1}[\tilde{f}_1]S_1T_1|\}\gamma^n[\tilde{f}_i]S_iT_i\rangle. \tag{8.143}$$

n_f and n_{f_1} are the respective dimensions of the representations S_n and

S_{n-1} (see Jahn and van Wieringen 1951) while

$$\langle \nu_1(\lambda_1\mu_1)L_1[f_1]l|\}[f]L, (\lambda\mu)\nu\rangle$$

and

$$\langle \tfrac{1}{2}\tfrac{1}{2}\gamma^{n-1}[\tilde{f}_1]S_1T_1|\}\ \gamma^n[\tilde{f}]ST\rangle$$

are the orbital and charge-spin fractional parentage coefficients.† Tables for these coefficients can be found in the articles by Jahn (1951) and Jahn and van Wieringen (1951). The weight factor $\sqrt{(n_{f_1}/n_f)}$ can also be found in these papers. A computer code for the calculation of fractional parentage coefficients is described by Braunschweig (1978).

The spin–isospin part of the above matrix element formula (8.143) is tabulated for $N=3$ and $N=4$ nucleons in Tables 8.7–8.10. These tables can, for example, be used for the calculation of the Gamow–Teller matrix element if the special formula (8.142) cannot be applied.

The SU(6)–SU(3) model has been applied for the explicit calculation of beta-decay matrix elements within the s–d-shell by Behrens *et al.* (1978). As also shown in this publication the main advantage of the SU(s)–SU(3) model in comparison with the shell model in j–j coupling is that the bulk of configurations in this scheme mixes very little for the low lying states if the residual interaction is switched on (see Hag and Parikh 1973; Feldmeier and Manakos 1977). That means in the SU(s)–SU(3) scheme we have always one main wave function component with an amplitude of about 70–90% in contrast to the situation in the j–j coupling scheme where many wave function components are more or less of the same order of magnitude. This fact, of course, simplifies the matrix element calculations in many cases since the relevant features for some special beta-transition can be extracted by a simple inspection instead of carrying out large computer calculations.

Beyond that the important smaller wave function components may be calculated in low order perturbation theory (see Feldmeier and Manakos 1977) as an alternative to the usually applied large scale matrix diagonalization.

8.1.1.5. Excited or deformed core

8.1.1.5.1. *Weak coupling.* The previous treatment of the nuclear matrix element calculation was based on the assumption of an inert spherical core with spin and isospin equal to zero. This picture of an inert core does not always, however, correspond to the true situation occurring in nature.

† It should be noted that

$$[l^{n-1}(\alpha_1 S_1 L_1)lSL\ ||\ l^n\alpha SL] = \sqrt{\left(\frac{n_{f_1}}{n_f}\right)}\langle \nu_1(\lambda_1\mu_1)L_1[f_1]l\ ||[f]L, (\lambda, \mu)\nu\rangle \times \langle \tfrac{1}{2}\tfrac{1}{2}\gamma^{n-1}[\tilde{f}_1]S_1T_1|\}\ \gamma^n[\tilde{f}]ST\rangle.$$

TABLE 8.7*a* *Spin–isospin part† of eqn (8.143) for the matrix element for 3 nucleons and for $s' = 0$*

$[\bar{f}]$ f / ST / i	$[111]$ $\frac{1}{2}\,\frac{1}{2}$	$[21]$ $\frac{1}{2}\,\frac{1}{2}$	$[21]$ $\frac{3}{2}\,\frac{1}{2}$	$[21]$ $\frac{1}{2}\,\frac{3}{2}$	$[3]$ $\frac{1}{2}\,\frac{1}{2}$	$[3]$ $\frac{3}{2}\,\frac{3}{2}$
$[111]$ $\frac{1}{2}\,\frac{1}{2}$	0·2041	−0·4082	0	−0·2887	0	0
$[21]$ $\frac{1}{2}\,\frac{1}{2}$	−0·4082	0·2041 0·2041	0	−0·2887 0·2887	0·4082	0
$[21]$ $\frac{3}{2}\,\frac{1}{2}$	0	0	0·4330 −0·1443	0	0	−0·2887
$[21]$ $\frac{1}{2}\,\frac{3}{2}$	0·2887	0·2887 −0·2887	0	0·3227 0·3227	0·2887	0
$[3]$ $\frac{1}{2}\,\frac{1}{2}$	0	0·4082	0	−0·2887	0·2041	0
$[3]$ $\frac{3}{2}\,\frac{3}{2}$	0	0	0·2887	0	0	0·2282

—— $f_1 = [11]$
– – – $f_1 = [2]$

$$\dagger \sum_{\alpha_1 S_1 T_1} (-1)^{S_i+\frac{1}{2}+S_f+s'} \begin{Bmatrix} \frac{1}{2} & S_f & S_1 \\ S_i & \frac{1}{2} & s' \end{Bmatrix} (-1)^{T_1+\frac{1}{2}+T_f+1} \begin{Bmatrix} \frac{1}{2} & T_f & T_1 \\ T_i & \frac{1}{2} & 1 \end{Bmatrix} \sqrt{2}$$

$\langle \frac{1}{2} || \Sigma^{(s')} || \frac{1}{2} \rangle \langle \frac{1}{2} || t || \frac{1}{2} \rangle \langle \gamma^n [\bar{f}_f] S_f T_f \{ [\frac{11}{22} \gamma^{n-1} [\bar{f}_1] \alpha_1 S_1 T_1 \rangle$
$\langle [\frac{11}{22} \gamma^{n-1} [\bar{f}_1] S_1 T_1] \} \gamma^n [\bar{f}_i] S_i T_i \rangle$.

TABLE 8.7*b* *Spin–isospin part of eqn (8.143) for the matrix element for 3 nucleons and for $s' = 1$*

$[\bar{f}]$ f / ST / i	$[111]$ $\frac{1}{2}\,\frac{1}{2}$	$[21]$ $\frac{1}{2}\,\frac{1}{2}$	$[21]$ $\frac{3}{2}\,\frac{1}{2}$	$[21]$ $\frac{1}{2}\,\frac{3}{2}$	$[3]$ $\frac{1}{2}\,\frac{1}{2}$	$[3]$ $\frac{3}{2}\,\frac{3}{2}$
$[111]$ $\frac{1}{2}\,\frac{1}{2}$	−0·3536	0	0·5000	−0·5000	0	0
$[21]$ $\frac{1}{2}\,\frac{1}{2}$	0	−0·3536 0·5893	−0·5000 −0·1667	−0·5000 −0·1667	0·4714	−0·3333
$[21]$ $\frac{3}{2}\,\frac{1}{2}$	−0·5000	0·5000 0·1667	0·5590 −0·1863	−0·4714	−0·1667	−0·3727
$[21]$ $\frac{1}{2}\,\frac{3}{2}$	0·5000	0·5000 0·1667	−0·4714	0·5590 −0·1863	−0·1667	−0·3727
$[3]$ $\frac{1}{2}\,\frac{1}{2}$	0	0·4714	0·1667	0·1667	0·5893	0·3333
$[3]$ $\frac{3}{2}\,\frac{3}{2}$	0	−0·3333	0·3727	0·3727	0·3333	0·2946

—— $f_1 = [11]$
– – – $f_1 = [2]$

TABLE 8.8a *Spin–isospin part of eqn (8.143) for the matrix element for 4 nucleons and for $s' = 0$*

$[\tilde{f}]_f$ ST i	[1111] 00	[211] 10	[211] 01	[211] 11	[22] 00	[22] 11	[22] 20	[22] 02	[31] 10	[31] 01	[31] 11	[31] 21	[31] 12	[4] 00	[4] 11	[4] 22
[1111] 00	0	0	−0·7071	0	0	0	0	0	0	0	0	0	0	0	0	0
[211] 10	0	0	0	−0·4082 0·2041	0	0	0	0	0	0	0·3536	0	0	0	0	0
[211] 01	0·7071	0	0·5773 0·1443	0	0·5000	0	0	−0·3536	0	−0·4330	0	0	0	0	0	0
[211] 11	0	0·4082 −0·2041	0	0·3333 0·0833	0	−0·2041	0	0	0·2041	0	0·1443	0	0·2041	0	0	0
[22] 00	0	0	−0·5000	0	0	0	0	0	0	0·5000	0	0	0	0	0	0
[22] 11	0	0	0	−0·2041	0	0·1667	0	0	−0·3333	0	0·1179	0	0·1667	0	0	0
[22] 20	0	0	0	0	0	0	0	0	0	0	0	−0·3162	0	0	0	0
[22] 02	0	0	0·3536	0	0	0	0	0·3873	0	0·3536	0	0	0	0	0	0
[31] 10	0	0	0	−0·2041	0	0·3333	0	0	0	0	−0·2357 0·1179	0	0	0	−0·3333	0
[31] 01	0	0	−0·4330	0	−0·5000	0	0	−0·3536	0	0·5773 0·1443	0	0	0	0·7071	0	0
[31] 11	0	−0·3536	0	0·1443	0	0·1179	0	0	0·2357 −0·1179	0	0	0	0·2357 −0·1179	0	0·2357	0
[31] 21	0	0	0	0	0	0	0·3162	0	0	0	0	−0·1291 0·2582	0	0	0	−0·2236
[31] 12	0	0	0	−0·2041	0	−0·1667	0	0	0	0	−0·2357 0·1179	0	0·2236 0·2236	0	0·1667	0
[4] 00	0	0	0	0	0	0	0	0	0	−0·7071	0	0	0	0	0	0
[4] 11	0	0	0	0	0	0	0	0	0·3333	0	0·2357	0	−0·1667	0	0·1667	0
[4] 22	0	0	0	0	0	0	0	0	0	0	0	0·2236	0	0	0	0·1732

– – – $f_1 = [111]$ —— $f_1 = [21]$ – · – $f_1 = [3]$

TABLE 8.8*b* *Spin–isospin part of eqn (8.143) for the matrix element for 4 nucleons and for $s' = 1$*

$[\tilde{f}]f$ / ST / i	[1111] 00	[211] 10	[211] 01	[211] 11	[22] 00	[22] 11	[22] 20	[22] 02	[31] 10	[31] 01	[31] 11	[31] 21	[31] 12	[4] 00	[4] 11	[4] 22
[1111] 00	0	0	0	0·7071	0	0	0	0	0	0	0	0	0	0	0	0
[211] 10	0	0	−0·7071 0·3536	−0·5773 −0·1443	0	0·3536	0	0	0	−0·3536	0·2500	−0·3536	0	0	0	0
[211] 01	0	−0·7071 0·3536	0	−0·5773 −0·1443	0	−0·3536	0	0	0·3536	0	−0·2500	0	0·3536	0	0	0
[211] 11	0·7071	0·5773 0·1443	0·5773 0·1443	0·4714 −0·2357	0	0	−0·3536	−0·3536	−0·1443	0·1443	0	−0·2887	0·2887	0	0	0
[22] 00	0	0	0	0	0	0·4082	0	0	0	0	0·5773	0	0	0	0	0
[22] 11	0	−0·3536	0·3536	0	0·4082	0	0·2887	0·2887	−0·1179	−0·1179	0·3333	0·2357	0·2357	0	0	0
[22] 20	0	0	0	−0·3536	0	0·2887	0	0	0	0	−0·2041	−0·3873	0	0	0	0
[22] 02	0	0	0	0·3536	0	0·2887	0	0	0	0	−0·2041	0	−0·3873	0	0	0
[31] 10	0	0	0·3536	0·1443	0	0·1179	0	0	0	−0·7071 −0·3536	−0·3333 0·4167	0·3536	0	0	−0·4714	0
[31] 01	0	−0·3536	0	−0·1443	0	0·1179	0	0	−0·7071 −0·3536	0	−0·3333 0·4167	0	0·3536	0	−0·4714	0
[31] 11	0	−0·2500	0·2500	0	0·5773	0·3333	−0·2041	−0·2041	0·3333 −0·4167	0·3333 −0·4167	0·2357 0·2357	−0·1667 −0·1667	−0·1667 −0·1667	0·4082	0·1667	−0·2887
[31] 21	0	−0·3536	0	0·2887	0	−0·2357	0·3873	0	0·3536	0	0·1667 0·1667	−0·1581 0·3162	−0·3536	0	−0·1179	−0·2739
[31] 12	0	0	0·3536	−0·2887	0	−0·2357	0	0·3873	0	0·3536	0·1667 0·1667	−0·3536	−0·1581 0·3162	0	−0·1179	−0·2739
[4] 00	0	0	0	0	0	0	0	0	0	0	0·4082	0	0	0	0·5773	0
[4] 11	0	0	0	0	0	0	0	0	0·4714	0·4714	0·1667	0·1179	0·1179	0·5773	0·3536	0·2041
[4] 22	0	0	0	0	0	0	0	0	0	0	−0·2887	0·2739	0·2739	0	0·2041	0·2121

– – – $f_1 = [111]$ —— $f_1 = [21]$ – · – $f_1 = [3]$

TABLE 8.9 *The factor* $\sqrt{\left(\frac{n_{f_1}}{n_f}\right)}$

3 nucleons

$[\tilde{f}]$ \ $[\tilde{f}_1]$	[2]	[11]
[3]	1	
[21]	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
[111]		1

4 nucleons

$[\tilde{f}]$ \ $[\tilde{f}_1]$	[3]	[21]	[111]
[4]	1		
[31]	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	
[22]		1	
[211]		$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$
[1111]			1

Since the core has collective degrees of freedom we can either have slow nuclear surface vibrations or static nuclear deformations. In the following we first intend to take into account the former kind of core excitations assuming the coupling between shell model and vibrational states to be weak. We then speak of the weak coupling model. Reviews about this model can be found in the articles by Alaga (1969), and Arima and Hamamoto (1971), and especially in the book by Eisenberg and Greiner (1970b).

The Hamiltonian which describes the single particle and the collective motion simultaneously is of the form

$$H = H_{sp} + H_{coll} + H_{int} \tag{8.144}$$

where H_{coll} is the collective and H_{sp} is the single particle Hamiltonian. The latter has been discussed extensively before. The interaction H_{int} between H_{sp} and H_{coll} is assumed to be weak. Therefore one can treat H_{sp} plus H_{coll} as the unperturbed Hamiltonian. Its solutions are then

given by

$$\phi(J'NRIM) = \sum_{M_J M_R} C(J'RI; M_J M_R)\phi(J'M_J)\phi(NRM_R) \quad (8.145)$$

where the single particle wave function is denoted by $\phi(J'M_J)$ and the collective one by $\phi(NRM_R)$. N is the total number of phonons and R their total angular momentum. The basic matrix element associated with the coupling $\langle J', 12; I\,|H_{\text{int}}|\,J, 00; I\rangle$ describes a process involving the emission or absorption of a vibrational quantum.

The wave function of the nuclear state in first order perturbation is then

$$\phi(IM) = a_0\phi(I00IM) + \sum_{J'} \alpha_{J'}\phi(J'12IM) \quad (8.146)$$

if $\phi(I00IM)$ denotes the wave function of the ground state with no quadrupole phonons present ($N = 0$).

Assuming that the core with quantum numbers N, R acts as a spectator during the transition as we also have always assumed before we obtain for the beta-transition matrix element† (see Edmonds 1964)

$$\langle I_f|\,|O_{KLs}|\,|I_i\rangle = \sqrt{\{(2I_i+1)(2I_f+1)\}} \sum_{J_f' J_i'} (-1)^{J_f'+R+I_i+K} \times a_{J_f'} a_{J_i'} \begin{Bmatrix} J_f' & I_f & R \\ I_i & J_i' & K \end{Bmatrix} \langle J_f'|\,|O_{KLs}|\,|J_i'\rangle. \quad (8.147)$$

The reduced transition matrix element $\langle J_f'|\,|O_{KLs}|\,|J_i'\rangle$ is the transition matrix element between the single- or many-particle shell model states discussed in Section 8.1.1.2 and 8.1.1.3.

† In the case of a core with isospin $T_c \neq 0$ within the isospin formalism we would have

$$\langle I_f T_f T_{3f}\left|\left|\sum_n \{O_{KLs} t_-\}_n\right|\right| I_i T_i T_{3i}\rangle = (-1)^{T_f - T_{3f}} \begin{pmatrix} T_f & 1 & T_i \\ -T_{3f} & -1 & T_{3i} \end{pmatrix} \sqrt{\{(2I_i+1)(2I_f+1)(2T_i+1)(2T_f+1)\}} \times \sum_{\substack{J_f' J_i' \\ T_f' T_i' T_c}} (-1)^{J_f'+R+I_i+K+T_f+T_c+T_i+1} a_{J_f'} a_{J_i'} \times \begin{Bmatrix} J_f' & I_f & R \\ I_i & J_i' & K \end{Bmatrix} \begin{Bmatrix} T_f' & T_f & T_c \\ T_i & T_i' & 1 \end{Bmatrix} \times \langle J_f' T_f'\left|\left|\sum_n \{O_{KLs} t_-\}_n\right|\right| J_i' T_i'\rangle.$$

It should be noted that this formula is also useful if $R = 0$ and $T_c \neq 0$, i.e. for an unexcited core with $T_c \neq 0$.

TABLE 8.10*a* *Nuclear structure factor*† X_N *for the Gamow–Teller matrix element (eqn 8.143) for 3 nucleons*

$[\tilde{f}]$ ST f / i	[111] $\frac{1}{2}\,\frac{1}{2}$	[21] $\frac{1}{2}\,\frac{1}{2}$	[21] $\frac{3}{2}\,\frac{1}{2}$	[21] $\frac{1}{2}\,\frac{3}{2}$	[3] $\frac{1}{2}\,\frac{1}{2}$	[3] $\frac{3}{2}\,\frac{3}{2}$
[111] $\frac{1}{2}\,\frac{1}{2}$	−4·2426	0	0	0	0	0
[21] $\frac{1}{2}\,\frac{1}{2}$	0	1·4142	−5·6571	−5·6571	0	0
[21] $\frac{3}{2}\,\frac{1}{2}$	0	5·6571	4·4721	−5·6571	0	0
[21] $\frac{1}{2}\,\frac{3}{2}$	0	5·6571	−5·6571	4·4721	0	0
[3] $\frac{1}{2}\,\frac{1}{2}$	0	0	0	0	6·6672	4·4451
[3] $\frac{3}{2}\,\frac{3}{2}$	0	0	0	0	4·4451	14·142

$$\dagger\, A_{\mathfrak{gR}_{101}} = (-1)^{T-T_{3f}} \begin{pmatrix} T_f & 1 & T_i \\ -T_{3f} & -1 & T_{3i} \end{pmatrix} \sqrt{\{3(2J_f+1)(2L+1)\}} \begin{Bmatrix} L & S_f & J_f \\ L & S_i & J_i \\ 0 & 1 & 1 \end{Bmatrix} X_N$$

That means in the case where the coefficients $a_{J'}$ are known the matrix elements of the above type can be evaluated without any difficulties, i.e. the problem reduces to the calculation of these coefficients. For a more detailed treatment of this calculation we should, however, refer to the publications mentioned above.

Examples for the application of the formulae given above are, for instance, given in the papers by Appel *et al.* (1974) and by Behrens *et al.* (1975).

8.1.1.5.2. Strong coupling and deformed nuclei. For nuclei with static deformations of the core the interaction between the collective mode and the motion of the valence nucleons is very strong. A treatment in first order perturbation is therefore not the appropriate one. Instead, the motion of the individual nucleons in a deformed potential has to be solved exactly, i.e. we arrive at a shell model for deformed potentials.

The Hamiltonian is now given by

$$H = H_{\text{coll}} + H_{\text{sp}} + H_{\text{int}}. \tag{8.148}$$

Usually the collective Hamiltonian includes rotation, vibration and the interaction between both modes. For the following we assume, however, that we are in the vibrational ground state.

TABLE 8.10*b* *Nuclear structure factor X_N for the Gamow–Teller matrix element (eqn 8.143) for 4 nucleons*

$[\tilde{f}]$ f / ST / i	[1111] 00	[211] 10	[211] 01	[211] 11	[22] 00	[22] 11	[22] 20	[22] 02	[31] 10	[31] 01	[31] 11	[31] 21	[31] 12	[4] 00	[4] 11	[4] 22
[1111] 00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[211] 10	0	0	0	−6	0	0	0	0	0	0	0	0	0	0	0	0
[211] 01	0	0	0	−6	0	0	0	0	0	0	0	0	0	0	0	0
[211] 11	0	6	6	0	0	0	0	0	0	0	0	0	0	0	0	0
[22] 00	0	0	0	0	0	4·8984	0	0	0	0	0	0	0	0	0	0
[22] 11	0	0	0	0	4·8984	0	7·7466	7·7466	0	0	0	0	0	0	0	0
[22] 20	0	0	0	0	0	7·7466	0	0	0	0	0	0	0	0	0	0
[22] 02	0	0	0	0	0	7·7466	0	0	0	0	0	0	0	0	0	0
[31] 10	0	0	0	0	0	0	0	0	0	−5·6572	3·4646	6·3254	0	0	0	0
[31] 01	0	0	0	0	0	0	0	0	−5·6572	0	3·4646	0	6·3254	0	0	0
[31] 11	0	0	0	0	0	0	0	0	−3·4646	−3·4646	8·4852	−7·7475	−7·7475	0	0	0
[31] 21	0	0	0	0	0	0	0	0	6·3254	0	7·7475	9·487	−7·072	0	0	0
[31] 12	0	0	0	0	0	0	0	0	0	6·3254	7·7475	−7·072	9·487	0	0	0
[4] 00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6·928	0
[4] 11	0	0	0	0	0	0	0	0	0	0	0	0	0	6·928	12·730	12·250
[4] 22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12·250	21·210

$s' = 1$

Then we have

$$H_{\text{coll}} = H_{\text{rot}} \tag{8.149}$$

with

$$H_{\text{rot}} = \sum_{n=1}^{3} \frac{L_n'}{2\theta_n} \tag{8.150}$$

where the L_n' are the three components of the angular momentum operator in the body fixed co-ordinate system (indicated by the prime) of the deformed core. If we consider the nucleus as a liquid drop with a spheroidal equilibrium shape the three moments of inertia θ_n are given by

$$\theta_n = \tfrac{4}{3}\theta_0 \sin^2(\gamma - \tfrac{2}{3}\pi n). \tag{8.151}$$

θ_0 is the moment of inertia parameter which is related (for the picture of an irrotational fluid) to the deformation β by† (R = nuclear radius)

$$\theta_0 = \frac{9}{8\pi} A M_N R^2 \beta^2. \tag{8.152}$$

γ is the triaxial deformation parameter. In the case of the axially symmetric rotator about the 3-axis we have $\theta_1 = \theta_2 = \theta_0$, i.e. $\gamma = 0$ or π.

The value $\gamma = 0$ yields a prolate (cigar shaped) ellipsoid and $\gamma = \pi$ an oblate ellipsoid.‡ It should also be mentioned that in truth θ_0 is not (as follows from the above equation) a constant for different excited rotational states of the nucleus but a function of the quantum numbers of these states like the spin (see Scharff-Goldhaber *et al.* 1976). Then the eigenvalues of H_{rot} are given by

$$\begin{aligned} H_{\text{rot}}\psi_{\text{rot}} &= \left[\frac{\mathbf{L}^2 - L_3'^2}{2\theta_0} + \frac{L_3'^2}{2\theta_3}\right]\psi_{\text{rot}} \\ &= \left[\frac{R(R+1) - K^2}{2\theta_0} + \frac{K^2}{2\theta_3}\right]\psi_{\text{rot}}. \end{aligned} \tag{8.153a}$$

† It should be noted that this formula gives very unrealistic values for the moment of inertia θ_0. Better results can be obtained by using a semiempirical relation

$$\theta_0 = \frac{3}{1225} A^{7/3}\beta^2 [\text{MeV}^{-1}]$$

(Bogdan *et al.* 1979).

‡ For a rigid rotator of ellipsoidal form we would have

$$\theta_1 = \theta_2 = \theta_0 = \tfrac{2}{5} A M_N R^2 \left(1 \pm \frac{1}{4}\sqrt{\left(\frac{5}{\pi}\right)}\beta\right)$$

$$\theta_3 = \tfrac{2}{5} A M_N R^2 \left(1 \mp \frac{1}{2}\sqrt{\left(\frac{5}{\pi}\right)}\beta\right)$$

(upper sign $\gamma = 0$, lower sign $\gamma = \pi$).

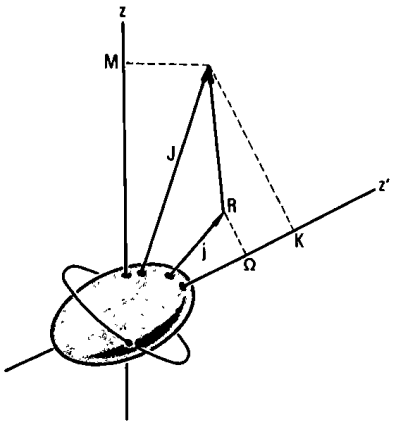


FIG. 8.1. Angular momentum vectors and co-ordinate system for deformed nuclei.

The rotational symmetry, i.e. $\gamma = 0$, leads to the momentum of inertia $\theta_3 = 0$, the reason being that in our model no rotation around the symmetry axis can be defined. Equivalently one could consider that fact from the viewpoint that for $\theta_3 \to 0$ an excitation of this component of rotation is associated with a very large energy. Thus we can write

$$H_{\text{rot}}\psi_{\text{rot}} = \frac{R(R+1)-K^2}{2\theta_0}\psi_{\text{rot}}. \qquad (8.153b)$$

The corresponding normalized wave function ψ_{rot} is given by

$$\psi_{\text{rot}} = \sqrt{\left\{\frac{2R+1}{16\pi^2(1+\delta_{K0})}\right\}}\{D^R_{MK}(\theta)+(-1)^{R+K}D^R_{M-K}(\theta)\}. \qquad (8.154)$$

The D^R_{MK} are the usual rotation matrices (see Edmonds 1964). θ symbolizes the three Euler angles relating the body fixed system to the laboratory system. R is the total angular momentum of the deformed core, K is the projection of R to the body fixed symmetry axis (3-axis) and M the projection of R to the z-axis in the laboratory system (see Fig. 8.1). The wave function, eqn (8.154), has to be invariant (with the exception of a phase factor) for a rotation $e^{-i\theta R_3} = e^{-i\theta K}$ about the symmetry axis. This implies that in the axially symmetric case we have $K = 0$, i.e. quantum mechanically no rotation about the 3-axis can be defined even for the rigid symmetric rotator. The form of the wave function (symmetrization) given above is chosen in such a way that the transformation from the body fixed to the laboratory co-ordinate system is unique (see, for example, Eisenberg and Greiner 1970*b*).

For further treatment we have now to consider the single-particle Hamiltonian H_{sp} for a deformed potential. For the sake of simplicity we first assume an oscillator potential.

Then we obtain (see Nilsson 1955)

$$H_{sp} = H_0 + H_{ls} + H_\delta \tag{8.155}$$

with

$$H_0 = \frac{\mathbf{p}'^2}{2M} + \tfrac{1}{2}M\omega^2 r'^2, \tag{8.156a}$$

$$H_{ls} = C\mathbf{l}'\mathbf{s}' + D\mathbf{l}'^2, \tag{8.156b}$$

and

$$H_\delta = -\delta M\omega_3^4 \sqrt{\left(\frac{\pi}{5}\right)} r'^2 Y_2^0(\hat{r}'). \tag{8.157}$$

This corresponds to an anisotropic oscillator where the equipotentials are ellipsoids with

$$\omega_{x'}^2 = \omega_{y'}^2 = \omega_0^2(1+\tfrac{2}{3}\delta) \tag{8.158}$$

$$\omega_{z'}^2 = \omega_0^2(1-\tfrac{4}{3}\delta). \tag{8.159}$$

In first order δ is related to the deformation β by†

$$\delta = \frac{3}{4}\sqrt{\left(\frac{5}{\pi}\right)}\beta. \tag{8.160}$$

As before the primes indicate that all the operators in eqns (8.156) and (8.157) are defined relative to the body fixed system. The term connected with the coefficient C is the usual spin orbit coupling term and that connected with the coefficient D is a term which interpolates between harmonic oscillator and square well potentials. The solutions of the well-known spherical harmonic oscillator, i.e. of H_0, have been used before (see eqn (8.33)). After including the spin-angular wave function (see eqn (8.16b)) they can be written as (see also Fig. 8.1)

$$\phi(j\Omega) = g_\nu(r', \kappa)\chi_\kappa^\Omega(\hat{r}') \tag{8.161}$$

where Ω is the projection of j to the body fixed symmetry axis (3-axis). The wave functions of eqn (8.161) are also the solutions of $H_0 + H_{ls}$ since C and D are constants.

The additional inclusion of the oscillator well anisotropy, on the other hand, leads to the consequence that the spin's j are no longer good quantum numbers. The single particle wave function can, however, then

† A compilation of the deformation parameters β deduced from different experiments for a range of nuclei was published by Löbner *et al.* (1970). In that article a discussion of various other definitions for the deformation parametrization was also given.

be written as a series of eigen functions $\phi(j\Omega)$ of the Hamiltonian H_0+H_{ls}, i.e.

$$\chi_\Omega(\mathbf{r}') = \sum_j C_{j\Omega}\phi(j\Omega) \tag{8.162}$$

and

$$H_{sp}\chi_\Omega(\mathbf{r}') = E_{N\Omega}\chi_\Omega(\mathbf{r}'). \tag{8.163}$$

It should be noted that the summation over j includes a summation over different l-values but such that the parity $(-1)^l$ has a definite value. The expansion coefficients† $C_{j\Omega}$ are normalized according to

$$\sum_j C_{j\Omega}^2 = 1. \tag{8.164}$$

Tables for these or related coefficients can, for example, be found in the paper by Nilsson (1955) and the book by Davidson (1968).

The oscillator quantum number $N=2(\nu-1)+l$ is usually assumed to be also a good quantum number‡ so that the single particle eigen functions of H_{sp} can be classified according to the quantum numbers Ω and N.

The oscillator constant ω_0 is usually chosen to be (see Section 8.1.1.2)

$$\omega_0 = 41A^{-1/3} \tag{8.165}$$

This corresponds to the oscillator parameter

$$b = 2.60\times10^{-3}A^{1/6}. \tag{8.166}$$

Of course, the adoption of the harmonic oscillator potential for H_0+H_{ls} is the simplest assumption. Other more refined potentials have, therefore, also been introduced, like, for example, the Woods–Saxon potential by Faessler and Sheline (1966). In that paper the wave functions χ_Ω are calculated as some linear combinations of Woods–Saxon spherical wave

† In the uncoupled representation (Nilsson 1955), where $\mathbf{l}$ and $\mathbf{s}$ are not coupled to j, χ_Ω is written as

$$\chi_\Omega = \sum_{l\Lambda} a_{l\Lambda}|Nl\Lambda\Sigma\rangle \quad \text{with} \quad \Omega = l+\Sigma.$$

(Note that $|NL\Lambda\Sigma\rangle = i^l(-1)^\nu\,|Nl\Lambda\Sigma\rangle_{\text{Nilsson}}$). The corresponding quantum numbers Λ and Σ are the projections of $\mathbf{l}$ and $\mathbf{s}$ to the body fixed symmetry axis. The coefficients $C_{j\Omega}$ are then related to the $a_{l\Lambda}$ by

$$C_{j\Omega} = \sum_{\Lambda\Sigma} C(l\tfrac{1}{2}j;\Lambda\Sigma)a_{l\Lambda}.$$

‡ That is not exactly true since the quadrupole field $r'^2Y_2^0$ mixes the quantum numbers N. However, in most cases this mixing is small and can be neglected (see, for example, Nilsson 1955; Bohr and Mottelson 1975; Eisenberg and Greiner 1970*b*).

functions (with the coefficients $C_{j\Omega}$) and the last ones as linear combinations of spherical harmonic oscillator wave functions (with the coefficients C_ν). Comprehensive tables can be also found in that publication.

The total wave function for one particle in a deformed well of $H = H_{\text{rot}} + H_{\text{sp}}$ can then be writen as† (see, for instance, Nilsson 1955; Davidson 1968; Eisenberg and Greiner 1970*b*; Bohr and Mottelson 1975)

$$\phi(JMK;\Omega) = \sqrt{\left\{\frac{2J+1}{16\pi^2(1+\delta_{K0})}\right\}}\{D^J_{MK}(\theta)\chi_\Omega(\mathbf{r}') + (-1)^{J+K+\Omega+\frac{1}{2}}\pi_\Omega D^J_{M-K}(\theta)\chi_{-\Omega}(\mathbf{r}')\} \quad (8.167)$$

where $\pi_\Omega = (-1)^l$ is the parity of the wave functions χ_Ω. The corresponding energy eigenvalues of

$$H = H_{\text{sp}} + \frac{(\mathbf{J}-\mathbf{j}')^2 + (J'_3 - j'_3)^2}{2\theta_0} + \frac{(J'_3 - j'_3)}{2\theta_3} \quad (8.168a)$$

are now (see eqn 8.153)

$$E_{JKN\Omega} = E_{N\Omega} + \frac{J(J+1)-(K-\Omega)^2}{2\theta_0} + \frac{(K-\Omega)^2}{2\theta_3} \quad (8.168b)$$

where J is the total angular momentum $\mathbf{J} = \mathbf{R} + \mathbf{j}$. The terms with $(\mathbf{j}'^2/2\theta_0) - (\mathbf{jJ}/\theta_0)$ have been regrouped and included in H_{int}. They are therefore not reflected in the eigenvalue given above.

From eqn (8.167) we see immediately that we always have $K = \Omega$ since the wave function has to be invariant for a rotation about the symmetry axis, i.e. about the 3-axis.

When we denote the different possible single particle states by χ_{Ω_0}, χ_{Ω_1}, χ_{Ω_2} etc. the ground state band is given by

$$K = \Omega_0 \quad (8.169)$$

with the angular momentum of the various rotational states

$$J = \Omega_0, \Omega_0 + 1, \Omega_0 + 2, \ldots \quad (8.170)$$

For the excited single particle states we have the analogous rotational bands

$$K - \Omega_n = 0 \qquad n = 1, 2, 3 \quad (8.171a)$$

$$J = |\Omega_n|, |\Omega_n| + 1, |\Omega_n| + 2, \ldots \quad (8.171b)$$

It remains to consider the interaction Hamiltonian H_{int}. The most impor-

† The unsymmetrized wave function could simply be written as

$$\phi(JMK;\Omega) = \psi_{\text{rot}}(\dot{\theta})\chi_\Omega(\mathbf{r}') = ND^J_{MK}(\theta)\chi_\Omega(\mathbf{r}').$$

The proper symmetrization, however, is necessary because the transformation of the wave function from the body fixed system to the laboratory system has to be unique (see, for example, Eisenberg and Greiner 1970*b*).

tant term of this Hamiltonian is the so-called Coriolis interaction, i.e.

$$H_{\text{int}} = H_{\text{Coriolis}} = -\frac{\mathbf{Jj}}{\theta_0}. \tag{8.172}$$

This interaction leads to a partial decoupling of the particle motion from the rotator, i.e. to the so-called decoupling. Generally it mixes two rotational bands with $\Delta K = \pm 1$ so that K is no longer a good quantum number (see, for example, Eisenberg and Greiner 1970*b*; Bohr and Mottelson 1975).

Since for our purpose of a calculation of beta-decay matrix elements this latter point is not of much relevance we will not give a more detailed discussion here, but refer to the literature cited above.

Let us now turn to the beta-decay transition matrix elements. These matrix elements are determined by the application of beta-decay operators† O^M_{KLs} given in eqns (8.6) and (8.14) on the wave function of eqn (8.167), i.e. especially on the single particle wave functions $\chi_\Omega(\mathbf{r}')$ in the body fixed system.

For that purpose we have to transform our beta-decay operators O^M_{KLs} to the body fixed system. Since the beta-decay operators O^M_{KLs} are tensor operators of rank K this transformation is easily carried out by

$$O^M_{KLs} = \sum_{M'} D^K_{MM'}(\theta)\{O^{M'}_{KLs}\}' \tag{8.173}$$

(see, for example, Edmonds 1964).

Then we obtain for the transition matrix element itself

$$\begin{aligned}
&\langle \phi(J_f M_f K_f; \Omega_f)|\, O^M_{KLs} t_- \,|(J_i M_i K_i; \Omega_i)\rangle \\
&= \frac{1}{16\pi^2}\sqrt{\left\{\frac{(2J_f+1)(2J_i+1)}{(1+\delta_{K_f0})(1+\delta_{K_i0})}\right\}}\sum_{M'}\Big\{\int (D^{J_f}_{M_f K_f}(\theta)\chi_{\Omega_f}(\mathbf{r}'))^+ \\
&\times D^K_{MM'}(\theta) O^{M'}_{KLs} D^{J_i}_{M_i K_i}(\theta)\chi_{\Omega_i}(\mathbf{r}')\,\mathrm{d}\theta\,\mathrm{d}^3r' + (-1)^{J_i+K_i+\Omega_i+\frac{1}{2}}\pi_{\Omega_i} \\
&\times \int (D^{J_f}_{M_f K_f}(\theta)\chi_{\Omega_f}(\mathbf{r}))^+ D^K_{MM'}(\theta) O^{M'}_{KLs} D^{J_i}_{M_i -K_i}(\theta)\chi_{-\Omega_i}(\mathbf{r}')\,\mathrm{d}\theta\,\mathrm{d}^3r' \\
&+ (-1)^{J_f+K_f+\Omega_f+\frac{1}{2}}\pi_{\Omega_f}\int (D^{J_f}_{M_f -K_f}(\theta)\chi_{-\Omega_f}(\mathbf{r}))^+ \\
&\times D^K_{MM'}(\theta) O^{M'}_{KLs} D^{J_i}_{M_i K_i}(\theta)\chi_{\Omega_i}(\mathbf{r}')\,\mathrm{d}\theta\,\mathrm{d}^3r' \\
&+ (-1)^{J_f+K_f+\Omega_f+J_i+K_i+\Omega_i+1}\pi_{\Omega_i}\pi_{\Omega_f}\int (D^{J_f}_{M_f -K_f}(\theta)\chi_{-\Omega_f}(\mathbf{r}'))^+ \\
&\times D^K_{MM'}(\theta) O^{M'}_{KLs} D^{J_i}_{M_i -K_i}(\theta)\chi_{-\Omega_i}(\mathbf{r}')\,\mathrm{d}\theta\,\mathrm{d}^3r'\Big\}.
\end{aligned} \tag{8.174}$$

† $O^M_{KLs} = j_L(qr)(1+\lambda\gamma_5)T^M_{KLs}(\hat{r})$ (in the case of the form factors)
or
$O^M_{KLs} = \left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r)(1+\lambda\gamma_5)T^M_{KLs}(\hat{r})$
(in the case of the form factor coefficients).

The integration of the three rotation matrices can, however, easily be carried out since (see Edmonds 1964)

$$\int (D^{J}_{M_f K_f}(\theta))^{+} D^{K}_{MM'}(\theta) D^{J}_{M_i K_i}(\theta)\, \mathrm{d}\theta = 8\pi^2 (-1)^{M_f - K_f} \times \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \begin{pmatrix} J_f & K & J_i \\ -K_f & M' & K_i \end{pmatrix}. \quad (8.175)$$

Thus we get

$$\begin{aligned} &\langle \phi(J_f M_f K_f; \Omega_f) |\, O^{M}_{KLs} t_{-} \,| \phi(J_i M_i K_i; \Omega_i) \rangle \\ &= \frac{1}{2} \sqrt{\left\{ \frac{(2J_f+1)(2J_i+1)}{(1+\delta_{K_f 0})(1+\delta_{K_i 0})} \right\}} (-1)^{J_f - M_f} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \\ &\times \sum_{M'} \left\{ (-1)^{J_f - K_f} \begin{pmatrix} J_f & K & J_i \\ -K_f & M' & K_i \end{pmatrix} \int \chi^{+}_{\Omega_f}(\mathbf{r}') O^{M'}_{KLs} \right. \\ &\times \chi_{\Omega_i}(\mathbf{r}')\, \mathrm{d}^3 r' + (-1)^{J_i + K_i + \Omega_i + \frac{1}{2} + J_f - K_f} \pi_{\Omega_i} \\ &\times \begin{pmatrix} J_f & K & J_i \\ -K_f & M' & -K_i \end{pmatrix} \int \chi^{+}_{\Omega_f}(\mathbf{r}') O^{M'}_{KLs} \chi_{-\Omega_i}(\mathbf{r}')\, \mathrm{d}^3 r' \\ &+ (-1)^{\Omega_f + \frac{1}{2}} \pi_{\Omega_f} \begin{pmatrix} J_f & K & J_i \\ K_f & M' & K_i \end{pmatrix} \int \chi^{+}_{-\Omega_f}(\mathbf{r}') O^{M'}_{KLs} \chi_{\Omega_i}(\mathbf{r}')\, \mathrm{d}^3 r' \\ &+ (-1)^{J_i + K_i + \Omega_f + \Omega_i + 1} \pi_{\Omega_i} \pi_{\Omega_f} \begin{pmatrix} J_f & K & J_i \\ K_f & M' & -K_i \end{pmatrix} \\ &\times \left. \int \chi^{+}_{-\Omega_f}(\mathbf{r}') O^{M'}_{KLs} \chi_{-\Omega_i}(\mathbf{r}')\, \mathrm{d}^3 r' \right\}. \end{aligned} \quad (8.176)$$

This formula can further be simplified because†

$$\int \chi^{+}_{\Omega_f}(\mathbf{r}') O^{M'}_{KLs} \chi_{\Omega_i}(\mathbf{r}')\, \mathrm{d}^3 r' = \sum_{j_f} \sum_{j_i} C_{j_f \Omega_f} C_{j_i \Omega_i} (-1)^{j_i - \Omega_f} \times \begin{pmatrix} j_f & K & j_i \\ -\Omega_f & M' & \Omega_i \end{pmatrix} \langle j_f | \,| O_{KLs} |\, | j_i \rangle \quad (8.177a)$$

and therefore

$$\int \chi^{+}_{-\Omega_f}(\mathbf{r}') O^{-M'}_{KLs} \chi_{-\Omega_i}(\mathbf{r}')\, \mathrm{d}^3 r' = (-1)^{K-1+2\Omega_f} \pi_{\Omega_i} \pi_{\Omega_f} \times \int \chi^{+}_{\Omega_f}(\mathbf{r}') O^{M'}_{KLs} \chi_{\Omega_i}(\mathbf{r}')\, \mathrm{d}^3 r'. \quad (8.177b)$$

† It should be noted that

$$\chi_{-\Omega} = \sum C_{j-\Omega} \phi(j - \Omega)$$

and

$$C_{j\Omega} = (-1)^{\frac{1}{2} - l} \pi_{\Omega} C_{j-\Omega}.$$

By first applying the last of the two formulae listed above (eqn (8.177*b*)) we arrive at

$$\begin{aligned}
&\langle\phi(J_fM_fK_f;\Omega_f)|\,O^M_{KLs}t_-\,|\phi(J_iM_iK_i;\Omega_i)\rangle \\
&\quad=\sqrt{\left\{\frac{(2J_f+1)(2J_i+1)}{(1+\delta_{K_f0})(1+\delta_{K_i0})}\right\}}(-1)^{J_f-M_f}\begin{pmatrix}J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix} \\
&\qquad\times\sum_{M'}\left\{(-1)^{J_f-K_f}\begin{pmatrix}J_f & K & J_i\\ -K_f & M' & K_i\end{pmatrix}\int\chi^+_{\Omega_f}(\mathbf{r}')O^{M'}_{KLs}\right. \\
&\qquad\times\chi_{\Omega_i}(\mathbf{r}')\,\mathrm{d}^3r'+(-1)^{\Omega_f+\frac{1}{2}}\pi_{\Omega_f}\begin{pmatrix}J_f & K & J_i\\ K_f & M' & K_i\end{pmatrix} \\
&\qquad\left.\times\int\chi^+_{-\Omega_f}(\mathbf{r})O^{M'}_{KLs}\chi_{\Omega_i}(\mathbf{r}')\,\mathrm{d}^3r'\right\}. \qquad (8.178)
\end{aligned}$$

That means two of the four terms in eqn (8.176) are equal to the other two. For the next step we can make use of eqn (8.177*a*).

Finally we then obtain for the reduced single particle matrix element in a deformed well†

$$\begin{aligned}
&\langle\phi(J_fK_f;\Omega_f)|\,|O_{KLs}t_-|\,|\phi(J_iK_i;\Omega_i)\rangle \\
&\quad=\sqrt{\left\{\frac{(2J_f+1)(2J_i+1)}{(1+\delta_{K_f0})(1+\delta_{K_i0})}\right\}}\sum_{j_fj_i}C_{j_f\Omega_f}C_{j_i\Omega_i} \\
&\qquad\times\left\{(-1)^{J_f-K_f+j_f-\Omega_f}\begin{pmatrix}J_f & K & J_i\\ -K_f & \Omega_f-\Omega_i & K_i\end{pmatrix}\right. \\
&\qquad\times\begin{pmatrix}j_f & K & j_i\\ -\Omega_f & \Omega_f-\Omega_i & \Omega_i\end{pmatrix}+\begin{pmatrix}J_f & K & J_i\\ K_f & -\Omega_f-\Omega_i & K_i\end{pmatrix} \\
&\qquad\left.\times\begin{pmatrix}j_f & K & j_i\\ \Omega_f & -\Omega_f-\Omega_i & \Omega_i\end{pmatrix}\right\}\langle j_f|\,|O_{KLs}|\,|j_i\rangle \qquad (8.179)
\end{aligned}$$

where $\Omega_f=\Omega_p$ and $\Omega_i=\Omega_n$. Usually we have more than one nucleon outside a deformed core.‡

† It should be noted that we have now a K selection rule

$$K\geqslant|K_f-K_i|$$

in addition to the usual angular momentum selection rule

$$|J_i-J_f|\leqslant K\leqslant J_i+J_f.$$

‡ The $\chi_{\Omega_{\text{many particle}}}$ can be written as

$$\chi_{\Omega_{\text{many particle}}}=\frac{1}{\sqrt{N!}}\sum(-1)^PP\chi^{(1)}_{\Omega_1}\ldots\chi^{(N)}_{\Omega_N}.$$

Here we have $\Omega_s=\sum_i\Omega_i$. P denotes the operator of permutation and $(-1)^P$ is the parity of the permutation. N is the number of nucleons outside the deformed core.

In the Nilsson approach the residual interaction between the valence nucleons is, however, neglected with the exception of a pairing effect. This latter interaction has the consequence that the individual nucleons interact with one another in such a way that their quantum numbers differ only in sign of Ω and pairs of nucleons with $\Omega_s = 0$ are always built up.

While in e–o and o–e nuclei (e = even, o = odd) the values $K = J$ are determined by the Ω-value of the final unpaired nucleon, giving $K = \Omega$ and originating a rotational band as described before in e–e nuclei, we have, on the other hand, $K = J = 0$, i.e. the last two nucleons are coupled to $K = \Omega^{(1)} - \Omega^{(2)} = 0$. In o–o nuclei for which the Ω quantum numbers for the last unpaired proton and neutron are respectively Ω_p and Ω_n, two values

$$K = |\Omega_p \pm \Omega_n| \tag{8.180}$$

are possible. If Ω_p and Ω_n do belong to the same orbit several choices are allowed, i.e.

$$K = 0 \qquad T = 0 \qquad J = 0, 2, 4, \ldots \tag{8.181a}$$

$$K = 0 \qquad T = 1 \qquad J = 1, 3, 5, \ldots \tag{8.181b}$$

$$K = 2\Omega \qquad T = 0 \qquad J = 2\Omega, 2\Omega + 1, \ldots \tag{8.181c}$$

The last of the three possible assignments is experimentally verified to hold for the ground states of o–o nuclei. While the matrix elements for beta-transitions from e–o to o–e nuclei where we have one valence nucleon are described by eqn (8.179) we have especially to consider the cases of a transition of an o–o to an e–e nucleus and vice versa where we have two valence nucleons.

Let us, for example, discuss a transition from an o–o to an e–e nucleus. For an o–o nucleus the wave function is (see Davidson 1968)

$$\begin{aligned}\phi(J_i M_i K_i; \Omega_i) = \sqrt{\left\{\frac{2J_i + 1}{16\pi^2(1+\delta_{K_i 0})}\right\}} \sum_{j_n j_p} C_{j_n \Omega_n} \\ \times C_{j_p \Omega_p} \{\phi_1(j_p \Omega_p)\phi_2(j_n K_i \mp \Omega_p) D^{J_i}_{M_i K_i}(\theta) + (-1)^{J_i - j_p - j_n} \\ \times \phi_1(j_p - \Omega_p)\phi_2(j_n - K_i \pm \Omega_p) D^{J_i}_{M_i - K_i}(\theta)\}\end{aligned} \tag{8.182}$$

and for an e–e nucleus ($K_f = \Omega_f = 0$)

$$\begin{aligned}\phi(J_f M_f K_f = 0; \Omega_f = 0) = \sqrt{\left(\frac{2J_f + 1}{64\pi^2}\right)} \sum_{j_p j'_p} C_{j_p \Omega_p} C'_{j'_p - \Omega_p} \\ \times [\{\phi_1(j_p \Omega_p)\phi_2(j'_p - \Omega_p) - \phi_1(j'_p - \Omega_p)\phi_2(j_p \Omega_p)\} \\ + (-1)^{J_f - j'_p - j_p} \{\phi_1(j_p - \Omega_p)\phi_2(j'_p \Omega_p) - \phi_1(j'_p \Omega_p)\phi_2(j'_p - \Omega_p)\}] D^{J_f}_{M_f 0}.\end{aligned} \tag{8.183}$$

After some calculations along the lines discussed before we then obtain

for the transition matrix element (see, for example, Berthier and Lipnick 1965)

$$\langle J_f M_f K_f = 0; \Omega_f = 0 \left| \left| \sum_{n=1,2} \{O^M_{KLs} t_-\} \right| \right| J_i M_i K_i; \Omega_i \rangle$$
$$= \frac{1}{2} \sqrt{\left\{ \frac{(2J_i+1)(2J_f+1)}{(1+\delta_{K,0})} \right\}} \begin{pmatrix} J_f & K & J_i \\ 0 & -K_i & K_i \end{pmatrix} [1+(-1)^{J_f}]$$
$$\times \sum_{j_n j_p} C_{j_n \Omega_n} C_{j_p - \Omega_p} (-1)^{j_p + \Omega_p} \begin{pmatrix} j_p & K & j_n \\ \Omega_p & -K_i & \Omega_n \end{pmatrix}$$
$$\times \langle j_p | | O_{KLs} | | j_n \rangle. \qquad (8.184)$$

where $\Omega_n = K_i \mp \Omega_p$.

Evidently the treatment of a transition of an e–e to an o–o nucleus is analogous.

In the foregoing section a short discussion of the methods needed for the calculation of matrix elements for deformed nuclei was presented. It was our aim to sketch the principles but not to go into every detail of collective nuclear models. Thus we have, for example, not included additionally vibrational excitations. More details about collective models can be found in the reviews by Davidson (1968), Eisenberg and Greiner (1970*b*), and Bohr and Mottelson (1975).

The formulae derived above are also the basis if more refined deformed models like, for instance, the asymmetric rotor model (see Bogdan *et al.* 1979), are used.

At the end of this section we would also like to refer to a number of publications where the calculation of beta-decay matrix elements for deformed nuclei has been described or practically applied (Bogdan 1965; Berthier and Lipnik 1965; Bogdan and Lipnik 1967; Bogdan 1967; Behrens and Bogdan 1970; van der Werf 1971; Bogdan *et al.* 1973*a*, 1973*b*; Bogdan *et al.* 1979; Bogdan *et al.* 1980).

Finally we can conclude that there is no more difficulty in the calculation of beta-decay matrix elements for deformed nuclei than for spherical nuclei, and sometimes even less.

8.1.1.6. Matrix elements in cartesian notation

In the preceding sections we have used the spherical polar co-ordinate system for the treatment of the beta-decay transition matrix elements. Because of the more or less spherical shape of all nuclei it is obvious that this co-ordinate system is that one which is the most suitable for explicit calculations. Most contributions on beta-decay published in the past two decades have, however, expressed their matrix elements in the cartesian notation (introduced by Konopinski and Uhlenbeck 1941). Thus we will

derive relations between the matrix elements in spherical polar co-ordinates and in rectangular cartesian co-ordinates. For that purpose we have first to express the spherical harmonics in cartesian co-ordinates, i.e.

$$rY_1^1(\theta,\phi)=\frac{1}{2}\sqrt{\left(\frac{3}{\pi}\right)}\left\{-\frac{1}{\sqrt{2}}(x+\mathrm{i}y)\right\} \tag{8.185a}$$

$$rY_1^0(\theta,\phi)=\frac{1}{2}\sqrt{\left(\frac{3}{\pi}\right)}z \tag{8.185b}$$

$$rY_1^{-1}(\theta,\phi)=\frac{1}{2}\sqrt{\left(\frac{3}{\pi}\right)}\left\{\frac{1}{\sqrt{2}}(x-\mathrm{i}y)\right\} \tag{8.185c}$$

$$r^2Y_2^0(\theta,\phi)=\sqrt{\left(\frac{5}{16\pi}\right)}(2z^2-x^2-y^2) \tag{8.185d}$$

$$r^2Y_2^{\pm1}(\theta,\phi)=\mp\sqrt{\left(\frac{15}{8\pi}\right)}(xz\pm\mathrm{i}yz) \tag{8.185e}$$

$$r^2Y_2^{\pm2}(\theta,\phi)=\sqrt{\left(\frac{15}{32\pi}\right)}(x^2-y^2\pm2\mathrm{i}xy) \tag{8.185f}$$

etc.

The first three spherical harmonics $Y_L^M(\theta,\phi)$ for $L=1$ can also be written as

$$rY_1^m(\theta,\phi)=\frac{1}{2}\sqrt{\left(\frac{3}{\pi}\right)}r_m \tag{8.186}$$

where the r_m are the spherical components† of $\mathbf{r}$. The three relations for $L=2$ can be expressed as linear combinations of the symmetric tensor

$$R_{ij}=r_ir_j-\tfrac{1}{3}\delta_{ij}\mathbf{r}^2 \tag{8.187}$$

with $r_1=x$, $r_2=y$ and $r_3=z$.

Then we have

$$\begin{aligned}r^2Y_2^0(\theta,\phi)&=\sqrt{\left(\frac{5}{16\pi}\right)}(2R_{33}-R_{11}-R_{22})\\&=3\sqrt{\left(\frac{5}{16\pi}\right)}R_{33}\end{aligned} \tag{8.188a}$$

$$r^2Y_2^{\pm1}(\theta,\phi)=\mp\sqrt{\left(\frac{15}{8\pi}\right)}(R_{13}\pm\mathrm{i}R_{23}) \tag{8.188b}$$

$$r^2Y_2^{\pm2}(\theta,\phi)=\sqrt{\left(\frac{15}{32\pi}\right)}(R_{11}-R_{22}\pm2\mathrm{i}R_{12}). \tag{8.188c}$$

† $r_{+1}=-\frac{1}{\sqrt{2}}(x+\mathrm{i}y)$

$r_0=z$

$r_{-1}=\frac{1}{\sqrt{2}}(x-\mathrm{i}y).$

Let us now consider the matrix elements which belong to the form factor coefficients (see eqn (8.13) and (8.14)). For the sake of simplicity we only select the cases with $N=0$ and $\rho=0$ ($I(k_e, m, n, 0; r)=1$). Starting with these matrix elements where K, L, s are lowest we obtain (see also eqns (6.27a) and (6.27b))

$$T^0_{000} = Y^0_0 = \frac{1}{\sqrt{(4\pi)}} \tag{8.189}$$

$$\gamma_5 T^0_{000} = \frac{1}{\sqrt{(4\pi)}} \gamma_5. \tag{8.190}$$

Thus by making use of eqn (8.14) and writing the results in the more symbolic notation of eqns (8.15a) and (8.15b) we get

$${}^{V}\mathfrak{M}^{(0)}_{000} = \int 1 \qquad \text{(Fermi matrix element)} \tag{8.191}$$

$${}^{A}\mathfrak{M}^{(0)}_{000} = \int \gamma_5. \tag{8.192}$$

Continuing with the next simplest case (see eqns (6.27a) and (b))

$$T^M_{101} = \mathbf{Y}^M_{10}\boldsymbol{\sigma}$$

$$= \frac{1}{\sqrt{(4\pi)}} \sigma_M \tag{8.193}$$

$$\gamma_5 T^M_{101} = \frac{1}{\sqrt{(4\pi)}} \alpha_M \tag{8.194}$$

where σ_M and α_M are the Mth spherical component of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$. Then we have†

$${}^{V}\mathfrak{M}^{(0)}_{101} = \int \boldsymbol{\alpha} \tag{8.195}$$

$${}^{A}\mathfrak{M}^{(0)}_{101} = \int \boldsymbol{\sigma} \qquad \text{(Gamow–Teller matrix element).} \tag{8.196}$$

As the next case we have to consider (see eqns (6.27a) and (b))

$$rT^0_{011} = \mathrm{i}r\mathbf{Y}^0_{01}\boldsymbol{\alpha}$$

$$= -\frac{\mathrm{i}}{\sqrt{(4\pi)}} \mathbf{r}\boldsymbol{\alpha} \tag{8.197}$$

$$r\gamma_5 T^0_{011} = -\frac{\mathrm{i}}{\sqrt{(4\pi)}} \mathbf{r}\boldsymbol{\sigma}. \tag{8.198}$$

† It should be noted that

$${}^{V}\mathfrak{M}^{(0)}_{101} = \frac{1}{\sqrt{(2J_i+1)}} \langle f \| \boldsymbol{\alpha} \| i \rangle$$

$${}^{A}\mathfrak{M}^{(0)}_{101} = \frac{1}{\sqrt{(2J_i+1)}} \langle f \| \boldsymbol{\sigma} \| i \rangle.$$

The corresponding matrix elements are then given by

$$^{V}\mathfrak{M}^{(0)}_{011} = -\int \mathrm{i}\frac{\boldsymbol{\alpha}\mathbf{r}}{R}$$

$$^{A}\mathfrak{M}^{(0)}_{011} = -\int \mathrm{i}\frac{\boldsymbol{\sigma}\mathbf{r}}{R}.$$

Proceeding further we go over to

$$rT^{M}_{1\,10} = \mathrm{i}rY^{M}_{1} = \frac{1}{2}\sqrt{\left(\frac{3}{\pi}\right)}\mathrm{i}r_{M}$$

$$\gamma_5 rT^{M}_{1\,10} = \frac{1}{2}\sqrt{\left(\frac{3}{\pi}\right)}\mathrm{i}\gamma_5 r_{M},$$

i.e.

$$^{V}\mathfrak{M}^{(0)}_{110} = \sqrt{3}\int \frac{\mathbf{ir}}{R}$$

$$^{A}\mathfrak{M}^{(0)}_{110} = \sqrt{3}\int \gamma_5 \frac{\mathbf{ir}}{R}.$$

A further example then is

$$T^{M}_{1\,11} = -\mathrm{i}\mathbf{Y}^{M}_{1}\boldsymbol{\alpha}$$
$$= -\mathrm{i}\sqrt{\left(\frac{3}{4\pi}\right)}\{\boldsymbol{\alpha}\otimes\mathbf{r}\}^{M}_{1}$$

We have, however

$$\{\boldsymbol{\alpha}\otimes\mathbf{r}\}^{M}_{1} = -\frac{\mathrm{i}}{\sqrt{2}}(\boldsymbol{\alpha}\times\mathbf{r})_{M}$$

and therefore

$$T^{M}_{111} = -\sqrt{\left(\frac{3}{8\pi}\right)}(\boldsymbol{\alpha}\times\mathbf{r})_{M}$$

$$\gamma_5 T^{M}_{111} = -\sqrt{\left(\frac{3}{8\pi}\right)}(\boldsymbol{\sigma}\times\mathbf{r})_{M},$$

i.e.

$$^{V}\mathfrak{M}^{(0)}_{111} = -\sqrt{\left(\frac{3}{2}\right)}\int\frac{\boldsymbol{\alpha}\times\mathbf{r}}{R}$$

$$^{A}\mathfrak{M}^{(0)}_{111} = -\sqrt{\left(\frac{3}{2}\right)}\int\frac{\boldsymbol{\sigma}\times\mathbf{r}}{R}.$$

A more complicated operator is the next one, where we have to consider

$$r^2 T^M_{121} = -r^2 \mathbf{Y}^1_{12}\boldsymbol{\alpha} = -r^2 \sum_M C(121; M-M)\alpha_M Y_2^{-M}. \tag{8.211}$$

For $M=0$ we have now

$$r^2 T^0_{121} = -r^2\left\{\sqrt{\left(\frac{3}{10}\right)}Y_2^{-1}\alpha_1 - \sqrt{\left(\frac{2}{5}\right)}Y_2^0\alpha_0 + \sqrt{\left(\frac{3}{10}\right)}Y_2^1\alpha_{-1}\right\}. \tag{8.212}$$

By expressing the spherical harmonics Y_2^M in cartesian co-ordinates (see eqns (8.188*a–c*)) and by replacing the spherical components α_M by the corresponding cartesian $\alpha_1, \alpha_2, \alpha_3$, we obtain

$$r^2 T^0_{121} = \frac{3}{\sqrt{(8\pi)}}\{R_{13}\alpha_1 + R_{23}\alpha_2 + R_{33}\alpha_3 - \tfrac{1}{3}(R_{11}+R_{22}+R_{33})\alpha_3\}. \tag{8.213}$$

This equation is, however, nothing else than

$$r^2 T^0_{121} = \frac{3}{\sqrt{(8\pi)}}\{(\mathbf{r}\boldsymbol{\alpha})r_3 - \tfrac{1}{3}r^2\alpha_3\}. \tag{8.214}$$

Similarly we have

$$r^2\gamma_5 T^0_{121} = \frac{3}{\sqrt{(8\pi)}}\{(\mathbf{r}\boldsymbol{\sigma})r_3 - \tfrac{1}{3}r^2\sigma_3\}. \tag{8.215}$$

The related matrix elements can therefore be expressed as

$${}^{\mathrm{V}}\mathfrak{M}^{(0)}_{121} = \frac{3}{\sqrt{2}}\int \frac{(\boldsymbol{\alpha}\mathbf{r})\mathbf{r} - \tfrac{1}{3}\boldsymbol{\alpha}r^2}{R^2} \tag{8.216}$$

$${}^{\mathrm{A}}\mathfrak{M}^{(0)}_{121} = \frac{3}{\sqrt{2}}\int \frac{(\boldsymbol{\sigma}\mathbf{r})\mathbf{r} - \tfrac{1}{3}\boldsymbol{\sigma}r^2}{R^2}. \tag{8.217}$$

The last case which we will explicitly consider is that with $K, L = 2$, $s = 0$. Here we have

$$r^2 T^0_{220} = -Y^0_2 = -\sqrt{\left(\frac{5}{16\pi}\right)}3R_{33} \tag{8.218}$$

$$r^2\gamma_5 T^0_{220} = -\sqrt{\left(\frac{5}{16\pi}\right)}3\gamma_5 R_{33}, \tag{8.219}$$

i.e.†

$$^{V}\mathfrak{M}^{(0)}_{220} = -\sqrt{\left(\frac{15}{2}\right)} \int \frac{R_{ij}}{R^2} \tag{8.220}$$

$$^{A}\mathfrak{M}^{(0)}_{220} = -\sqrt{\left(\frac{15}{2}\right)} \int \gamma_5 \frac{R_{ij}}{R^2}. \tag{8.221}$$

Without giving the explicit proof we will additionally list the other relations between matrix elements in both co-ordinate systems, but it is evident that the derivation would follow the lines outlined above and could therefore easily, but often lengthily, be carried out.

As far as special matrix elements are concerned the following relations are of importance (see, for example, Bühring 1963*b*; Bühring and Schülke 1965; Behrens and Jänecke 1969)

$$^{V}\mathfrak{M}^{(0)}_{211} = \frac{\sqrt{3}}{2} \int \frac{\mathrm{i}A_{ij}}{R} \tag{8.222}$$

$$^{A}\mathfrak{M}^{(0)}_{211} = \frac{\sqrt{3}}{2} \int \frac{\mathrm{i}B_{ij}}{R} \tag{8.223}$$

$$^{V}\mathfrak{M}^{(0)}_{221} = -\frac{\sqrt{5}}{2} \int \gamma_5 \frac{\mathrm{i}T_{ij}}{R^2} \tag{8.224}$$

$$^{A}\mathfrak{M}^{(0)}_{221} = -\frac{\sqrt{5}}{2} \int \frac{\mathrm{i}T_{ij}}{R^2} \tag{8.225}$$

$$^{V}\mathfrak{M}^{(0)}_{321} = -\frac{1}{2}\sqrt{\left(\frac{5}{6}\right)} \int \frac{A_{i_1 \ldots j_3}}{R^2} \tag{8.226}$$

$$^{A}\mathfrak{M}^{(0)}_{321} = -\frac{1}{2}\sqrt{\left(\frac{5}{6}\right)} \int \frac{B_{i_1 \ldots j_3}}{R^2}. \tag{8.227}$$

As we will see later on some general relations are needed in addition:

$$^{V}\mathfrak{M}^{(0)}_{KK0} = \sqrt{\left\{\frac{(2K+1)!!}{K!}\right\}} \int \frac{\mathrm{i}^K R_{i_1 \ldots i_K}}{R^K} \tag{8.228}$$

$$^{V}\mathfrak{M}^{(0)}_{KK-11} = \frac{1}{K!}\sqrt{\left\{\frac{(2K-1)!!}{(K-1)!}\right\}} \int \frac{\mathrm{i}^{K-1} A_{i_1 \ldots i_K}}{R^{K-1}} \tag{8.229}$$

$$^{A}\mathfrak{M}^{(0)}_{KK1} = -\frac{1}{K!}\sqrt{\left\{\frac{(2K+1)!!}{K!}\right\}}\sqrt{\left(\frac{K}{K+1}\right)} \int \frac{\mathrm{i}^{K+1} T_{i_1 \ldots i_K}}{R^K} \tag{8.230}$$

† The factor $\sqrt{\frac{2}{3}}$ introduced in eqn (8.220) has historical reasons. Then our definition of $\int R_{ij}$ agrees with that of Konopinski and Uhlenbeck (1941) and of Morita (1973). Namely, for $K=2$ the cartesian operator has more components than the corresponding spherical one. Thus, the normalization factor N is determined by the requirement

$$N^2 r^4 \sum_M \overset{*}{T}{}^M_{220} T^M_{220} = \sum_{ij} \overset{*}{R}_{ij} R_{ij}.$$

$$^{A}\mathfrak{M}^{(0)}_{K+1\,K1} = -\frac{1}{(K+1)!}\sqrt{\left\{\frac{(2K+1)!!}{K!}\right\}}\int\frac{i^K B_{i_1\ldots i_{K+1}}}{R^K}. \quad (8.231)$$

The cartesian tensor operators occurring in those matrix elements, but not mentioned up to now, are given by

$$B_{ij} = \delta_i r_j + \delta_j r_i - \tfrac{2}{3}\delta_{ij}(\boldsymbol{\sigma}\mathbf{r}) \quad (8.232)$$

$$A_{ij} = \alpha_i r_j + \alpha_j r_i - \tfrac{2}{3}\delta_{ij}(\boldsymbol{\alpha}\mathbf{r}) \quad (8.233)$$

$$T_{ij} = (\boldsymbol{\sigma}\times\mathbf{r})_i r_j + (\boldsymbol{\sigma}\times\mathbf{r})_j r_i \quad (8.234)$$

It should be noted that all the matrices, i.e. 1 (unit matrix), $\boldsymbol{\sigma}$, $\boldsymbol{\alpha}$, γ_5 etc. at the right side of the above equations are 4×4 matrices. The parity selection rule $\pi_i\pi_f = \pm1$ differs always between $^{V}\mathfrak{M}^{(0)}_{KLs}$ and $^{A}\mathfrak{M}^{(0)}_{KLs}$.

Summarizing this section we could say that the cartesian notation has the advantage that relativistic and non-relativistic nuclear matrix elements can immediately be recognized from the symbol for the matrix element (the relativistic ones are always those connected with γ_5 or $\boldsymbol{\alpha}$). It is very often this reason why matrix elements are denoted with the cartesian symbols. For explicit calculations, however, the cartesian co-ordinate system is completely impractical and should therefore never be used for computations.

8.1.2. Finite size nucleons and induced interactions

As indicated in Section 6.2 the hadron current is influenced by the presence of strong interactions, although up to now we have assumed point-like nucleons. One effect caused by the strong interaction is, however, that the nucleons have a finite size and an internal structure. Then the hadron current for the decay of a neutron (eqn (6.4)) will be modified such that other and new fundamental couplings will be introduced (induced). In the impulse approximation treatment, which we will further assume to be valid, these terms will naturally also make contributions to the beta-decays of complicated nuclei. To proceed further we now have to look for the most general form which the hadron current of a neutron (a finite size nucleon) can have.

Because the lepton current (see eqn (6.3)) is a sum of a Lorentz vector and axial vector, Lorentz invariance of the whole interaction density requires that the nucleon current transforms also like a vector, axial vector or a sum of both. It can then be shown that the most general possible nucleon current (for beta-decay) takes the form† (see, for example, Bernstein 1968, p. 64)

$$J^{+}_{\mu} = \langle p|\, V_\mu(0) + A_\mu(0)\,|n\rangle \quad (8.235)$$

† and

$$\sigma_{\mu\nu} = -\tfrac{1}{2}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

$$\sigma_{\mu\nu}q_\nu = \sigma_{\mu1}q_1 + \sigma_{\mu2}q_2 + \sigma_{\mu3}q_3 + \sigma_{\mu4}q_4.$$

with

$$\langle p|\, V_\mu(0)\,|n\rangle = \mathrm{i}\,\langle \bar{u}_p[\gamma_\mu - f_M\sigma_{\mu\nu}q_\nu + \mathrm{i}f_S q_\mu]u_n\rangle \tag{8.236}$$

$$\langle p|\, A_\mu(0)\,|n\rangle = \mathrm{i}\,\langle \bar{u}_p[\lambda\gamma_\mu\gamma_5 - f_T\sigma_{\mu\nu}\gamma_5 q_\nu + \mathrm{i}f_P\gamma_5 q_\mu]u_n\rangle. \tag{8.237}$$

The quantities f_M, f_S, f_T and f_P are functions of q^2. Because of the low four momentum transfer in beta-decay this q^2 dependence can, however, be neglected. The f_M is usually called the weak magnetic, f_S the induced scalar, f_T the induced tensor and f_P the induced pseudoscalar coupling constant.

It should also be pointed out that the form of the above equations, eqns (8.236) and (8.237), is based on the essential hypothesis that the u are the free particle Dirac spinors and satisfy the Dirac equation† (see, for example, Bernstein 1968) i.e. that the nucleons are on their mass shell.

$$[\mathrm{i}\gamma_\mu p_\mu + M_N]u = 0. \tag{8.238}$$

This is, on the other hand, just the assumption which we make in the standard impulse approximation treatment. $q = q_f - p_i = p_p - p_n$ is, as before, the nucleon four momentum transfer. It can also be expressed in terms of gradient operators acting on both the initial and final nuclear wave functions, i.e.

$$\mathbf{q} = -\mathrm{i}(\nabla_f - \nabla_i) = -\mathrm{i}(\nabla_p - \nabla_n) \tag{8.239}$$

$$q_0 = \mathrm{i}\left(\frac{\partial}{\partial t}\right)_f - \mathrm{i}\left(\frac{\partial}{\partial t}\right)_i = W_f - W_i = W_p - W_n. \tag{8.240}$$

The matrix element $(\phi_p\,|\mathbf{q}|\,\phi_n)$ is then given by

$$(\phi_p\,|\mathbf{q}|\,\phi_n) = i\{(\phi_p\,|\nabla_n|\,\phi_n) + (\nabla_p\phi_p\,|\,\phi_n)\}, \tag{8.241}$$

i.e. we have the symmetrized form $\mathbf{q} = \mathrm{i}(\vec{\nabla} + \overleftarrow{\nabla})$.

Later on we have to handle matrix elements of a type where they are mutliplied by other operators on which the gradient operators are not acting. Thus we have to consider them in a little more detail. Let us assume $\mathbf{A}$ to be an arbitrary vector operator and write

$$\nabla(\phi_f^*\mathbf{A}\phi_i) = \phi_f^*\mathbf{A}(\nabla\phi_i) + \phi_f^*(\nabla\mathbf{A})\phi_i + (\nabla\phi_f^*)\mathbf{A}\phi_i. \tag{8.242}$$

The brackets indicate those functions on which the gradient operator acts.

It then follows that

$$\int\{\phi_f^*\mathbf{A}(\nabla\phi_i) + (\nabla\phi_f^*)\mathbf{A}\phi_i\}\,\mathrm{d}\tau = \int\nabla(\phi_f^*\mathbf{A}\phi_i)\mathrm{d}\tau - \int\phi_f^*(\nabla\mathbf{A})\phi_i\,\mathrm{d}\tau. \tag{8.243}$$

† If that condition is not fulfilled we have 12 vector and 12 axial vector terms. This follows from the fact that generally 12 independent Lorentz vectors (and axial vectors) can be built up from the numerical tensors $\delta_{\mu\nu}$ and $e_{\mu\nu\lambda\sigma}$ and from the γ-matrices (see Bernstein 1968, p. 64). We shall come back to this point later on.

By applying Gauss's theorem we can replace the volume integral $\int \nabla(\phi_f^* \mathbf{A} \phi_i)\, d\tau$ by a surface integral, i.e.

$$\int_V \nabla(\phi_f^* \mathbf{A} \phi_i)\, d\tau = \int_S (\phi_f^* \mathbf{A} \phi_i)\, d\sigma. \tag{8.244}$$

If ϕ_f and ϕ_i are bound state wave functions the surface integral is zero because the wave functions vanish at infinity. Thus we have verified that

$$-i \langle \phi_p | (\nabla_p - \nabla_n) \mathbf{A} | \phi_n \rangle = -i \langle \phi_p | (\nabla \mathbf{A}) | \phi_n \rangle. \tag{8.245}$$

That means the matrix element on the left side of the above equation, where the gradient operators act on the nucleon wave functions only but not on **A**, is transformed into another one where now ∇ acts on **A** but not on the nucleon wave functions. Because of the static Coulomb field of the nucleus gauge invariance requires that

$$\frac{\partial}{\partial \chi_\mu} \rightarrow \frac{\partial}{\partial \chi_\mu} - ieA_\mu. \tag{8.246}$$

The quantity $A_\mu = (\mathbf{A}, i\phi)$ is the four vector of the electromagnetic field. In our case of the static potential of the nuclear charge we have

$$\mathbf{A} = 0, \qquad e\phi = V(r) \simeq \frac{\alpha Z}{R} U(r) \tag{8.247}$$

where we assume approximately that the protons are moving in the electric potential of the whole nucleus (i.e. residual Coulomb interactions are neglected).

Our nucleon currents can now be written as

$$\langle p | V_\mu(0) | n \rangle = i \langle \bar{\phi}_p | [\gamma_\mu + i f_M \sigma_{\mu\nu} (\nabla_\nu - ieA_\nu) + f_S (\nabla_\mu - ieA_\mu)] | \phi_n \rangle \tag{8.248}$$

and

$$\langle p | A_\mu(0) | n \rangle = i \langle \bar{\phi}_p | [\lambda \gamma_\mu \gamma_5 + i f_T \sigma_{\mu\nu} \gamma_5 (\nabla_\nu - ieA_\nu) + f_P \gamma_5 (\nabla_\mu - ieA_\mu)] | \phi_n \rangle. \tag{8.249}$$

The three space components of ∇_μ have to be applied in the sense extensively discussed before. The time component ∇_4 requires, however, special consideration. In free-nucleon kinematics as assumed in the impulse approximation we would have (see eqn (8.240))

$$\nabla_4 = -q_0 = \sum_{i=1}^{A} \left\{ \frac{p_n^2}{2M_n} - \frac{p_p^2}{2M_p} + M_n - M_p \right\}_i. \tag{8.250}$$

In our following treatment, however, we take ∇_4 as

$$\nabla_4 = -i \sum_{i=1}^{A} \left\{ \left(\frac{\partial}{\partial t} \right)_p - \left(\frac{\partial}{\partial t} \right)_n \right\}_i = \Delta = W_0 \tag{8.251a}$$

i.e. as the total transition energy W_0. It is clear that we have here made a small step beyond the impulse approximation treatment because W_0 is not entirely determined by the difference between the kinetic energies of initial and final nucleons.

In order to be consistent we then also take

$$\nabla_4 + \frac{\alpha Z}{R} U(r) = W_0 + W_c \tag{8.251b}$$

where W_c is the Coulomb energy difference between the initial and final nuclear state.

The choice of eqns (8.251) has the advantage of reproducing results obtained by other methods like the Foldy–Wouthuysen transformation to the total (lepton plus nuclear) Hamiltonian (see, for example, Rose and Osborn 1954; Huffacker and Greuling 1963; Blin-Stoyle 1973) the reason being that q_0 in nucleon kinematics is here the same as the q_0 in lepton kinematics. A more detailed discussion of that point can, for instance, be found in an article by Serot (1978). It should also be noted that taking q_0 as the total transition energy fits in naturally with the prescription to apply the operators derived for independent decaying nucleons on the bound state wave functions.

In that context is also worth mentioning what happens in the case of β^+-decay and electron capture where we need the hermitian conjugate current of J_μ^+ (see Section 6.1). Assuming the coupling constants f_M, f_S, f_T and f_P are real (time reversal invariance) we then obtain

$$\langle n|\, V_\mu^+(0)\,|p\rangle = \mathrm{i}\,\langle \bar{u}_n[\gamma_\mu + \mathrm{i}f_M\sigma_{\mu\nu}(\nabla_\nu + \mathrm{i}eA_\nu) - f_S(\nabla_\mu + \mathrm{i}eA_\mu)]u_p\rangle \tag{8.252}$$

and

$$\langle n|\, A_\mu^+(0)\,|p\rangle = \mathrm{i}\,\langle \bar{u}_n[\lambda\gamma_\mu\gamma_5 - \mathrm{i}f_T\sigma_{\mu\nu}\gamma_5(\nabla_\nu + \mathrm{i}eA_\nu) + f_P\gamma_5(\nabla_\mu + \mathrm{i}eA_\mu)]u_p\rangle. \tag{8.253}$$

The coupling constants for β^--decay, β^+-decay and electron capture are therefore related as follows (Behrens and Bühring 1974):

β^--decay		β^+-decay		EC	
f_M	$\rightarrow$	f_M	$\rightarrow$	f_M	(8.254*a*)
f_S	$\rightarrow$	$-f_S$	$\rightarrow$	$-f_S$	(8.254*b*)
f_T/λ	$\rightarrow$	$-f_T/\lambda$	$\rightarrow$	$-f_T/\lambda$	(8.254*c*)
f_P/λ	$\rightarrow$	f_P/λ	$\rightarrow$	f_P/λ	(8.254*d*)
Z	$\rightarrow$	$-Z$	$\rightarrow$	$-Z$	(8.254*e*)
(W_c)	$\rightarrow$	$(-W_c)$	$\rightarrow$	$(-W_c)$	
W_0	$\rightarrow$	W_0	$\rightarrow$	$W_0' = W_0 + W_x$.	(8.254*f*)

It should be noted that (8.254*e*) and (8.254*f*) only apply to formulae expressing form factors in terms of coupling constants and nuclear matrix elements, but not to the formulae expressing the observables in terms of form factors (for that point see Section 7.4).

The Coulomb energy difference between the initial and final nuclear state W_c is essentially equal to the Coulomb displacement energy (see Jänecke 1969). For a homogeneously charged sphere we would have

$$W_c = \frac{6}{5}\frac{\alpha Z}{R}. \tag{8.255}$$

In the following we assume throughout this value for W_c.

After this short diversion let us now come back to the β^--decay. The different terms in eqns (8.236) and (8.237) can be rewritten in the following way.†

Time component, i.e. $\mu = 4$:

$$\gamma_4\gamma_4 = 1 \tag{8.256a}$$

$$\mathrm{i}f_M\gamma_4\sigma_{4\nu}\nabla_\nu = -\mathrm{i}f_M\beta\boldsymbol{\alpha}\nabla \tag{8.256b}$$

$$f_S\gamma_4\left(\nabla_4 + \frac{\alpha Z}{R}U(r)\right) = -f_S\beta\left(W_0 + \frac{6}{5}\frac{\alpha Z}{R}\right) \tag{8.256c}$$

$$\lambda\gamma_4\gamma_4\gamma_5 = \lambda\gamma_5 \tag{8.256d}$$

$$\mathrm{i}f_T\gamma_4\sigma_{4\nu}\gamma_5\nabla_\nu = -\mathrm{i}f_T\beta\boldsymbol{\sigma}\nabla \tag{8.256e}$$

$$f_P\gamma_4\gamma_5\left(\nabla_4 + \frac{\alpha Z}{R}U(r)\right) = -f_P\beta\gamma_5\left(W_0 + \frac{6}{5}\frac{\alpha Z}{R}\right). \tag{8.256f}$$

Space components, i.e. $\mu = 1, 2, 3$:

$$\mathrm{i}\gamma_4\gamma_\mu = -\alpha_\mu \tag{8.257a}$$

$$-f_M\gamma_4\sigma_{\mu\nu}\nabla_\nu = -f_M\beta(\boldsymbol{\alpha}\times\nabla)_\mu \qquad \nu \neq 4 \tag{8.257b}$$

$$-f_M\gamma_4\sigma_{\mu 4}\left(\nabla_4 + \frac{\alpha Z}{R}U(r)\right) = -f_M\beta\alpha_\mu\left(W_0 + \frac{6}{5}\frac{\alpha Z}{R}\right) \tag{8.257c}$$

$$\mathrm{i}f_S\gamma_4\nabla_\mu = -\mathrm{i}f_S\beta\nabla_\mu \tag{8.257d}$$

† We have

$$\sigma_{4\mu} = -\sigma_{\mu 4} = \alpha_\mu \qquad \mu \neq 4$$

$$\sigma_{\mu u} = 0$$

$$\sigma_{kl} = -\mathrm{i}\alpha_k\alpha_l = \sigma_m \qquad k, l, m = 1, 2, 3$$

$$\sigma_{km} = -\sigma_l$$

$$\sum_n \sigma_{kn}\nabla_n = \sigma_{kl}\nabla_l + \sigma_{km}\nabla_m$$

$$= \sigma_m\nabla_l - \sigma_l\nabla_m$$

$$= -(\boldsymbol{\sigma}\times\nabla)_k.$$

$$i\lambda\gamma_4\gamma_\mu\gamma_5 = -\lambda\sigma_\mu \tag{8.257e}$$

$$-f_T\gamma_4\sigma_{\mu\nu}\gamma_5\nabla_\nu = -f_T\beta(\boldsymbol{\alpha}\times\boldsymbol{\nabla})_\mu \qquad \nu \neq 4 \tag{8.257f}$$

$$-f_T\gamma_4\sigma_{\mu 4}\gamma_5\left(\nabla_4 + \frac{\alpha Z}{R}U(r)\right) = -f_T\beta\sigma_\mu\left(W_0 + \frac{6}{5}\frac{\alpha Z}{R}\right) \tag{8.257g}$$

$$if_P\gamma_4\gamma_5\nabla_\mu = -if_P\beta\gamma_5\nabla_\mu. \tag{8.257h}$$

As we have seen in Section 8.1.1.1 the nuclear transition matrix elements for point like nucleons contain spherical tensor operators of the form† (see eqn (8.6))

$$1j_K(qr)T^M_{KK0}(\hat{r}) = j_0^V j_K(qr)i^K Y^M_K(\hat{r}) \tag{8.258a}$$

$$\lambda\gamma_5 j_K(qr)T^M_{KK0}(\hat{r}) = j_0^A j_K(qr)i^K Y^M_K(\hat{r}) \tag{8.258b}$$

$$1j_L(qr)T^M_{KL1}(\hat{r}) = (-\mathbf{j}^V)(-1)^{L-K+1}j_L(qr)i^L\mathbf{Y}^M_{KL}(\hat{r}) \tag{8.258c}$$

$$\lambda\gamma_5 j_L(qr)T^M_{KL1}(\hat{r}) = (-\mathbf{j}^A)(-1)^{L-K+1}j_L(qr)i^L\mathbf{Y}^M_{KL}(\hat{r}). \tag{8.258d}$$

We recall that the spherical Bessel functions $j_L(qr)$ and the spherical tensor operators $T^M_{KLs}(\hat{r})$ have their origin from our special expansion in multipole form factors. For that expansion no use has been made, however, of the fact that the nucleon operators are just $\boldsymbol{\alpha}$, $\lambda\boldsymbol{\sigma}$, 1 or $\lambda\gamma_5$, but only that they behave like scalars or vectors, respectively, under rotations. Therefore we can replace these nucleon operators by the more complicated ones for finite size nucleons, i.e. we have to make the following substitutions (Behrens and Bühring 1971):

$$1 \to j_0^V = 1 - if_M\beta\boldsymbol{\alpha}\boldsymbol{\nabla} - f_S\beta\left(W_0 + \frac{6}{5}\frac{\alpha Z}{R}\right) \tag{8.259a}$$

$$\lambda\gamma_5 \to j_0^A = \lambda\gamma_5 - if_T\beta\boldsymbol{\sigma}\boldsymbol{\nabla} - f_P\beta\gamma_5\left(W_0 + \frac{6}{5}\frac{\alpha Z}{R}\right) \tag{8.259b}$$

$$\boldsymbol{\alpha} \to -\mathbf{j}^V = \boldsymbol{\alpha} + f_M\beta(\boldsymbol{\sigma}\times\boldsymbol{\nabla}) + f_M\beta\boldsymbol{\alpha}\left(W_0 + \frac{6}{5}\frac{\alpha Z}{R}\right) + if_S\beta\boldsymbol{\nabla} \tag{8.259c}$$

$$\lambda\boldsymbol{\sigma} = -\mathbf{j}^A = \lambda\boldsymbol{\sigma} + f_T\beta(\boldsymbol{\alpha}\times\boldsymbol{\nabla}) + f_T\beta\boldsymbol{\sigma}\left(W_0 + \frac{6}{5}\frac{\alpha Z}{R}\right) + if_P\beta\gamma_5\boldsymbol{\nabla}. \tag{8.259d}$$

Following our earlier discussion (see eqns (8.241)–(8.245)) the gradient

† The corresponding nuclear current for point like nucleons can be written as

$$\left.\begin{aligned} j_0^V &= \gamma_4\gamma_4 = 1 \\ j_0^A &= \lambda\gamma_5 \\ \mathbf{j}^V &= i\gamma_4\gamma_\mu = -\boldsymbol{\alpha} \\ \mathbf{j}^A &= i\lambda\gamma_4\gamma_\mu\gamma_5 = -\lambda\boldsymbol{\sigma} \end{aligned}\right\} \mu = 1, 2, 3.$$

operators ∇ have to be applied on the $j_K(qr)\ Y_K^M(\hat{r})$ or on the $j_L(qr)\ \mathbf{Y}_{KL}^M(\hat{r})$.

This can be done by means of gradient and related formulae as given, for instance, by Edmonds (1964).

Specialized to our demand these formulae become

$$-\mathrm{i}^{K+1}\boldsymbol{\alpha}\nabla j_K(qr)\,Y_K^M(\hat{r}) = \sqrt{\left(\frac{K}{2K+1}\right)}\,qj_{K-1}(qr)T_{KK-11}^M(\hat{r}) - \sqrt{\left(\frac{K+1}{2K+1}\right)}\,qj_{K+1}(qr)T_{KK+11}^M(\hat{r}) \quad (8.260)$$

$$(-1)^{L-K+1}\mathrm{i}^{L+1}\nabla\times j_L(qr)\mathbf{Y}_{KL}^M(\hat{r}) = \begin{cases} -\sqrt{\left(\frac{K}{2K+1}\right)}\,qj_K(qr)T_{KK0}^M(\hat{r}) & L=K-1 \\ 0 & L=K \\ \sqrt{\left(\frac{K+1}{2K+1}\right)}\,qj_K(qr)T_{KK0}^M(\hat{r}) & L=K+1 \end{cases} \quad (8.261)$$

$$(-1)^{L-K+1}\mathrm{i}^{L}\boldsymbol{\alpha}[\nabla\times j_L(qr)\mathbf{Y}_{KL}^M(\hat{r})] = \begin{cases} \sqrt{\left(\frac{K+1}{2K+1}\right)}\,qj_K(qr)T_{KK1}^M(\hat{r}) & L=K-1 \\ \sqrt{\left(\frac{K+1}{2K+1}\right)}\,qj_{K-1}(qr)T_{KK-11}^M(\hat{r}) + \\ +\sqrt{\left(\frac{K}{2K+1}\right)}\,qj_{K+1}(qr)T_{KK+11}^M(\hat{r}) & L=K \\ \sqrt{\left(\frac{K}{2K+1}\right)}\,qj_K(qr)T_{KK1}^M(\hat{r}) & L=K+1. \end{cases} \quad (8.262)$$

Straightforward application of the above formulae leads to the final expressions for the form factors in terms of nuclear matrix elements (Stech and Schülke 1964; Bühring and Schülke 1965; Behrens and Bühring 1971), i.e. $(q^2=\mathbf{q}^2)$

$${}^{\mathrm{V}}F_{KK0}(\mathbf{q}^2) = {}^{\mathrm{V}}\mathfrak{M}_{KK0}(\mathbf{q}^2) + \frac{f_M}{R}\left[\sqrt{\{K(2K+1)\}}\,{}^{\mathrm{C}}\mathfrak{M}_{KK-11}(\mathbf{q}^2) - \sqrt{\left\{\frac{K+1}{(2K+3)^2(2K+1)}\right\}}(qR)^2\,{}^{\mathrm{C}}\mathfrak{M}_{KK+11}(\mathbf{q}^2)\right] - \frac{f_S}{R}\{W_0R+\tfrac{6}{5}\alpha Z\}\,{}^{\mathrm{C}}\mathfrak{M}_{KK0}(\mathbf{q}^2) \quad (8.263a)$$

$$
{}^{\mathrm{A}}F_{KK0}(\mathbf{q}^2) = \lambda\, {}^{\mathrm{A}}\mathfrak{M}_{KK0}(\mathbf{q}^2) + \frac{f_{\mathrm{T}}}{R}\left[\sqrt{\{K(2K+1)\}}\, {}^{\mathrm{D}}\mathfrak{N}_{KK-11}(\mathbf{q}^2) - \sqrt{\left\{\frac{K+1}{(2K+3)^2(2K+1)}\right\}}(qR)^2\, {}^{\mathrm{D}}\mathfrak{N}_{KK+11}(\mathbf{q}^2)\right] - \frac{f_{\mathrm{P}}}{R}\{W_0R + \tfrac{6}{5}\alpha Z\}\, {}^{\mathrm{D}}\mathfrak{N}_{KK0}(\mathbf{q}^2) \quad (8.263b)
$$

$$
{}^{\mathrm{V}}F_{KK1}(\mathbf{q}^2) = {}^{\mathrm{V}}\mathfrak{M}_{KK1}(\mathbf{q}^2) + \frac{f_{\mathrm{M}}}{R}\left[\sqrt{\{(K+1)(2K+1)\}}\, {}^{\mathrm{D}}\mathfrak{N}_{KK-11}(\mathbf{q}^2) + \sqrt{\left\{\frac{K}{(2K+3)^2(2K+1)}\right\}}(qR)^2\, {}^{\mathrm{D}}\mathfrak{N}_{KK+11}(\mathbf{q}^2) + (W_0R + \tfrac{6}{5}\alpha Z)\, {}^{\mathrm{C}}\mathfrak{N}_{KK1}(\mathbf{q}^2)\right] \quad (8.263c)
$$

$$
{}^{\mathrm{A}}F_{KK1}(\mathbf{q}^2) = \lambda\, {}^{\mathrm{A}}\mathfrak{M}_{KK1}(\mathbf{q}^2) + \frac{f_{\mathrm{T}}}{R}\left[\sqrt{\{(K+1)(2K+1)\}}\, {}^{\mathrm{C}}\mathfrak{N}_{KK-11}(\mathbf{q}^2) + \sqrt{\left\{\frac{K}{(2K+3)^2(2K+1)}\right\}}(qR)^2\, {}^{\mathrm{C}}\mathfrak{N}_{KK+11}(\mathbf{q}^2) + (W_0R + \tfrac{6}{5}\alpha Z)\, {}^{\mathrm{D}}\mathfrak{N}_{KK1}(\mathbf{q}^2)\right] \quad (8.263d)
$$

$$
-{}^{\mathrm{V}}F_{KK-11}(\mathbf{q}^2) = {}^{\mathrm{V}}\mathfrak{M}_{KK-11}(\mathbf{q}^2) + \frac{f_{\mathrm{M}}}{R}\left[\sqrt{\left\{\frac{K+1}{(2K+1)^3}\right\}}(qR)^2\, {}^{\mathrm{D}}\mathfrak{N}_{KK1}(\mathbf{q}^2) + (W_0R + \tfrac{6}{5}\alpha Z)\, {}^{\mathrm{C}}\mathfrak{N}_{KK-11}(\mathbf{q}^2)\right] - \frac{f_{\mathrm{S}}}{R}\sqrt{\left\{\frac{K}{(2K+1)^3}\right\}} \times (qR)^2\, {}^{\mathrm{C}}\mathfrak{N}_{KK0}(\mathbf{q}^2) \quad (8.263e)
$$

$$
-{}^{\mathrm{A}}F_{KK-11}(\mathbf{q}^2) = \lambda\, {}^{\mathrm{A}}\mathfrak{M}_{KK-11}(\mathbf{q}^2) + \frac{f_{\mathrm{T}}}{R}\left[\sqrt{\left\{\frac{K+1}{(2K+1)^3}\right\}}(qR)^2\, {}^{\mathrm{C}}\mathfrak{N}_{KK1}(\mathbf{q}^2) + (W_0R + \tfrac{6}{5}\alpha Z)\, {}^{\mathrm{D}}\mathfrak{N}_{KK-11}(\mathbf{q}^2)\right] - \frac{f_{\mathrm{P}}}{R}\sqrt{\left\{\frac{K}{(2K+1)^3}\right\}} \times (qR)^2\, {}^{\mathrm{D}}\mathfrak{N}_{KK0}(\mathbf{q}^2) \quad (8.263f)
$$

$$
-{}^{\mathrm{V}}F_{KK+11}(\mathbf{q}^2) = {}^{\mathrm{V}}\mathfrak{M}_{KK+11}(\mathbf{q}^2) + \frac{f_{\mathrm{M}}}{R}\left[\sqrt{\left\{\frac{K(2K+3)^2}{2K+1}\right\}}\, {}^{\mathrm{D}}\mathfrak{N}_{KK1}(\mathbf{q}^2) + (W_0R + \tfrac{6}{5}\alpha Z)\, {}^{\mathrm{C}}\mathfrak{N}_{KK+11}(\mathbf{q}^2)\right] + \frac{f_{\mathrm{S}}}{R}\sqrt{\left\{\frac{(K+1)(2K+3)^2}{2K+1}\right\}}\, {}^{\mathrm{C}}\mathfrak{N}_{KK0}(\mathbf{q}^2) \quad (8.263g)
$$

$$
-{}^{\mathrm{A}}F_{KK+11}(\mathbf{q}^2) = \lambda\, {}^{\mathrm{A}}\mathfrak{M}_{KK+11}(\mathbf{q}^2) + \frac{f_{\mathrm{T}}}{R}\left[\sqrt{\left\{\frac{K(2K+3)^2}{2K+1}\right\}}\, {}^{\mathrm{C}}\mathfrak{N}_{KK1}(\mathbf{q}^2) + (W_0R + \tfrac{6}{5}\alpha Z)\, {}^{\mathrm{D}}\mathfrak{N}_{KK+11}(\mathbf{q}^2)\right] + \frac{f_{\mathrm{P}}}{R}\sqrt{\left\{\frac{(K+1)(2K+3)^2}{2K+1}\right\}}\, {}^{\mathrm{D}}\mathfrak{N}_{KK0}(\mathbf{q}^2). \quad (8.263h)
$$

In addition to the matrix elements introduced earlier in Sections 8.1.1.1 and 8.1.1.2, new types of the following form now occur:

$$^{C}\mathfrak{M}_{KK0}(\mathbf{q}^2) = \sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\frac{(2K+1)!!}{(qR)^K}\langle\phi_f||j_K(qr)\beta T_{KK0}(\hat{r})||\phi_i\rangle \quad (8.264a)$$

$$^{D}\mathfrak{M}_{KK0}(\mathbf{q}^2) = \sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\frac{(2K+1)!!}{(qR)^K}\langle\phi_f||j_K(qr)\beta\gamma_5 T_{KK0}(\hat{r})||\phi_i\rangle \quad (8.264b)$$

$$^{C}\mathfrak{M}_{KL1}(\mathbf{q}^2) = \sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\frac{(2L+1)!!}{(qR)^L}\langle\phi_f||j_L(qr)\beta T_{KL1}(\hat{r})||\phi_i\rangle \quad (8.264c)$$

$$^{D}\mathfrak{M}_{KL1}(\mathbf{q}^2) = \sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\frac{(2L+1)!!}{(qR)^L}\langle\phi_f||j_L(qr)\beta\gamma_5 T_{KL1}(\hat{r})||\phi_i\rangle. \quad (8.264d)$$

Their single particle values analogous to those given in eqns (8.21*a–d*) are written as (Behrens and Bühring 1971)

$$^{C}\mathfrak{M}_{KK0}(\mathbf{q}^2) = \frac{\sqrt{2}}{\sqrt{(2J_i+1)}}\frac{(2K+1)!!}{(qR)^K}\left\{-G_{KK0}(\kappa_f,\kappa_i)\int_0^\infty g_f(r,\kappa_f)j_K(qr)\right.$$
$$\times g_i(r,\kappa_i)r^2\,dr + \mathrm{sign}(\kappa_f)\mathrm{sign}(\kappa_i)G_{KK0}(-\kappa_f,-\kappa_i)\int_0^\infty f_f(r,\kappa_f)$$
$$\left.\times j_K(qr)f_i(r,\kappa_i)r^2\,dr\right\} \quad (8.265a)$$

$$^{D}\mathfrak{M}_{KL1}(\mathbf{q}^2) = \frac{\sqrt{2}}{\sqrt{(2J_i+1)}}\frac{(2L+1)!!}{(qR)^L}\left\{-G_{KL1}(\kappa_f,\kappa_i)\int_0^\infty g_f(r,\kappa_f)j_L(qr)\right.$$
$$\times g_i(r,\kappa_i)r^2\,dr + \mathrm{sign}(\kappa_f)\mathrm{sign}(\kappa_i)G_{KL1}(-\kappa_f,-\kappa_i)\int_0^\infty f_f(r,\kappa_f)$$
$$\left.\times j_L(qr)f_i(r,\kappa_i)r^2\,dr\right\} \quad (8.265b)$$

$$^{D}\mathfrak{M}_{KK0}(\mathbf{q}^2) = \frac{\sqrt{2}}{\sqrt{(2J_i+1)}}\frac{(2K+1)!!}{(qR)^K}\left\{-\mathrm{sign}(\kappa_i)G_{KK0}(\kappa_f,-\kappa_i)\int_0^\infty g_f(r,\kappa_f)\right.$$
$$\times j_K(qr)f_i(r,\kappa_i)r^2\,dr + \mathrm{sign}(\kappa_f)G_{KK0}(-\kappa_f,\kappa_i)\int_0^\infty f_f(r,\kappa_f)$$
$$\left.\times j_K(qr)g_i(r,\kappa_i)r^2\,dr\right\} \quad (8.265c)$$

$$^{C}\mathfrak{M}_{KL1}(\mathbf{q}^2) = \frac{\sqrt{2}}{\sqrt{(2J_i+1)}}\frac{(2L+1)!!}{(qR)^L}\left\{-\mathrm{sign}(\kappa_i)G_{KL1}(\kappa_f,-\kappa_i)\right.$$
$$\times\int_0^\infty g_f(r,\kappa_f)j_L(qr)f_i(r,\kappa_i)r^2\,dr + \mathrm{sign}(\kappa_f)G_{KL1}(-\kappa_f,\kappa_i)$$
$$\left.\times\int_0^\infty f_f(r,\kappa_f)j_L(qr)g_i(r,\kappa_i)r^2\,dr\right\}. \quad (8.265d)$$

The geometrical coefficients have been defined by eqn (6.141) and are listed in Table 8.2. By comparing the above formulae with eqns (8.19) and (8.21) we see that the $\mathfrak{M}_{KL1}(\mathbf{q}^2)$ and the $\mathfrak{N}_{KL1}(\mathbf{q}^2)$ differ by the Dirac matrix β by which the operators of the latter ones are additionally multiplied.† Therefore we are not going to list the explicit expressions for the $\mathfrak{N}_{KL1}(\mathbf{q}^2)$ in cartesian notation as well.

From the formulae discussed above we see immediately that in the non-relativistic limit the non-relativistic matrix elements $\mathfrak{M}_{KL1}(\mathbf{q}^2)$ and $\mathfrak{N}_{KL1}(\mathbf{q}^2)$ are related by

$$^{C}\mathfrak{N}_{KK0}(\mathbf{q}^2) = -\,^{V}\mathfrak{M}_{KK0}(\mathbf{q}^2) \tag{8.266a}$$

$$^{D}\mathfrak{N}_{KL1}(\mathbf{q}^2) = -\,^{A}\mathfrak{M}_{KL1}(\mathbf{q}^2). \tag{8.266b}$$

This is obviously a consequence of the fact that β is diagonal in terms of the submatrices and thus connects the large components with the large components and the small components with the small components of the wave functions. Then, if we forget the small components, as we do in the non-relativistic limit, we can simply replace β by -1.

In the case of the relativistic matrix elements, on the other hand, no simple relation exists between the $\mathfrak{M}_{KL1}(\mathbf{q}^2)$ and the $\mathfrak{N}_{KL1}(\mathbf{q}^2)$. The procedure for the explicit evaluation of this type of matrix elements has been outlined in Section 8.1.1.2.

The results of this section demonstrate very clearly that it is only the definition of the form factors in terms of nuclear matrix elements and coupling constants which is modified by introducing finite size nucleons, i.e. by introducing the so-called induced interaction, while the expressions for the observables in terms of form factors remain unchanged.

For practical applications we need additionally the corresponding formulae for the form factor coefficients $F^{(N)}_{KLs}(k_e, m, n, \rho)$. The relation between form factors and form factor coefficients is given by the eqns (6.159) and (6.160). The procedure follows that used in Section 8.1.1.1 (see, for example, eqn (8.11)). Now we have, however, the situation that

† Explicitly we have

$$^{V}\mathfrak{M}_{KK0} = \int O_{KK0} \qquad ^{C}\mathfrak{N}_{KK0} = \int \beta O_{KK0}$$

$$^{A}\mathfrak{M}_{KK0} = \int \gamma_5 O_{KK0} \qquad ^{D}\mathfrak{N}_{KK0} = \int \beta\gamma_5 O_{KK0}$$

$$^{V}\mathfrak{M}_{KL1} = \int O_{KLs} \qquad ^{C}\mathfrak{N}_{KL1} = \int \beta O_{KLs}$$

$$^{A}\mathfrak{M}_{KL1} = \int \gamma_5 O_{KLs} \qquad ^{D}\mathfrak{N}_{KL1} = \int \beta\gamma_5 O_{KLs}.$$

the order L' of the spherical Bessel functions in some $\mathfrak{N}_{KL's'}(\mathbf{q}^2)$ is different from the order L in the $J(q)$. Specifically in the integral (see eqn (6.159))

$$F^{(N)}_{KLs}(k_e, m, n, \rho) = \int_0^\infty J(q) F_{KLs}(q^2) q^2 \, dq \tag{8.267}$$

we have additional terms of two types. One where $L' = L+1$

$$\begin{aligned}
\int_0^\infty J(q)\mathfrak{N}_{KL's'}(\mathbf{q}^2)q^2\,dq &= \frac{2}{\pi}(2L+3)\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\int_0^\infty\int_0^\infty\left(\frac{r'}{R}\right)^{L+2N} \\
&\times I(k_e, m, n, \rho; r')j_L(qr')r'^2\,dr'(qR)\int\phi_f^+(r_1\ldots r_A)\sum_{j=1}^{A}\{j_{L+1}(qr) \\
&\times O_{KL's't_-}\}_j\phi_i(r_1\ldots r_A)\,d^3r_1\ldots d^3r_A q^2\,dq \\
&= (2L+3)R\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\int\int\ldots\int\phi_f^+(r_1\ldots r_A)\sum_{j=1}^{A}\left\{\left(-\frac{\partial}{\partial r}+\frac{L}{r}\right)\right. \\
&\times\left.\left[\left(\frac{r}{R}\right)^{L+2N}I(k_e, m, n, \rho; r)\right]O_{KL's't_-}\right\}_j\phi_i(r_1\ldots r_A)\,d^3r_1\ldots d^3r_A,
\end{aligned} \tag{8.268}$$

and the other one where $L' = L-1$

$$\begin{aligned}
\int_0^\infty J(q)\mathfrak{N}_{KL's'}(\mathbf{q}^2)q^2\,dq &= \frac{2}{\pi}\frac{1}{2L+1}\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\int_0^\infty\int_0^\infty\left(\frac{r'}{R}\right)^{L+2N} \\
&\times I(k_e, m, n, \rho; r')j_L(qr')r'^2\,dr'(qR)\int\phi_f^+(r_1\ldots r_A)\sum_{j=1}^{A} \\
&\times\{j_{L-1}(qr)O_{KL's't_-}\}_j\phi_i(r_1\ldots r_A)\,d^3r_1\ldots d^3r_A q^2\,dq \\
&= \frac{R}{2L+1}\sqrt{\left(\frac{4\pi}{2J_i+1}\right)}\int\int\ldots\int\phi_f^+(r_1\ldots r_A)\sum_{j=1}^{A}\left\{\left(\frac{\partial}{\partial r}+\frac{L+1}{r}\right)\right. \\
&\times\left.\left[\left(\frac{r}{R}\right)^{L+2N}I(k_e, m, n, \rho; r)\right]O_{KL's't_-}\right\}_j\phi_i(r_1\ldots r_A)\,d^3r_1\ldots d^3r_A.
\end{aligned} \tag{8.269}$$

The differential operators $\partial/\partial r$ occurring in these formulae act only on $(r/R)^{L+2N}I(k_e, m, n, \rho; r)$, and not on any other function under the integral. Thus we now obtain two additional types of radial functions having their origin in the expansion of electron radial wave function:

$$\begin{aligned}
&\left(-\frac{\partial}{\partial r}+\frac{L}{r}\right)\left[\left(\frac{r}{R}\right)^{L+2N}I(k_e, m, n, \rho; r)\right] \\
&\qquad = -\left(\frac{r}{R}\right)^{L+2N-1}\frac{2NI(k_e, m, n, \rho; r)+rI'(k_e, m, n, \rho; r)}{R}
\end{aligned} \tag{8.270a}$$

$$\left(\frac{\partial}{\partial r}+\frac{L1}{r}\right)\left[\left(\frac{r}{R}\right)^{L+2N} I(k_e, m, n, \rho; r)\right]=\left(\frac{r}{R}\right)^{L+2N-1}$$
$$\times\frac{(2L+1+2N)I(k_e, m, n, \rho; r)+rI'(k_e, m, n, \rho; r)}{R} \qquad (8.270b)$$
$$\left(I'=\frac{\partial I}{\partial r}\right).$$

It is therefore no problem to write down also the corresponding formulae for the form factor coefficients as a function of the induced terms (see Behrens and Bühring 1971).

At the end of this section we would like also to refer to some other publications where a derivation of the relevant formulae for the induced terms in impulse approximation treatment is given. Derivations based on the application of the Foldy–Wouthuysen transformation to the whole interaction Hamiltonian (i.e. of a method different from ours) are published by Rose and Osborn (1954), Huffacker and Greuling (1963), Blin-Stoyle and Nair (1966), Eisenberg and Greiner (1970a), and Blin-Stoyle (1973). Treatments which follow more the methods applied here can be found in articles by Stech and Schülke (1964), Bühring and Schülke (1965), Behrens and Bühring (1971), and Walecka (1975). In that context it is especially worth mentioning the detailed treatment by Serot (1978) who extended his analysis through order $(v/c)^2$ nucleon where v is the nucleon velocity.

As noted earlier all the beta-decay observables are functions of the form factors where all dynamic nuclear properties are contained in. These form factors (or form factor coefficients) are the parameters which in the ideal case are determined by beta-decay experiments. For a theoretical calculation, on the other hand, the form factors have to be expressed in terms of nuclear matrix elements. This involves approximations like the impulse approximation.

Of course, we need also numerical values for the coupling constants λ, f_M, f_S, f_T and f_P. Up to now, however, we have not discussed the question if and how special values for these coupling constants can be obtained. For the present we have only assumed that they are real numbers (time reversal invariance), but because of the importance of that point we will give a detailed discussion later on under the heading symmetries. At this point we should only mention that for certain types of coupling constants, i.e. f_M, f_S and f_P, numerical values can be obtained on the basis of symmetry relations like the conserved vector current (CVC) theory and the partially conserved axial vector current (PCAC) theory. The axial vector coupling constant λ will be especially considered under the heading exchange effects, in the following section.

8.2. Exchange effects

8.2.1. Renormalization of the coupling constants

In order to express the form factors in terms of nuclear matrix elements up to now we have assumed that the nucleons inside the nucleus interact with leptons in the same way as do free nucleons, i.e. we have made use of the impulse approximation treatment. The question, however, arises, does a nucleon inside a nucleus differ in its properties from a free nucleon and especially does the coupling of this nucleon to the lepton field deviate from the free nucleon value. Considering the free neutron decay we find for the vector coupling constant

$$f_V = 1 \tag{8.271}$$

and for the axial vector coupling constant

$$\lambda = 1{\cdot}25. \tag{8.272}$$

That means the first coupling constant is not renormalized while the latter coupling constant is remarkably influenced by the strong interaction.

It is, therefore, expected that the axial coupling constant λ is additionally modified if the virtual pion field around a nucleon is changed when other nucleons are in the near vicinity of that one under consideration. For the vector coupling constant f_V we, on the other hand, intuitively expect that renormalization effects of the surrounding nuclear matter are not of such importance. If we restrict ourselves to the form factor coefficients ${}^{V}F^{(0)}_{000}$ and ${}^{A}F^{(0)}_{101}$ which are governing the allowed transitions (see Section 14.2) it can indeed be shown that f_V is not renormalized within a nucleus and is always exactly equal to 1. The reason is that the vector current V_μ satisfies a continuity equation of the form

$$\frac{\partial V_\mu}{\partial x_\mu} = 0$$

which is usually called the conserved vector current (CVC) theory (this theory and their consequences are extensively discussed in Section 10.2). In the case of the axial current A_μ no such strong continuity equation exists but only an appropriate one of the form

$$\frac{\partial A_\mu}{\partial x_\mu} = m_\pi^2 f_\pi \phi_{\pi^*} \tag{8.273}$$

where m_π is the pion mass, f_π the charged pion decay constant ($f_\pi = 0{\cdot}932 m_\pi$) and ϕ_{π^*} the pion field.

This equation is usually called the partially conserved axial vector

current (PCAC) theory (it is discussed in Section 10.3 in detail) and does not forbid any renormalization by nuclear matter. Thus, we will treat this renormalization of the weak axial coupling constant λ in the following. In impulse approximation the axial vector form factor coefficient ${}^{A}F_{101}^{(0)}$ was given by (see eqn (8.139*b*))

$$\begin{aligned} {}^{A}F_{101}^{(0)} &= \mp\lambda\, {}^{A}\mathfrak{M}_{101}^{(0)} \\ &= \mp\frac{\lambda}{\sqrt{(2J_i+1)}}\langle f\Big|\Big|\sum_{n=1}^{A}\{\boldsymbol{\sigma}t_{\mp}\}_n\Big|\Big|i\rangle \end{aligned} \tag{8.274}$$

(upper sign β^--decay, lower sign β^+-decay). Inside the nucleus the nucleons interact via various meson exchanges. Thus we would have to add other two (or more) body matrix elements which are a direct consequence of these different exchange modes. In this treatment we take all these exchange contributions into account in a more global form by introducing a new effective coupling constant λ_{Ae} such that

$${}^{A}F_{101}^{(0)} = \mp\lambda_{Ae}\, {}^{A}\mathfrak{M}_{101}^{(0)} \tag{8.275}$$

whereby

$$\lambda_{Ae} = \lambda(1+\delta) \tag{8.276}$$

expresses the renormalization of λ by an amount δ caused by all exchange effects.

As we will see later on this method is analogous to the macroscopic treatment of an ordinary electric dipole surrounded by a medium with the dielectric constant ε. The more explicit method writing down the various exchange matrix elements and summing up these different terms, will be discussed in the Section 8.2.2.

For ${}^{A}F_{101}^{(0)}$ we are only interested in the space part $\mathbf{A}$ of the axial vector current (see eqn (6.25)). In the non-relativistic limit this current between nuclear states is simply given by (see eqns (8.259*d*) and (9.23))

$$\mathbf{j}_A = -\left(\lambda\boldsymbol{\sigma}\frac{1}{2M_N} + f_P(\boldsymbol{\sigma}\mathbf{q})\mathbf{q}\right)t_{\mp}. \tag{8.277}$$

In the static limit we have $A_0 = 0$ and eqn (8.273) reduces to

$$\nabla\mathbf{A} = m_\pi^2 f_\pi \phi_{\pi^*}. \tag{8.278}$$

On the other hand, from eqn (8.277) it follows:†

$$\nabla\mathbf{A} = \nabla\mathbf{j}_A = \mathrm{i}\left[\lambda(\boldsymbol{\sigma}\mathbf{q}) + \frac{1}{2M_N} f_P(\boldsymbol{\sigma}\mathbf{q})\mathbf{q}^2\right]t_{\mp}. \tag{8.279}$$

† The divergence of the current $\mathbf{A}$ is obtained as (see eqn (6.16))

$$\langle f|\frac{\partial A_\mu(x)}{\partial x_\mu}|i\rangle = \frac{\partial}{\partial x_\mu}\mathrm{e}^{\mathrm{i}(p_{f_\mu}-p_{i_\mu})x_\mu}\langle f|A_\mu(0)|i\rangle = -\mathrm{i}\mathbf{q}\langle f|\mathbf{A}|i\rangle - \mathrm{i}\Delta\langle f|A_0|i\rangle$$

i.e. $\nabla\mathbf{A} = -\mathrm{i}\mathbf{q}\mathbf{A}$.

Thus, we get

$$i\left[\lambda(\boldsymbol{\sigma}\mathbf{q})+\frac{1}{2M_N}f_p(\boldsymbol{\sigma}\mathbf{q})\mathbf{q}^2\right]t_{\mp}=m_\pi^2 f_\pi \phi_{\pi^*}(\mathbf{q}). \tag{8.280}$$

This expression offers us a relation between the pion field of a nucleon and the axial vector coupling constant together with the pseudoscalar coupling constant.

The pion field itself for a point-like nucleon located at a position x_i in free space has the usual Yukawa form

$$\phi_{\pi^*}(x)=-\frac{g_r\sqrt{2}}{8\pi M_N}\boldsymbol{\sigma}\nabla\left(\frac{\exp(-m_\pi|\mathbf{x}-\mathbf{x}_i|)}{|\mathbf{x}-\mathbf{x}_i|}\right)t_{\pm} \tag{8.282}$$

where g_r is the pion nucleon coupling constant ($g_r^2/4\pi=14{\cdot}3$).

In momentum space we have to take the Fourier transform and obtain

$$\phi_{\pi^*}(\mathbf{q})=\frac{ig_r}{\sqrt{2}\,M_N}\frac{\boldsymbol{\sigma}\mathbf{q}}{\mathbf{q}^2+m_\pi^2}t_{\pm} \tag{8.283a}$$

or

$$\phi_{\pi^*}(\mathbf{q})=\frac{ig_r}{\sqrt{2}\,M_N}\frac{\boldsymbol{\sigma}\mathbf{q}}{m_\pi^2}\left[1-\frac{\mathbf{q}^2}{(\mathbf{q}^2+m_\pi^2)}\right]t_{\pm}. \tag{8.283b}$$

By inserting this expression into eqn (8.280) we then get

$$\lambda=\frac{g_r f_\pi}{\sqrt{2}\,M_N} \tag{8.284}$$

$$\begin{aligned} f_P&=-\frac{g_r f_\pi}{\sqrt{2}\,M_N}\frac{2M_N}{\mathbf{q}^2+m_\pi^2}\\ &=-\frac{2M_N\lambda}{\mathbf{q}^2+m_\pi^2}. \end{aligned} \tag{8.285}$$

The first expression (eqn (8.284)) is the so-called Goldberger–Treiman relation (Goldberger and Treiman 1958). The second formula relates the pseudoscalar coupling constant to the axial vector one (these equations are discussed in a little more detail in Section 10.3).

Thus, PCAC relates the axial vector current to the pion field operator through eqn (8.280). As a consequence the coupling constants λ and f_P come out to be a function of the pion decay amplitude.

The relations given above are only valid for a free nucleon. If the nucleon is embedded in nuclear matter they are modified. In the following we will, therefore, study how they are modified since then we also obtain the renormalization of λ and f_P in parallel. This topic has been treated by Ericson and co-workers in detail (Ericson *et al.* 1973; Delorme and Ericson 1976; Delorme *et al.* 1976; Ericson and Bernabeu 1977).

Similar results were also obtained by Rho (1974), and Ohta and Wakamatsu (1974), by using different approaches. We shall, however, not go into much detail in the following but rather refer to the references listed above. A complete discussion of the whole problem can be found in the comprehensive article by Delorme *et al.* (1976). A very good survey is also given in some chapters of *Mesons in Nuclei* (Rho and Wilkinson 1979), especially in those written by Blin–Stoyle (1979) and Ericson (1979).

Besides, the essential features can be shown by discussing a classical analogue, i.e. the field of a static electric dipole embedded in a medium with dielectric constant ε. This analogy can immediately be seen if the relevant differential equations and their solutions for the pion field and the static electric dipole field are considered. The pion field (see eqn (8.282)) is a solution of the Klein–Gordon equation

$$(-\Box + m_\pi^2)\phi_{\pi^*}(\mathbf{x}) = (-\nabla^2 + m_\pi^2)\phi_{\pi^*}(\mathbf{x}) = j_{\pi^*}(\mathbf{x}) \tag{8.286}$$

where the source term $j_{\pi^*}(\mathbf{x})$ reads as

$$j_{\pi^*}(\mathbf{x}) = -\frac{g_r\sqrt{2}}{2M_N}\boldsymbol{\sigma}\nabla\delta(\mathbf{x}-\mathbf{x}_i)t_\pm. \tag{8.287}$$

In the classical electromagnetic theory we have an analogous equation (Poisson equation) for the electrostatic potential $V(\mathbf{x})$ (see, for example, Jackson 1962)

$$-\Box V(\mathbf{x}) = -\nabla^2 V(\mathbf{x}) = \frac{1}{\varepsilon_0}\rho(\mathbf{x}) \tag{8.288}$$

where $\rho(\mathbf{x})$ is the source charge distribution. For the particular case of an electric dipole $\mathbf{d}$ located at $\mathbf{x}_1$ the source term has the same form as given in eqn (8.287) (see, for example, Jackson 1962)

$$\rho(\mathbf{x}) = -\mathbf{d}\nabla\delta(\mathbf{x}-\mathbf{x}_1). \tag{8.289}$$

The solution of the Poisson equation with this source term, i.e. the potential created by a static dipole, is then obtained as

$$\begin{aligned} V(\mathbf{x}) &= -\frac{1}{4\pi\varepsilon_0}\mathbf{d}\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}_1|}\right) \\ &= -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{dr}}{r^3} \end{aligned} \tag{8.290}$$

with $\mathbf{r} = \mathbf{x} - \mathbf{x}_1$.

The difference between this electrostatic potential and the pion field of eqn (8.232) arises from the pion mass which is responsible for the exponential decay of the pion field. In the case of the pion field we have also to replace the dipole by an axial dipole.

If we now embed the dipole in a medium with dielectric constant ε analogously to the embedding of a nucleon in nuclear matter, the source term of the Poisson equation $\rho_e(\mathbf{x})$ is changed to

$$\rho_\varepsilon(\mathbf{x}) = -\mathbf{d}\nabla\delta(\mathbf{x}-\mathbf{x}_1) - \nabla\mathbf{P}(\mathbf{x}) \tag{8.291}$$

where the polarization density is

$$\mathbf{P} = \alpha\varepsilon_0\mathbf{E} = -\alpha\varepsilon_0\nabla V(\mathbf{x}) \tag{8.292}$$

and α is the electric polarizability coefficient ($\alpha = \varepsilon - 1$) and $\mathbf{E}$ the electric field.

The corresponding differential equation now reads as

$$-(1+\alpha)\nabla^2(\mathbf{x}) = -\frac{1}{\varepsilon_0}\mathbf{d}\nabla\delta(\mathbf{x}-\mathbf{x}_1) \tag{8.293}$$

with the solution

$$V(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\frac{1}{1+\alpha}\frac{\mathbf{dr}}{r^3}. \tag{8.294}$$

For the pion field the analogous Klein–Gordon equation for a nucleus in nuclear matter is,† if correlations are not taken into account (see, for example, Delorme *et al.* 1976),

$$-[(1+\alpha_0)\nabla^2 + m_\pi^2]\phi_{\pi^*}(\mathbf{x}) = j_{\pi^*}(\mathbf{x}). \tag{8.295}$$

Here, the pion propagator is modified but the source term (pionic vertex) is left unchanged. The corresponding solution $\phi_{\pi^*}(\mathbf{x})$ for the pion field is then given by

$$\phi_{\pi^*}(\mathbf{x}) = -\frac{1}{1+\alpha_0}\frac{g_r\sqrt{2}}{8\pi M_N}\boldsymbol{\sigma}\nabla\left(\frac{\exp(-m'_\pi|\mathbf{x}-\mathbf{x}_i|)}{|\mathbf{x}-\mathbf{x}_i|}\right)t_\pm \tag{8.296}$$

where $m'_\pi = m_\pi/(1+\alpha_0)^{1/2}$.

Thus for a nucleon in nuclear matter the strength of the pion field is altered by a factor $1/(1+\alpha_0)$ and the range is changed ($m_\pi \to m'_\pi$).

In momentum space we have now

$$\phi_{\pi^*}(\mathbf{q}) = \frac{ig_r}{\sqrt{2}\,M_N}\frac{\boldsymbol{\sigma}\mathbf{q}}{(1+\alpha_0)\mathbf{q}^2 + m_\pi^2}t_\pm \tag{8.297}$$

with the consequence that λ is not modified but f_P reads now as

$$f_P = -\frac{2M_N(1+\alpha_0)}{(1+\alpha_0)\mathbf{q}^2 + m_\pi^2}\lambda. \tag{8.298}$$

† If $\alpha_0 = \alpha_0(x)$, we have

$$\{-\nabla[1+\alpha_0(x)]\nabla + m_\pi^2\}\phi_{\pi^*}(x) = j_{\pi^*}(x).$$

The nuclear matter is, however, not of the type of a homogenously distributed matter but rather of a more granular form. We have, therefore, correlations between the surrounding nucleons (short range, Pauli principle). Classically, the effect of these correlations can be simulated by a hole, which surrounds the electric dipole embedded in a dielectric medium (of dielectric constant ε).

Then, the potential outside the cavity has the form

$$V = -\frac{1}{4\pi\varepsilon_0}\frac{1}{1+\alpha}\frac{1+\alpha}{1+\frac{2}{3}\alpha}\frac{\mathbf{dr}}{r^3}. \tag{8.299}$$

For small α this corresponds to a potential of a dipole of a strength $d(1+\frac{1}{3}\alpha)$. By introducing an effective α_e given by

$$\alpha_e = \frac{\alpha}{1-\frac{1}{3}\alpha} \tag{8.300}$$

we obtain

$$V = -\frac{1}{4\pi\varepsilon_0}\frac{1}{1+\alpha_e}\frac{(1+\frac{1}{3}\alpha_e)\mathbf{dr}}{r^3}. \tag{8.301}$$

This is a solution of the Poisson equation

$$-(1+\alpha_e)\nabla^2 = -\frac{1}{\varepsilon_0}(1+\tfrac{1}{3}\alpha_e)\mathbf{d}\nabla\delta(\mathbf{x}-\mathbf{x}_1). \tag{8.302}$$

In the case of the pion field it can be shown that short range correlations and antisymmetrization effects modify the Klein–Gordon equation completely analogously (Ericson *et al.* 1973; Delorme and Ericson 1976; Delorme *et al.* 1976), i.e.

$$[-\nabla(1+\alpha_s(\mathbf{x}))\nabla + m_\pi^2]\phi_{\pi^+}(\mathbf{x}) = (1+\tfrac{1}{3}\alpha_s(\mathbf{x}))j_{\pi^+}(\mathbf{x}). \tag{8.303}$$

In our infinite and homogeneous nuclear medium the polarizability coefficient is a constant $\alpha_s = \alpha_s(\mathbf{x})$. Then, the solution is greatly simplified.

Thus, the nuclear medium affects the virtual pion in the following two ways. First, it modifies the pion propagator by the 'dielectric constant' $\varepsilon = 1+\alpha_s$ of the nuclear matter. Secondly, it renormalizes the pion nucleon coupling constant g_r by the Lorentz–Lorenz factor

$$\eta_s = 1+\tfrac{1}{3}\alpha_s \tag{8.304}$$

and replaces the 'polarizability' α_0 by an effective value α_s

$$\alpha_s = \frac{\alpha_0}{1-\frac{1}{3}\alpha_0}. \tag{8.305}$$

For a point source in infinite and homogeneous nuclear matter eqn

(8.303) again has the usual Yukawa type solution which now in momentum space reads as

$$\phi_{\pi^*}(\mathbf{q}) = \frac{i\eta_s g_r}{\sqrt{2}\, M_N} \frac{\boldsymbol{\sigma}\mathbf{q}}{(1+\alpha_s)\mathbf{q}^2 + m_\pi^2} t_\pm. \tag{8.306}$$

If we insert this expression into eqn (8.280) we obtain a modified relation between the axial vector coupling constant λ (or the pseudoscalar coupling constant f_P) and the strong interaction coupling constant. It now looks like

$$\lambda_{Ae} = \eta_s \frac{g_r f_\pi}{\sqrt{2}\, M_N} \tag{8.307}$$

$$f_{P_{Ae}} = -\frac{2M_N(1+\alpha_s)}{(1+\alpha_s)\mathbf{q}^2 + m_\pi^2} \eta_s \lambda. \tag{8.308}$$

That means the axial vector coupling constant λ is renormalized to

$$\lambda_{Ae} = \eta_s \lambda \tag{8.309}$$

and the pseudoscalar coupling constant to

$$f_{P_{Ae}} = \eta_s(1+\alpha_s) \frac{\mathbf{q}^2 + m_\pi^2}{(1+\alpha_s)\mathbf{q}^2 + m_\pi^2} f_P. \tag{8.310}$$

By inserting the numerical value $\alpha_s = -0{\cdot}75$ (Ericson 1979) we obtain

$$\frac{\lambda_{Ae}}{\lambda} = 0{\cdot}75 \tag{8.311}$$

$$\frac{f_{PAe}}{f_P} = 0{\cdot}19 \tag{8.312}$$

Both constants are quenched, λ by an amount of ~25% and f_P by the large amount of ~81%. Indeed, there is some indication (see Section 14.2.2.1) that a quenching of such type can be observed.

It should, however, be noted that the quenching is further modified if one assumes a more realistic model of a finite nucleus instead of the infinite nuclear matter model. For that purpose eqn (8.303) can also be solved for a model of one active nucleon distributed in an infinitely thin spherical shell. This distribution may simulate that of a valence nucleon. The distribution of the other nucleons is then assumed to be uniform in a sphere of radius R (Delorme *et al.* 1976).

The solution of this model shows that the renormalization of λ is strongly dependent on the location of the source nucleon and on the nuclear radius. If the source nucleon is close to the surface, the quenching is very small and nearly independent of the nuclear radius R. If on the

other hand, the source nucleon is located in the centre of the nucleus the quenching is strongly dependent on the nuclear radius R; for small values of R the quenching is less pronounced (for $R \to 0\ \lambda_{Ae}/\lambda \to 1$) while for large R the renormalization reaches the nuclear matter value (Delorme *et al.* 1976).

For an observation of the Lorentz–Lorenz quenching, transitions where peripheral valence nucleons are involved should therefore not be considered. Instead, such a transition should take place in the inner region of the nucleus.

Finally, we will also give the relation between the axial current and the pion field in co-ordinate space. The Klein–Gordon equation, eqn (8.303), can also be written as

$$(-\nabla^2 + m_\pi^2)\phi_{\pi^*}(\mathbf{x}) = j_{\pi^*}(\mathbf{x}) + \nabla \mathbf{P}^{\pm}(\mathbf{x}) \tag{8.313}$$

with the polarization density

$$\mathbf{P}^{\pm}(\mathbf{x}) = \alpha_s \nabla \phi_{\pi^*}(\mathbf{x}) - \tfrac{1}{3}\alpha_s \frac{g_r \sqrt{2}}{2M_N} \boldsymbol{\sigma} \delta(\mathbf{x} - \mathbf{x}_i) t_{\pm}. \tag{8.314}$$

Inserting eqn (8.287) and transferring the ∇^2 term to the right-hand side we get

$$m_\pi^2 \phi_{\pi^*}(\mathbf{x}) = \nabla \left[-\frac{g_r \sqrt{2}}{2M_N} \boldsymbol{\sigma} \delta(\mathbf{x} - \mathbf{x}_i) t_{\pm} + \nabla \phi_{\pi^*}(\mathbf{x}) + \mathbf{P}^{\pm}(\mathbf{x}) \right]. \tag{8.315}$$

By making use of the PCAC relation of eqn (8.273) and of the Goldberger–Treiman relation of eqn (8.284), the axial vector current can then be expressed as

$$\mathbf{A}(\mathbf{x}) = -\lambda \boldsymbol{\sigma} \delta(\mathbf{x} - \mathbf{x}_i) t_{\pm} + f_\pi \nabla \phi_{\pi^*}(\mathbf{x}) + f_\pi \mathbf{P}(\mathbf{x}) \tag{8.316}$$

or

$$\mathbf{A}(\mathbf{x}) = -\lambda (1 + \tfrac{1}{3}\alpha_s) \boldsymbol{\sigma} \delta(\mathbf{x} - \mathbf{x}_i) t_{\pm} + f_\pi (1 + \alpha_s) \nabla \phi_{\pi^*}(\mathbf{x}). \tag{8.317}$$

Classically, eqn (8.316) has an analogue which allows a good physical interpretation of this expresion. The quantity $-\lambda \boldsymbol{\sigma} \delta(\mathbf{x} - \mathbf{x}_i) t_{\pm}$ corresponds to a distribution $\mathbf{D}_0(\mathbf{x})$ of permanent dipoles in a medium, $\mathbf{P}(\mathbf{x})$ to that of induced dipoles $\mathbf{P}_c(\mathbf{x})$ and $\nabla \phi_{\pi^*}(\mathbf{x})$ to an electric field $\mathbf{E}$. The quantity $\mathbf{A}(\mathbf{x})$ is therefore the axial analogue of the displacement vector

$$\mathbf{D}(\mathbf{x}) = \mathbf{D}_0(\mathbf{x}) + \mathbf{E}(\mathbf{x}) + \mathbf{P}_c(\mathbf{x}) \tag{8.318}$$

of a polarizable medium which has permanent dipoles (pyroelectric medium).

Quantum mechanically, eqn (8.137) can be interpreted by the diagrams shown in Fig. 8.2.

For $\alpha_s = 0$, the first term corresponds to the usual axial coupling of a

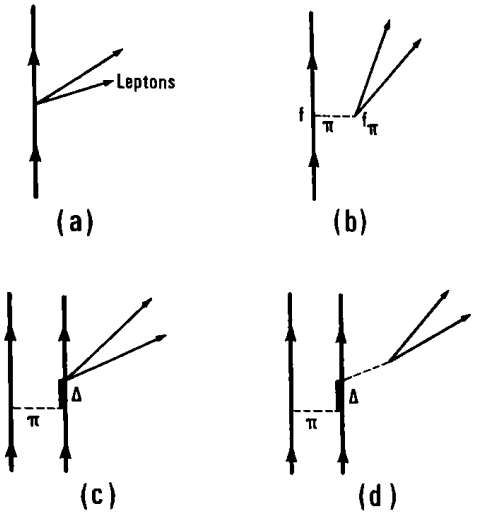


FIG. 8.2. Diagrams representing (a) usual nucleon-lepton coupling; (b) induced pseudoscalar interaction; (c) and (d) effect of nuclear matter on the axial vector current.

nucleon to the lepton field as shown in Fig. 8.2(*a*). The second term represents the induced pseudoscalar interaction which has its origin in the interaction of the lepton pair via the exchange of a pion (see diagram (*b*)). For $\alpha_s \neq 0$ the axial current is modified by diagrams like those shown in (*c*) and (*d*).

It should also be mentioned that the renormalization scheme sketched above has recently been more generalized by making use of the Landau–Migdal Fermi liquid theory. The quenching is then a function of the spin–isospin polarizability expressed by the Landau–Migdal parameter g' (see Oset and Rho 1979; Mukhopadhyay *et al.* 1979).

However, it should be pointed out that another important contribution to the renormalization comes from core polarization effects involving a great number of highly excited shell model configurations. Since these effects can also produce a relatively large quenching they have been suggested as the main mechanism by some authors (Shimizu *et al.* 1974; Ejiri and Fujita 1978; Arima and Hyuga 1979). It has, however, been argued (Oset and Rho 1979) that other tensor correlations may cancel the usual core polarization contributions (Rho 1974).

It should therefore always be kept in mind that other processes could interfere with the quenching caused by one pion and heavier meson

exchange. Thus, for a comparison other processes have also to be considered.

8.2.2. Exchange matrix elements

In impulse approximation it was assumed that the nucleons inside the nucleus interact as free particles (see Section 8.1). Here, the internucleon interaction through the mesonic field was, however, not taken into account. These mesonic interactions manifest themselves by a number of many-body interaction effects which are usually called exchange effects. Thus, in addition to the single particle terms in the hadron (or nuclear) current given by eqn (8.1) two, three, etc., body terms will now appear. If we restrict ourselves to the static (i.e. allowed) approximation we then have

$$V_\mu(\mathbf{x}) = \mathrm{i}\sum_n \bar{\psi}(\mathbf{x})\gamma_\mu\{t_\mp\}_n\psi(\mathbf{x}) + \sum_{n<m} V_\mu^{(2)}(\mathbf{r}_n-\mathbf{x}, \mathbf{r}_m-\mathbf{x}) + (\text{three or many body terms})$$

$$A_\mu(\mathbf{x}) = \mathrm{i}\lambda\sum_n \bar{\psi}(\mathbf{x})\gamma_\mu\gamma_5\{t_\mp\}_n\psi(\mathbf{x}) + \sum_{n<m} A_\mu^{(2)}(\mathbf{r}_n-\mathbf{x}, \mathbf{r}_m-\mathbf{x}) + (\text{three or many body terms}).$$

A general discussion of all possible two body terms has been given by Foldy and Lock (1979) by making use of a phenomenological approach, but, we do not intend to present here a systematic and comprehensive treatment of meson exchange currents.

Excellent recent reviews of that topic can, however, be found in the book *Mesons in Nuclei* (Rho and Wilkinson 1979), for example, the chapters by Chemtob, Delorme and Blin-Stoyle. Our purpose is more to elucidate some essential points by treating special cases. Thus, as before, we limit ourselves to considering those form factor coefficients which are the most important ones for allowed transitions (see Section 14.2), i.e. ${}^{\mathrm{V}}F_{000}^{(0)}$ and ${}^{\mathrm{A}}F_{101}^{(0)}$.

In contrast to the foregoing section we now consider the two body exchange terms explicitly.

As already mentioned before we have no exchange contributions to the vector form factor coefficient ${}^{\mathrm{V}}F_{000}^{(0)}$ because the CVC theory forbids the existence of such terms in this case (see the detailed discussion in Section 10.2). The form factor coefficient ${}^{\mathrm{V}}F_{000}^{(0)}$ can therefore be exactly calculated in impulse approximation by applying eqn (8.140). For the axial current, on the other hand, no such strong conservation law exists (see Section 10.3). Thus, exchange terms do contribute in principle. Neglecting the induced terms for the moment the axial form factor

coefficient ${}^{A}F^{(0)}_{101}$ can be related to the one-body matrix element ${}^{A}\mathfrak{M}^{(0)}_{101}$ (Gamow–Teller matrix element) in impulse approximation by eqn (8.274). ${}^{A}\mathfrak{M}^{(0)}_{101}$ is the Gamow–Teller matrix element. For symmetry reasons we can now assume that the exchange operators transform under rotation in isospin space, rotation and inversion in ordinary space in the same way as the Gamow–Teller operator $\sum_n \{\boldsymbol{\sigma} t_{\mp}\}_n$ (see eqn (8.274)). This latter operator is an axial vector in ordinary space and an isovector in isospin space. Of course, transformation under time reversal (see Section 10.1) has also to be chosen as time reversal invariant.

The two body exchange operators have therefore to be constructed as isovectors and as axial vectors (Osborn and Foldy 1950; Bell and Blin-Stoyle 1958; Chemtob and Rho 1971; Chemtob 1979). The players in the game at our disposal are the co-ordinates of the two nucleons $\mathbf{x}_1, \mathbf{x}_2$, their momenta $\mathbf{p}_1, \mathbf{p}_2$, the Pauli spin matrices $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ and the isospin matrices $\mathbf{t}(1), \mathbf{t}(2)$. In isospin space the following three isovector operators can be constructed

$$[\mathbf{t}(1)\times\mathbf{t}(2)]_{\mp} \tag{8.319a}$$

$$[t_{\mp}(1)+t_{\mp}(2)] \tag{8.319b}$$

$$[t_{\mp}(1)-t_{\mp}(2)]. \tag{8.319c}$$

In ordinary space usually relative and centre of mass co-ordinates are used

$$\mathbf{r}_{12}=\mathbf{x}_1-\mathbf{x}_2, \qquad \mathbf{R}=\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_2) \tag{8.320}$$

Operators depending on $\mathbf{R}$ (translationally non-invariant terms) or on the momenta $\mathbf{p}_1, \mathbf{p}_2$ (non-local terms) which are allowed but usually small will not be considered further. The following seven axial vectors can then be built up (Osborn and Foldy 1950):

$$(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2) \tag{8.321a}$$

$$(\boldsymbol{\sigma}_1\pm\boldsymbol{\sigma}_2) \tag{8.321b}$$

$$[(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\mathbf{r}_{12}]\mathbf{r}_{12}/r_{12}^2 \tag{8.321c}$$

$$[(\boldsymbol{\sigma}_1\pm\boldsymbol{\sigma}_2)\mathbf{r}_{12}]\mathbf{r}_{12}/r_{12}^2 \tag{8.321d}$$

$$[(\boldsymbol{\sigma}_1\mathbf{r}_{12})(\boldsymbol{\sigma}_2\times\mathbf{r}_{12})+(\boldsymbol{\sigma}_1\times\mathbf{r}_{12})(\boldsymbol{\sigma}_2\mathbf{r}_{12})]/r_{12}^2. \tag{8.321e}$$

The complete two-body exchange operators are obtained by combining the spin co-ordinate parts with the isospin parts in all possible ways, with the restriction that they are symmetric under interchange of all co-ordinates of the two nucleons involved. It is then straightforward to construct the most general (translational invariant) expression for the

corresponding two-body exchange† term (Bell and Blin-Stoyle 1958; Chemtob and Rho 1971; Chemtob 1979):

$$\begin{aligned}{}^{A}\mathfrak{M}^{(0)}_{101}(1,2) = \frac{1}{\sqrt{(2J_i+1)}}\langle f\Big|\Big|\sum_{i\neq j}\{&[(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j)g_{\mathrm{I}}+T^{(\times)}_{ij}g_{\mathrm{II}}]\\ &\times[\mathbf{t}(i)\times\mathbf{t}(j)]_{\mp}+[(\boldsymbol{\sigma}_i-\boldsymbol{\sigma}_j)h_{\mathrm{I}}+T^{(-)}_{ij}h_{\mathrm{II}}]\\ &\times[t_{\mp}(i)-t_{\mp}(j)]+[(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)j_{\mathrm{I}}+T^{(+)}_{ij}j_{\mathrm{II}}]\\ &\times[t_{\mp}(i)+t_{\mp}(j)]\}\Big|\Big|i\rangle \end{aligned} \tag{8.322}$$

where

$$T^{\odot}_{ij} = \left[\frac{(\boldsymbol{\sigma}_i\odot\boldsymbol{\sigma}_j)(\mathbf{r}_{ij}\mathbf{r}_{ij})}{r^2_{ij}} - \tfrac{1}{3}\boldsymbol{\sigma}_i\odot\boldsymbol{\sigma}_j\right]$$

with $\odot = \pm, \times$.

The functions g_{I}, g_{II}, h_{I}, h_{II}, j_{I} and j_{II} are scalar functions of r_{ij} which are real because of time reversal invariance. These functions have then to be determined by considering special mechanisms of pion and heavier meson exchange as shown in Fig. 8.3. The different diagrams shown in Fig. 8.3 correspond to the following processes (a) recoil, (b) pair, (c) non-Born, (d) non-Born and (e) heavy vector meson exchange (Chemtob 1979). The following results for the radial functions g, h and j have explicitly been evaluated by Chemtob and Rho (1971) by taking into account these different diagrams.

(1) *One pion exchange*

(a) Recoil

$$h_{\mathrm{I}} = j_{\mathrm{I}} = -\tfrac{1}{4}g_{\mathrm{I}} = -\frac{1}{3\pi}f^2_{\pi NN}\left[K_0(x_\pi) - \frac{K_1(x_\pi)}{x_\pi}\right] \tag{8.323a}$$

$$h_{\mathrm{II}} = j_{\mathrm{II}} = \tfrac{1}{2}g_{\mathrm{II}} = -\frac{1}{\pi}f^2_{\pi NN}K_2(x_\pi) \tag{8.323b}$$

where $K_l(x_\pi)$ are modified Bessel functions.

(b) Pair

$$g_{\mathrm{I}} = \frac{2}{3}\frac{m_\pi}{M_{\mathrm{N}}}f^2_{\pi NN}Y_0(x_\pi) \tag{8.324a}$$

$$g_{\mathrm{II}} = -\frac{m_\pi}{M_{\mathrm{N}}}f^2_{\pi NN}Y_2(x_\pi). \tag{8.324b}$$

(c) Non-Born

$$g_{\mathrm{I}} = \tfrac{2}{3}\xi_\pi\alpha(0)Y_0(x_\pi) \tag{8.324c}$$

$$g_{\mathrm{II}} = -\xi_\pi\alpha(0)Y_2(x_\pi) \tag{8.324d}$$

$$h_{\mathrm{I}} = j_{\mathrm{I}} = -\tfrac{1}{6}\xi_\pi\gamma(0)Y_0(x_\pi) \tag{8.324e}$$

$$h_{\mathrm{II}} = j_{\mathrm{II}} = -\tfrac{1}{2}\xi_\pi\gamma(0)Y_2(x_\pi). \tag{8.324f}$$

† Wave function renormalization effects due to presence of virtual π's, Δ's etc. have also to be taken into account.

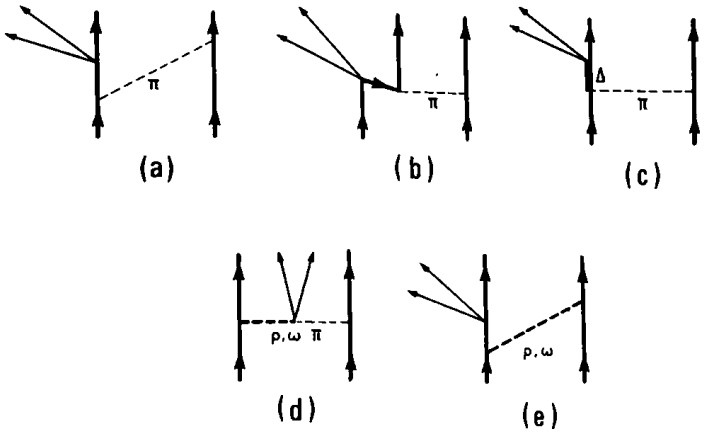


FIG. 8.3. Axial vector exchange diagrams corresponding to the following processes: (a) recoil; (b) pair; (c) non-Born; (d) non-Born; (e) heavy vector meson exchange.

Here, we have†

$$Y_0(x) = \frac{e^{-x}}{x} \tag{8.325}$$

$$Y_2(x) = \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) Y_0(x) \tag{8.326}$$

$$x_\pi = m_\pi r_{ij} \tag{8.327}$$

$$\xi_\pi = \frac{1}{8\pi} \frac{g_r}{\lambda} \frac{m_\pi}{M_N} m_\pi^2 \tag{8.328}$$

$$f_{\pi NN}^2 = \frac{1}{4\pi} \left(g_r \frac{m_\pi}{2M_N}\right)^2 \simeq 0{\cdot}08 \tag{8.329}$$

where m_π is the pion mass, g_r is the pion nuclear coupling constant ($g_r^2/4\pi = 14{\cdot}3$), and $\alpha(0)$, $\gamma(0)$ are combinations of the invariant pion-nucleon scattering amplitudes evaluated at $q^2 = 0$ ($\alpha(0) = \frac{1}{4}\gamma(0) = 0{\cdot}578 m_\pi^{-2}$).

(2) *Heavy meson exchange*

Recoil

$$\rho \text{ meson: } j_I = -h_I = -\left(\frac{g_{\rho NN}}{2\pi}\right)^2 K_0(x_\rho) \tag{8.330a}$$

$$\omega \text{ meson: } j_I = h_I = -\left(\frac{g_{\omega NN}}{2\pi}\right)^2 K_0(x_\omega) \tag{8.330b}$$

† $2\pi^2 m_\pi Y_0(x_\pi) = \int d\mathbf{q} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{q^2 + m_\pi^2}$.

where $x_\rho = m_\rho r_{ij}$, $x_\omega = m_\omega r_{ij}$ and $g_{\rho NN}$, $g_{\omega NN}$ are the corresponding meson nucleon coupling constants. It should, however, be pointed out that these latter results are not as reliable as those for one pion exchange.

If exchange effects are included the form factor coefficient ${}^{A}F^{(0)}_{101}$ then generally reads as

$$ {}^{A}F^{(0)}_{101} = \mp\lambda[{}^{A}\mathfrak{M}^{(0)}_{101}(1) + {}^{A}\mathfrak{M}^{(0)}_{101}(1,2)] \tag{8.331} $$

whereby the first term denotes the usual one-body part and the second one the two-body part discussed above. In view of the interest in these terms, a number of detailed calculations of exchange contributions to the axial vector form factor coefficient ${}^{A}F^{(0)}_{101}$, i.e. of the term ${}^{A}\mathfrak{M}^{(0)}_{101}(1,2)$, have been carried out recently (Barroso and Blin-Stoyle 1975; Blin-Stoyle 1975; Jaus 1976; Khanna *et al.* 1978; Towner and Khanna 1979; Bargholtz 1979) with the result that exchange effects of an order of about 0–25% could be expected.

However, it should be noted that a direct comparison between calculated exchange effects and experimentally determined deviations of Gamow–Teller matrix elements from their one-body values is difficult.

Other competing processes which have already been mentioned have also to be taken into account, like core polarization (Shimizu *et al.* 1974) and relativistic effects in the nuclear wave functions (Friar 1973, 1979). Then, one is confronted with various contributing terms with a high degree of cancellation (see, for example, Towner and Khanna 1979). Nevertheless, today we find a surprisingly good agreement between theory and experiment (see Section 14.2.2.2).

Up to now we have only treated the special case of the allowed transitions. In principle, all types of form factors $F_{KLs}(q^2)$, i.e. those for forbidden transitions too, can, however, be represented as a sum of 1, 2, 3 and more body matrix elements (with exception of some vector form factors like, for instance, ${}^{V}F_{000}(q^2)$). The one-body terms, which can be calculated in impulse approximation, are the most important ones. They have been extensively discussed before in Section 8.1. Besides, two-body terms for every type of form factor $F_{KLs}(q^2)$ can explicitly be found in the article by Delorme (1979). However, since calculations of exchange effects are very complicated and a topic on its own, we have not given more details here. As we stressed earlier, they are a little beyond the scope of this book. Once again, for details we refer readers to the reviews mentioned above.

8.2.3. Off-mass shell effects

In Section 8.1.2 we considered the most general form of the nuclear current, but we had restricted our discussion to the case of free nucleons

which are strictly on the mass shell, i.e. to nucleons where the spinors $u(p)$ are solutions of the free particle Dirac equation (standard impulse approximation treatment)

$$[i\gamma_\mu p_\mu + M_N]u(p) = 0. \tag{8.332}$$

Now, the nucleons within a nucleus are bounded and are therefore off the mass shell. Since the binding energy B is always small compared with the nuclear mass M_N the off-mass shell effects are also small (of the order B/M_N). Thus, in most cases they can be neglected. There exists, however, one exception, and that is the second class current problem (see Section 10.4) where off-mass shell effects are of a certain importance. In the following we shall now particularly discuss that aspect of the off-mass shell behaviour of the bound nucleons. If off-mass shell effects are admitted, relativistic covariance allows more complicated expressions than those given in eqns (8.236) and (8.237). In this case the vector and axial vector current consist of 12 terms each and has the form (see, for example, Bernstein 1968; Delorme and Rho 1971)

$$\begin{aligned}\langle p|\, V_\mu(0)\,|n\rangle = i\Big\langle \bar{u}_p(p')\Big\{ \sum_{i=V,M,S} [g_i O^i_\mu + F_i(i\gamma_\lambda p'_\lambda + M_N)O^i_\mu \\ + G_i O^i_\mu(i\gamma_\lambda p_\lambda + M_N) + H_i(i\gamma_\lambda p'_\lambda + M_N)O^i_\mu \\ \times (i\gamma_\lambda p_\lambda + M_N)]\Big\} u_n(p)\Big\rangle\end{aligned} \tag{8.333}$$

where $O^V_\mu = \gamma_\mu$, $O^M_\mu = \sigma_{\mu\nu}q_\nu$, and $O^S_\mu = iq_\mu$.

$$\begin{aligned}\langle p|\, A_\mu(0)\,|n\rangle = i\Big\langle \bar{u}_p(p')\Big\{ \sum_{i=A,T,P} [g_i O^i_\mu + F_i(i\gamma_\lambda p'_\lambda + M_N)O^i_\mu \\ + G_i O^i_\mu(i\gamma_\lambda p_\lambda + M_N) + H_i(i\gamma_\lambda p'_\lambda + M_N)O^i_\mu \\ \times (i\gamma_\lambda p_\lambda + M_N)]\Big\} u_n(p)\Big\rangle\end{aligned} \tag{8.334}$$

where $O^A_\mu = \gamma_\mu\gamma_5$, $O^T_\mu = \sigma_{\mu\nu}\gamma_5 q_\nu$ and $O^P_\mu = iq_\mu\gamma_5$.

On the mass shell the terms associated with the coupling constants† F_i, G_i and H_i vanish. If time reversal invariance is assumed to be valid they are real (see Section 10.1). The CVC theory requires that second class currents (see Section 10.4) are absent for vector currents. Therefore, we do not treat vector currents further and only consider axial currents. The 12 independent covariants of eqn (8.334) have, however, no definite behaviour under a G-parity transformation (see Section 10.4). By reordering these 12 terms we can construct 12 other independent terms with

† In principle, the F_i, G_i and H_i are form factors depending on q^2, p'^2 and p^2 (Delorme and Rho 1971). However, in beta-decay it is certainly justified to ignore this form factor behaviour and take them as constants.

definite G transformation properties. They are listed† below together with their on-shell limits (Delorme and Rho 1971)

(a) First class axial covariants under a G transformation together with their on-shell limits

$$i\gamma_\mu\gamma_5 \to i\gamma_\mu\gamma_5 \tag{8.335a}$$

$$q_\mu\gamma_5 \to q_\mu\gamma_5 \tag{8.335b}$$

$$iP_\mu(\gamma P)\gamma_5 \to 0 \tag{8.335c}$$

$$iq_\mu(\gamma q)\gamma_5 \to -2M_N q_\mu\gamma_5 \tag{8.335d}$$

$$i\sigma_{\mu\nu}P_\nu\gamma_5 \to -2iM_N\gamma_\mu\gamma_5 + q_\mu\gamma_5 \tag{8.335e}$$

$$iq_\mu\sigma_{\lambda\nu}P_\lambda q_\nu\gamma_5 \to -(4M_N^2+q^2)q_\mu\gamma_5 \tag{8.335f}$$

$$i\varepsilon_{\mu\nu\rho\sigma}P_\nu q_\rho\gamma_5 \to iq^2\gamma_\mu\gamma_5 + 2M_N q_\mu\gamma_5. \tag{8.335g}$$

(b) Second class axial covariants under a G transformation together with their on-shell limits

$$i\sigma_{\mu\nu}q_\nu\gamma_5 \to i\sigma_{\mu\nu}q_\nu\gamma_5 \tag{8.336a}$$

$$P_\mu\gamma_5 \to i\sigma_{\mu\nu}q_\nu\gamma_5 \tag{8.336b}$$

$$iP_\mu(\gamma q)\gamma_5 \to -2iM_N\sigma_{\mu\nu}q_\nu\gamma_5 \tag{8.336c}$$

$$iq_\mu(\gamma P)\gamma_5 \to 0 \tag{8.336d}$$

$$iP_\mu\sigma_{\lambda\nu}P_\lambda q_\nu\gamma_5 \to -i(4M_N^2+q^2)\sigma_{\mu\lambda}q_\lambda\gamma_5. \tag{8.336e}$$

If first-class axial currents are considered, off-mass shell effects introduce corrections of the order $1/M_N$ only, which are of the same order as the usually also neglected relativistic effects. On the other hand, if we look at the second class currents we find that the free nucleon results are drastically modified.

We now have five terms.‡ The simplest allowable but not uniquely specified way is now to take the two terms

$$\Gamma_\mu = i[-f_T\sigma_{\mu\nu}q_\nu\gamma_5 + if_T'P_\mu\gamma_5] \tag{8.337}$$

into account but neglect the other three. It can be shown (Delorme and Rho 1971) that the two terms of eqn (8.337) are the only ones which survive if the second class axial current is conserved, but even if this current is not conserved this is the simplest combination of covariants which can be chosen. In order to be able to make any predictions one is forced to restrict the consideration to the above form, otherwise things would be much too diffuse and complicated. For a further treatment we have to consider the non-relativistic one-body current which corresponds to the operator of eqn (8.337) sandwiched between the two nuclear Dirac

† The simplified notation (γP) is used for $\gamma_\nu P_\nu$.

‡ If the dependence of the F_i, G_i and H_i on p^2, p'^2 is neglected. Otherwise we would have even more terms.

states. It is obtained as (Delorme and Rho 1971)

$$S_0 = -(f_T + f'_T) \sum_n \{\boldsymbol{\sigma}\mathbf{q} t_\mp\}_n \tag{8.338}$$

$$\mathbf{S} = W_0 f_T \sum_n \{\boldsymbol{\sigma} t_\mp\}_n. \tag{8.339}$$

Here and in the following $\mathbf{q}$ has to be taken in lepton kinematics, i.e. $\mathbf{q} = -(\mathbf{p}_e + \mathbf{p}_\nu)$. S_0 and $\mathbf{S}$ denote the second class current part of the time and space components of the axial currents, respectively.

As shown in the foregoing two sections mesonic exchange effects are also of importance if nucleons are taken as bound. It can be expected that they are of the same order of magnitude as the off-mass shell effects just discussed. For our purpose it suffices to take into account parts (*b*) and (*d*) of Fig. 8.3. Then one obtains for the corresponding exchange contributions (Delorme and Rho 1971; Kubodera *et al.* 1973) the following result, which is, however, only valid for allowed transitions

$$S_0(1,2) = -\lambda_s \mathbf{q} \boldsymbol{\Sigma} \tag{8.340}$$

$$\mathbf{S}(1,2) = W_0 f'_T \sum_n \{\boldsymbol{\sigma} t_\mp\}_n + \lambda_s \boldsymbol{\Lambda} \tag{8.341}$$

$$\begin{aligned}
\boldsymbol{\Lambda} = \sum_{j<k} & \{\tfrac{1}{2}[t_\mp(j) + t_\mp(k)](\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_k) + \tfrac{1}{2}[t_\mp(j) - t_\mp(k)] \\
& \times (\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_k)(1 + 2P^\sigma_{jk})\} Y_0(x_\pi) + \tfrac{3}{2}\Big\{[t_\mp(j) + t_\mp(k)] T^{(+)}_{jk} \\
& + [t_\mp(j) - t_\mp(k)] T^{(-)}_{jk} (1 - P^\sigma_{jk}) + \frac{\mathrm{i}}{r^2_{jk}} [t_\mp(j) + t_\mp(k)] \\
& \times [(\boldsymbol{\sigma}_j \times \mathbf{r}_{jk})(\boldsymbol{\sigma}_k \mathbf{r}_{jk}) + (\boldsymbol{\sigma}_k \times \mathbf{r}_{jk})(\boldsymbol{\sigma}_j \mathbf{r}_{jk})]\Big\} Y_2(x_\pi) \\
& + \frac{3}{m_\pi r_{jk}} \mathrm{i}\{[t_\mp(j) - t_\mp(k)](\mathbf{p}_j(\boldsymbol{\sigma}_k \mathbf{r}_{jk}) + \mathbf{p}_k(\boldsymbol{\sigma}_k \mathbf{r}_{jk}) \\
& + \mathbf{p}_k(\boldsymbol{\sigma}_j \mathbf{r}_{jk})) + [t_\mp(j) + t_\mp(k)](\mathbf{p}_k(\boldsymbol{\sigma}_k \mathbf{r}_{jk}) - \mathbf{p}_j(\boldsymbol{\sigma}_j \mathbf{r}_{jk}))\} \\
& \times \left(1 + \frac{1}{x_\pi}\right) Y_0(x_\pi),
\end{aligned} \tag{8.342}$$

$$\begin{aligned}
\boldsymbol{\Sigma} = \frac{3M_N}{m_\pi} \sum_{j<k} & [t_\mp(k)\boldsymbol{\sigma}_k \mathbf{r}_{jk} + t_\mp(j)\boldsymbol{\sigma}_j \mathbf{r}_{jk}] - (\mathbf{r}_j + \mathbf{r}_k) \\
& \times \frac{1}{r_{jk}} [t_\mp(j)\boldsymbol{\sigma}_j \mathbf{r}_{jk} - t_\mp(k)\boldsymbol{\sigma}_k \mathbf{r}_{jk}] \left(1 + \frac{1}{x_\pi}\right) Y_0(x_\pi).
\end{aligned} \tag{8.343}$$

For the notations see eqns (8.139), (8.321), (8.322), (8.325), (8.326), (8.327) and (8.329).

Quantities not used before are†

$$P^{\sigma}_{jk} = \tfrac{1}{2}(1 + \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k) \tag{8.344}$$

$$\lambda_s = \frac{m_\pi^3 g_r^2}{24\pi M_N^2}\left(f'_T - \frac{g_{\omega NN} F_\omega}{g_r m_\omega^2}\right) \tag{8.345}$$

with m_ω equal to the mass of the ω meson ($\sim$784 MeV), $g_{\omega NN}$ to the ω nuclear coupling constant ($g_{\omega NN} \sim 6{\cdot}82$) and F_ω to the $\omega \to \pi e \nu$ lifetime.

As far as allowed transitions are concerned the important axial form factor coefficients, which are the sensitive ones for induced tensor interactions, i.e. second class currents, are ${}^{A}F^{(0)}_{101}$ and ${}^{A}F^{(0)}_{110}$ (see Section 14.2.1.2, especially eqns (14.95) and (14.100)).

By combining eqns (8.339) and (8.341) and by making use of the methods discussed in Section 8.1.2 we then can easily derive that

$${}^{A}F^{(0)}_{101} = \mp\lambda\, {}^{A}\mathfrak{M}^{(0)}_{101} + \frac{1}{R}\{\zeta(W_0 R \pm \tfrac{6}{5}\alpha Z) + \lambda_s J R\}\, {}^{A}\mathfrak{M}^{(0)}_{101} \tag{8.346}$$

(upper sign β^--decay, lower sign β^+-decay) whereby J is the matrix element ratio

$$J = \frac{\langle f \| \mathbf{\Lambda} \| i \rangle}{\langle f \| \sum_n \{t_\mp \boldsymbol{\sigma}\}_n \| i \rangle}. \tag{8.347}$$

Both matrix elements are reduced with respect to both spin and isospin. Here, the Coulomb interaction has been additionally taken into account (term of $6\alpha Z/5R$).

From eqns (8.338) and (8.340) we obtain similarly

$${}^{A}F^{(0)}_{110} = \pm\lambda\, {}^{A}\mathfrak{M}^{(0)}_{110} - \frac{\sqrt{3}}{R}(\zeta + \lambda_s L)\, {}^{A}\mathfrak{M}^{(0)}_{101} \tag{8.348}$$

whereby

$$L = \frac{\langle f \| \mathbf{\Sigma} \| i \rangle}{\langle f \| \sum_n \{t_\mp \boldsymbol{\sigma}\}_n \| i \rangle}. \tag{8.349}$$

The induced pseudoscalar term is neglected in the above equation. ζ is equal to the sum of f_T and f'_T

$$\zeta = f_T + f'_T. \tag{8.350}$$

One can summarize the above result‡ to the statement that in ${}^{A}F^{(0)}_{101}$ and ${}^{A}F^{(0)}_{110}$ we have to make the replacements (see for comparison eqns (8.263*b*) and (8.263*f*))

$${}^{A}F^{(0)}_{101} \qquad f_T \to \zeta + \frac{\lambda_s J R}{W_0 R \pm \frac{6}{5}\alpha Z} \tag{8.351}$$

$${}^{A}F^{(0)}_{110} \qquad f_T \to \zeta + \lambda_s L \tag{8.352}$$

† λ_s in our notation corresponds to the λ in the notation of Kubodera *et al.* (1973).
‡ For L, J and ζ we have adopted the notation of Kubodera *et al.* (1973).

if off-mass shell and exchange effects are additionally included. That means we are now dealing with effective coupling constants which are transition and phenomenon dependent and not with a fundamental constant as before in the case of f_T. The two matrix element ratios involved, J and L, have then to be calculated.

Notwithstanding of the complexity of the calculus, J and L have been evaluated for several decays in light nuclei for which reliable wave functions exist (Kubodera *et al.* 1973; Kubodera *et al.* 1977; Oka and Kubodera 1980).

Experiments in order to determine ζ and λ_s together with their corresponding results will be discussed later on in Section 14.2.3.

9

OTHER REPRESENTATIONS OF THE NUCLEAR CURRENT IN TERMS OF FORM FACTORS

9.1. Cartesian invariant form factors

In Section 6.2 the expansion of the nuclear current in terms of multipole form factors used in this book has been described in detail. Particularly in the elementary particle theory (EPT), however, the so-called cartesian approach is very often chosen. This latter method was, on the other hand, first extended to beta-decay also by Kim and Primakoff (1965). A detailed discussion of that treatment is, for example, given in the article by Kim and Primakoff (1979). Therefore, in addition, we will give a sketch of this formalism in the following.

Let us start with the special example of a transition between two nuclei with $J_i^\pi = J_f^\pi = 0^+$ or $J_i^\pi = J_f^\pi = 0^-$. The vector current matrix element

$$\langle 0^\pm | \, V_\mu(0) \, | 0^\pm \rangle \tag{9.1}$$

must then be a Lorentz four vector. For the axial vector current matrix element we would have in this case

$$\langle 0^\pm | \, A_\mu(0) \, | 0^\pm \rangle = 0. \tag{9.2}$$

Since the spins of the nuclei are zero the vector current matrix element can only be a function of the two four momenta p_i and p_f. There are two other independent four vectors that can be built up from these two momenta, i.e.

$$q = p_f - p_i \qquad P = p_f + p_i$$

and one Lorentz scalar† q^2.

Thus the most general expression in terms of invariant cartesian form factors can then simply be written as

$$\langle 0^\pm | \, V_\mu(0) \, | 0^\pm \rangle = F_V(q^2) P_\mu - F_S(q^2) q_\mu. \tag{9.3}$$

In our multipole expansion approach, on the other hand, we would have (see eqns (6.24) and (6.25))

$$\langle 0^\pm | \, V_0(0) \, | 0^\pm \rangle = {}^{\mathrm{V}}F_{000}(\mathbf{q}^2) \tag{9.4}$$

$$\langle 0^\pm | \, \mathbf{V}(0) \, | 0^\pm \rangle = -\frac{\mathbf{q}R}{3} {}^{\mathrm{V}}F_{011}(\mathbf{q}^2). \tag{9.5}$$

† q^2 and P^2 are both Lorentz scalars. They are, however, not independent since both are functions of Pq.

Since in the Breit system one has (see eqns (5.41) and (5.42))

$$\mathbf{P}=0 \qquad P_0=W_f+W_i\simeq 2M_A \tag{9.6}$$

and

$$q_0=-\Delta \tag{9.7}$$

we obtain the relation

$${}^{V}F_{000}(\mathbf{q}^2)=2M_A F_V(q^2)+\Delta F_S(q^2) \tag{9.8}$$

$${}^{V}F_{011}(\mathbf{q}^2)=\frac{3}{R}F_S(q^2). \tag{9.9}$$

In case the parity changes, i.e. for a transition $0^+\to 0^-$ or $0^-\to 0^+$, the above considerations remain valid with the exception that vector and axial vector terms have to be interchanged. That means

$$F_V\to F_A \quad \text{and} \quad F_S\to F_P \tag{9.10}$$

$${}^{V}F_{000}\to {}^{A}F_{000} \qquad {}^{V}F_{011}\to {}^{A}F_{011} \tag{9.11}$$

and

$$\langle 0^{\mp}|\, V_\mu(0)\,|0^{\pm}\rangle=0. \tag{9.12}$$

Another simple example is the one where the spins and parities of the initial and final nucleus are

$$J_i^{\pi}=J_f^{\pi}=\tfrac{1}{2}^{+} \quad \text{or} \quad J_i^{\pi}=J_f^{\pi}=\tfrac{1}{2}^{-}.$$

The expansion in invariant cartesian form factors takes here, as previously discussed (see eqns (8.236) and (8.237)), the well known form (for a derivation see, for instance, Bernstein 1968)

$$\langle \tfrac{1}{2}^{\pm}|\, V_\mu(0)\,|\tfrac{1}{2}^{\pm}\rangle = i\bar{\phi}_f(p_f)[F_V(q^2)\gamma_\mu - F_M(q^2)\sigma_{\mu\nu}q_\nu + iF_S(q^2)q_\mu]\phi_i(p_i) \tag{9.13}$$

$$\langle \tfrac{1}{2}^{\pm}|\, A_\mu(0)\,|\tfrac{1}{2}^{\pm}\rangle = i\bar{\phi}_f(p_f)[F_A(q^2)\gamma_\mu\gamma_5 - F_T(q^2)\sigma_{\mu\nu}\gamma_5 q_\nu + iF_P(q^2)\gamma_5 q_\mu]\phi_i(p_i). \tag{9.14}$$

$\phi_i(p_i)$ and $\phi_f(p_f)$ are spinor wave functions (see eqn (6.31)) and obey the free particle Dirac equation, i.e. $(i\gamma p+M_A)\phi=0$.

As shown in Section 6.2 (see eqns (6.30) to (6.44)) in detail we can, therefore, easily evaluate the above equations (9.13) and (9.14) more explicitly, and obtain

$$\langle \phi_f(p_f)|\, V_0(0)\,|\phi_i(p_i)\rangle = N(\chi^{M_f})^{+}\{F_V(q^2)+\frac{1}{4(2M_A)^2}(\mathbf{P}^2-\mathbf{q}^2)$$

$$\times F_V(q^2)-\frac{\mathrm{i}}{2(2M_A)^2}F_V(q^2)(\mathbf{P}\times\mathbf{q})\boldsymbol{\sigma}-\frac{1}{2M_A}F_M(q^2)\mathbf{q}^2$$

$$-\frac{\mathrm{i}}{2M_A}F_M(q^2)(\mathbf{P}\times\mathbf{q})\boldsymbol{\sigma}-F_S(q^2)q_0+\frac{1}{4(2M_A)^2}F_S(q^2)q_0(\mathbf{P}^2-\mathbf{q}^2)$$

$$-\frac{\mathrm{i}}{2}\frac{1}{(2M_A)^2}F_S(q^2)q_0(\mathbf{P}\times\mathbf{q})\boldsymbol{\sigma}\Big\}\chi^{M_i} \tag{9.15}$$

$$\langle\phi_f(p_f)|\,A_0(0)\,|\phi_i(p_i)\rangle = N(\chi^{M_f})^+\Big\{-\frac{1}{2M_A}F_A(q^2)(\mathbf{P}\boldsymbol{\sigma})$$

$$-\frac{q_0}{2M_A}F_P(q^2)(\mathbf{q}\boldsymbol{\sigma})-F_T(q^2)(\mathbf{q}\boldsymbol{\sigma})+\frac{1}{4}\frac{1}{(2M_A)^2}F_T(q^2)$$

$$\times[(\mathbf{P}\mathbf{q})(\boldsymbol{\sigma}\mathbf{P}+\boldsymbol{\sigma}\mathbf{q})-(\boldsymbol{\sigma}\mathbf{q})(\mathbf{P}^2+\mathbf{q}^2)]\Big\}\chi^{M_i} \tag{9.16}$$

$$\langle\phi_f(p_f)|\,\mathbf{V}(0)\,|\phi_i(p_i)\rangle = N(\chi^{M_f})^+\Big\{\frac{1}{2M_A}F_V(q^2)\mathbf{P}+\frac{\mathrm{i}}{2M_A}F_V(q^2)(\boldsymbol{\sigma}\times\mathbf{q})$$

$$+\mathrm{i}F_M(q^2)(\boldsymbol{\sigma}\times\mathbf{q})-\frac{1}{2M_A}F_M(q^2)q_0\mathbf{q}-\frac{\mathrm{i}}{4M_A}F_M(q^2)q_0(\boldsymbol{\sigma}\times\mathbf{P})$$

$$-F_S(q^2)\mathbf{q}+\frac{1}{4(2M_A)^2}F_S(q^2)\mathbf{q}(\mathbf{P}^2-\mathbf{q}^2)-\frac{\mathrm{i}}{2(2M_A)^2}F_S(q^2)\mathbf{q}$$

$$\times((\mathbf{P}\times\mathbf{q})\boldsymbol{\sigma})-\frac{\mathrm{i}}{2(2M_A)^2}F_M(q^2)\mathbf{P}((\mathbf{P}\times\mathbf{q})\boldsymbol{\sigma})-\frac{\mathrm{i}}{4(2M_A)^2}$$

$$\times F_M(q^2)(\mathbf{P}^2+\mathbf{q}^2)(\boldsymbol{\sigma}\times\mathbf{q})+\frac{\mathrm{i}}{2(2M_A)^2}F_M(q^2)(\mathbf{P}\mathbf{q})(\boldsymbol{\sigma}\times\mathbf{P})\Big\}\chi^{M_i} \tag{9.17}$$

$$\langle\phi_f(p_f)|\,\mathbf{A}(0)\,|\phi_i(p_i)\rangle = N(\chi^{M_f})^+\Big\{-F_A(q^2)\boldsymbol{\sigma}+\frac{1}{2(2M_A)^2}$$

$$\times F_A(q^2)\mathbf{P}^2\boldsymbol{\sigma}-\frac{1}{4(2M_A)^2}F_A(q^2)(\mathbf{P}^2+\mathbf{q}^2)\boldsymbol{\sigma}-\frac{i}{2(2M_A)^2}$$

$$\times F_A(q^2)(\mathbf{P}\times\mathbf{q})-\frac{1}{2(2M_A)^2}F_A(q^2)[(\boldsymbol{\sigma}\mathbf{P})\mathbf{P}-(\boldsymbol{\sigma}\mathbf{q})\mathbf{q}]$$

$$+\frac{1}{2M_A}F_T(q^2)[(\mathbf{P}\mathbf{p})\boldsymbol{\sigma}-\mathbf{q}(\boldsymbol{\sigma}\mathbf{P})]-F_T(q^2)q_0\boldsymbol{\sigma}$$

$$+\frac{1}{2(2M_A)^2}F_T(q^2)q_0\mathbf{P}^2\boldsymbol{\sigma}-\frac{1}{4(2M_A)^2}F_T(q^2)q_0(\mathbf{P}^2+\mathbf{q}^2)\boldsymbol{\sigma}$$

$$-\frac{\mathrm{i}}{2(2M_A)^2}F_T(q^2)q_0(\mathbf{P}\times\mathbf{q})-\frac{1}{2(2M_A)^2}F_T(q^2)q_0[(\boldsymbol{\sigma}\mathbf{P})\mathbf{P}$$

$$-(\boldsymbol{\sigma}\mathbf{q})\mathbf{q}]-\frac{1}{2M_A}F_P(q^2)(\boldsymbol{\sigma}\mathbf{q})\mathbf{q}\Big\}\chi^{M_i}. \tag{9.18}$$

In the above formulae we have always made the approximations $W_i + M_{A_i} \simeq 2M_A$, $W_f + M_{A_f} = 2M_A$ and $(W_i + M_{A_i})(W_f + M_{A_f}) = (2M_A)^2$. That means the formulae are exact only in the order

$$\frac{\mathbf{q}^2}{(2M_A)^2} \quad \text{or} \quad \frac{\mathbf{P}^2}{(2M_A)^2}. \quad \text{Here,} \quad M_A = \tfrac{1}{2}(M_{A_i} + M_{A_f})$$

(nuclear mass) and

$$q_0 = W_f - W_i \simeq -\Delta + \frac{\mathbf{Pq}}{2M_A}.$$

The χ^M are the simple two component spinors ($\chi^{+1/2} = \binom{1}{0}$, $\chi^{-1/2} = \binom{0}{1}$). The factor N is determined by the normalization of the free particle Dirac spinor wave functions. From eqn (6.31) it follows that

$$N = 1 - \frac{1}{4}\frac{\mathbf{P}^2 + \mathbf{q}^2}{(2M_A)^2}. \tag{9.19}$$

Since our multipole expansion is exact in the Breit system (see Section 5.2) only, the suitable system for a comparison of both expansions is therefore the Breit one. Then in the Breit system we have (since $\mathbf{P} = 0$)

$$\langle \phi_f(p_f) | \, V_0 \, | \phi_i(p_i) \rangle = (\chi^{M_f})^+ \left\{ F_V(q^2)\left(1 - \frac{\mathbf{q}^2}{2(2M_A)^2}\right) - \frac{1}{2M_A} F_M(q^2)\mathbf{q}^2 + F_S(q^2)\Delta \right\} \chi^{M_i} \tag{9.20}$$

$$\langle \phi_f(p_f) | \, A_0 \, | \phi_i(p_i) \rangle = (\chi^{M_f})^+ \left\{ -F_T(q^2)(\boldsymbol{\sigma}\mathbf{q}) + \frac{\Delta}{2M_A} \times F_P(q^2)\left(1 - \frac{\mathbf{q}^2}{4(2M_A)^2}\right)(\boldsymbol{\sigma}\mathbf{q}) \right\} \chi^{M_i} \tag{9.21}$$

$$\langle \phi_f(p_f) | \, \mathbf{V} \, | \phi_i(p_i) \rangle = (\chi^{M_f})^+ \left\{ \frac{\mathrm{i}}{2M_A} F_V(q^2)(\boldsymbol{\sigma} \times \mathbf{q}) + \mathrm{i}F_M(q^2) \times \left(1 - \frac{\mathbf{q}^2}{2(2M_A)^2}\right)(\boldsymbol{\sigma} \times \mathbf{q}) + \frac{1}{2M_A} F_M(q^2)\Delta\mathbf{q} - F_S(q^2)\mathbf{q} \right\} \chi^{M_i} \tag{9.22}$$

$$\langle \phi_f(p_f) | \, \mathbf{A} \, | \phi_i(p_i) \rangle = (\chi^{M_f})^+ \left\{ -F_A(q^2)\boldsymbol{\sigma} + \frac{1}{2(2M_A)^2} \times F_A(q^2)(\boldsymbol{\sigma}\mathbf{q})\mathbf{q} + F_T(q^2)\Delta\boldsymbol{\sigma} - \frac{1}{2(2M_A)^2} \times F_T(q^2)\Delta(\boldsymbol{\sigma}\mathbf{q})\mathbf{q} - \frac{1}{2M_A} F_P(q^2)(\boldsymbol{\sigma}\mathbf{q})\mathbf{q} \right\} \chi^{M_i}. \tag{9.23}$$

Terms with $(\mathbf{q}/2M_A)^n$ where $n > 2$ have been omitted.

In our multipole expansion (see eqns (6.24) and (6.25)), on the other hand, we have for the case $J_i^\pi = J_f^\pi = \frac{1}{2}^+$

$$\langle \tfrac{1}{2}^+ | \, V_0 \, | \tfrac{1}{2}^+ \rangle = {}^{\mathrm{V}}F_{000}(\mathbf{q}^2) \tag{9.24}$$

$$\langle \tfrac{1}{2}^+ | \, A_0 \, | \tfrac{1}{2}^+ \rangle = \frac{R}{3} \sum_M (-1)^{\frac{1}{2}-M_f} \sqrt{6} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -M_f & M & M_i \end{pmatrix} q_M^* \, {}^{\mathrm{A}}F_{110}(\mathbf{q}^2) \tag{9.25}$$

$$\langle \tfrac{1}{2}^+ | \, \mathbf{V} \, | \tfrac{1}{2}^+ \rangle = -\frac{\mathbf{q}R}{3} {}^{\mathrm{V}}F_{011}(\mathbf{q}^2) - \frac{R}{\sqrt{3}} \sum_M (-1)^{\frac{1}{2}-M_f} \times \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -M_f & M & M_i \end{pmatrix} \mathrm{i}(\mathbf{e} \times \mathbf{q})_M^* \, {}^{\mathrm{V}}F_{111}(\mathbf{q}^2) \tag{9.26}$$

$$\langle \tfrac{1}{2}^+ | \, \mathbf{A} \, | \tfrac{1}{2}^+ \rangle = \sum_M (-1)^{\frac{1}{2}-M_f} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -M_f & M & M_i \end{pmatrix} \times \left\{ {}^{\mathrm{A}}F_{101}(\mathbf{q}^2) \sqrt{2} \, \mathbf{e}_M^* - \frac{R^2}{5} [(\mathbf{qe}) \mathbf{q}_M^* - \tfrac{1}{3} \mathbf{q}^2 \mathbf{e}_M^*] \, {}^{\mathrm{A}}F_{121}(\mathbf{q}^2) \right\}. \tag{9.27}$$

$\mathbf{e}$ is a vector with the unit components e_1, e_2, e_3.

In eqns (9.20) to (9.23) the σ-operators (Pauli matrices) are the only ones which act on the free particle spinors χ^M. Since, however,

$$(\chi_f^{M_f})^+ \boldsymbol{\sigma} (\chi_i^{M_i}) = \sum_M (-1)^{\frac{1}{2}-M_f} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -M_f & M & M_i \end{pmatrix} \underbrace{\langle \tfrac{1}{2} | \, \boldsymbol{\sigma} \, | \tfrac{1}{2} \rangle}_{\sqrt{6}} \mathbf{e}_M^* \tag{9.28}$$

we can simplify these equations further and compare them directly term by term with eqns (9.24) to (9.27).

Then we obtain for the relations between our multipole and the invariant cartesian form factors in the case $J_i^\pi = J_f^\pi = \frac{1}{2}^+$

$${}^{\mathrm{V}}F_{000}(\mathbf{q}^2) = F_{\mathrm{V}}(q^2) + F_{\mathrm{S}}(q^2) \Delta \tag{9.29a}$$

$${}^{\mathrm{V}}F_{011}(\mathbf{q}^2) = \frac{3}{R} \left\{ F_{\mathrm{S}}(q^2) - \frac{\Delta}{2M_{\mathrm{A}}} F_{\mathrm{M}}(q^2) \right\} \tag{9.29b}$$

$${}^{\mathrm{V}}F_{111}(\mathbf{q}^2) = -\frac{3\sqrt{2}}{R} \left\{ \frac{F_{\mathrm{V}}(q^2)}{2M_{\mathrm{A}}} + F_{\mathrm{M}}(q^2) \right\} \tag{9.29c}$$

$${}^{\mathrm{A}}F_{110}(\mathbf{q}^2) = \frac{3}{R} \left\{ -F_{\mathrm{T}}(q^2) + \frac{\Delta}{2M_{\mathrm{A}}} F_{\mathrm{P}}(q^2) \right\} \tag{9.29d}$$

$${}^{\mathrm{A}}F_{101}(\mathbf{q}^2) = \sqrt{3} \left\{ -F_{\mathrm{A}}(q^2) + F_{\mathrm{T}}(q^2) \Delta - \frac{\mathbf{q}^2}{6M_{\mathrm{A}}} F_{\mathrm{P}}(q^2) \right\} \tag{9.29e}$$

$${}^{\mathrm{A}}F_{121}(\mathbf{q}^2) = \frac{15}{R^2} \sqrt{\left(\frac{2}{3}\right)} \frac{F_{\mathrm{P}}(q^2)}{2M_{\mathrm{A}}}. \tag{9.29f}$$

The terms $\Delta/2M_{\mathrm{A}}$ and $\mathbf{q}^2/8M_{\mathrm{A}}^2$ have always been neglected against 1.

As the next example we will consider a transition $J_i^\pi = 1^\pm$ to $J_f^\pi = 0^\pm$. In that case we get for the cartesian expansion (see, for example, Kim and Primakoff 1979; Delorme 1979)

$$\langle 0^\pm | \, V_\lambda \, | 1^\pm \rangle = -\frac{F_M(q^2)}{2M_A} \varepsilon_{\lambda\mu\nu\rho} \xi_\mu q_\nu P_\rho \tag{9.30}$$

$$\langle 0^\pm | \, A_\lambda \, | 1^\pm \rangle = F_A(q^2)\xi_\lambda + \frac{F_P(q^2)}{2M_A} q_\lambda (\xi q) + \frac{F_T(q^2)}{2M_A} P_\lambda (\xi q). \tag{9.31}$$

$\varepsilon_{\lambda\mu\nu\rho}$ is the completely antisymmetric tensor defined with $\varepsilon_{1234} = +1$ (Levi–Civita symbol). For details see the next section.

ξ_λ is the polarization four vector† of the spin-one nucleus in the initial state and given by (see, for example, Hagedorn 1963; Pilkuhn 1979)

$$\xi_\lambda = \left(\mathbf{e} + \frac{\mathbf{p}_i (\mathbf{e}\mathbf{p}_i)}{M_{A_i}(W_i + M_{A_i})}, \quad \mathrm{i} \frac{(\mathbf{e}\mathbf{p}_i)}{M_{A_i}} \right). \tag{9.32}$$

The eqns (9.30) and (9.31) can more explicitly be written as‡

$$\langle 0^\pm | \, V_0 \, | 1^\pm \rangle = \frac{F_M(q^2)}{2M_A} \boldsymbol{\xi}(\mathbf{q} \times \mathbf{P}) \tag{9.33}$$

$$\langle 0^\pm | \, A_0 \, | 1^\pm \rangle = F_A(q^2)\xi_0 + \frac{F_P(q^2)}{2M_A} (\boldsymbol{\xi}\mathbf{q} - \xi_0 q_0) q_0 + \frac{F_T(q^2)}{2M_A} (\boldsymbol{\xi}\mathbf{q} - \xi_0 q_0) P_0 \tag{9.34}$$

$$\langle 0^\pm | \, \mathbf{V} \, | 1^\pm \rangle = -\frac{F_M(q^2)}{2M_A} \{ \mathrm{i}(\boldsymbol{\xi} \times \mathbf{q}) P_0 - \mathrm{i}(\boldsymbol{\xi} \times \mathbf{P}) q_0 + \mathrm{i}(\mathbf{q} \times \mathbf{P}) \xi_0 \} \tag{9.35}$$

$$\langle 0^\pm | \, \mathbf{A} \, | 1^\pm \rangle = F_A(q^2)\boldsymbol{\xi} + \frac{F_P(q^2)}{2M_A} (\boldsymbol{\xi}\mathbf{q} - \xi_0 q_0)\mathbf{q} + \frac{F_T(q^2)}{2M_A} (\boldsymbol{\xi}\mathbf{q} - \xi_0 q_0)\mathbf{P}. \tag{9.36}$$

In the Breit system (see Section 5.2) the above formulae reduce to

$$\langle 0^\pm | \, V_0 \, | 1^\pm \rangle = 0 \tag{9.37}$$

† If the initial nucleus is in the rest system and polarized or aligned we have

$$\xi_\lambda = (\mathbf{e}, 0) \quad \text{with} \quad \mathbf{e} = \{ \tilde{a}_{-1} e_{-1}, \tilde{a}_0 e_0, \tilde{a}_{+1} e_{+1} \}$$

where the $\tilde{a}_{M_i}$ are the coefficients of the expansion into the pure states $\mathbf{e}_M$.

‡ $q_0 = W_f - W_i \simeq \left(\Delta - \frac{\mathbf{P}\mathbf{q}}{2M_A} \right)$

$P_0 = W_f + W_i \simeq 2M_A \left(1 + \frac{\mathbf{P}^2 + \mathbf{q}^2}{8M_A^2} \right)$.

$$\langle 0^{\pm}|\, A_0\,|1^{\pm}\rangle = -\frac{F_A(q^2)}{2M_A}(\mathbf{eq}) - \frac{F_P(q^2)}{2M_A}\Delta\left[\mathbf{eq}\left(1-\frac{\Delta}{2M_A}\right) + \frac{\mathbf{q}^2(\mathbf{eq})}{8M_A^2}\right] + F_T(q^2)\left[\mathbf{eq}\left(1-\frac{\Delta}{2M_A}\right)+\frac{\mathbf{q}^2(\mathbf{eq})}{8M_A^2}\right] \tag{9.38}$$

$$\langle 0^{\pm}|\, \mathbf{V}\,|1^{\pm}\rangle = -F_M(q^2)\left[\mathrm{i}(\mathbf{e}\times\mathbf{q}) + \mathrm{i}\frac{(\mathbf{q}\times\mathbf{e})(\mathbf{eq})}{8M_A^2}\right]\left(1+\frac{\mathbf{q}^2}{8M_A^2}\right) \tag{9.39}$$

$$\langle 0^{\pm}|\, \mathbf{A}\,|1^{\pm}\rangle = F_A(q^2)\left[\mathbf{e}+\frac{\mathbf{q}(\mathbf{eq})}{8M_A^2}\right] + \frac{F_P(q^2)}{2M_A}\mathbf{q} \times\left[(\mathbf{eq})\left(1-\frac{\Delta}{2M_A}\right)+\frac{\mathbf{q}^2(\mathbf{eq})}{8M_A^2}\right]. \tag{9.40}$$

Our multipole expansion (Stech–Schülke formalism) for a $1^{\pm}$ to $0^{\pm}$ transition leads, on the other hand, to the following results:

$$\langle 0^{\pm}|\, V_0\,|1^{\pm}\rangle = 0 \tag{9.41}$$

$$\langle 0^{\pm}|\, A_0\,|1^{\pm}\rangle = -\frac{R}{\sqrt{3}}\,{}^{\mathrm{A}}F_{110}(\mathbf{q}^2)(\mathbf{eq}) \tag{9.42}$$

$$\langle 0^{\pm}|\, \mathbf{V}\,|1^{\pm}\rangle = \frac{R}{\sqrt{6}}\,{}^{\mathrm{V}}F_{111}(\mathbf{q}^2)\mathrm{i}(\mathbf{e}\times\mathbf{q}) \tag{9.43}$$

$$\langle 0^{\pm}|\, \mathbf{A}\,|1^{\pm}\rangle = -{}^{\mathrm{A}}F_{101}(\mathbf{q}^2)\mathbf{e} + \frac{R^2}{5\sqrt{2}}\,{}^{\mathrm{A}}F_{121}(\mathbf{q}^2)\left[(\mathbf{qe})\mathbf{q}-\frac{\mathbf{q}^2}{3}\mathbf{e}\right]. \tag{9.44}$$

By comparing eqns (9.37) to (9.40) with eqns (9.41) to (9.44) we then obtain

$${}^{\mathrm{V}}F_{111}(\mathbf{q}^2) = -\frac{\sqrt{6}}{R}F_M(q^2) \tag{9.45}$$

$${}^{\mathrm{A}}F_{110}(\mathbf{q}^2) = \frac{\sqrt{3}}{R}\left\{-F_T(q^2)+\frac{\Delta}{2M_A}F_P(q^2)+\frac{F_A(q^2)}{2M_A}\right\} \tag{9.46}$$

$${}^{\mathrm{A}}F_{101}(\mathbf{q}^2) = -F_A(q^2)-\frac{\mathbf{q}^2}{6M_A}F_P(q^2) \tag{9.47}$$

$${}^{\mathrm{A}}F_{121}(\mathbf{q}^2) = \frac{5\sqrt{2}}{R^2}\frac{F_P(q^2)}{2M_A}. \tag{9.48}$$

The terms $\Delta/2M_A$ and $\mathbf{q}^2/8M_A^2$ have, as before, been neglected if they occur in expressions like

$$\left(1-\frac{\Delta}{2M_A}+\frac{\mathbf{q}^2}{8M_A^2}\right).$$

Thus the relations given above are exact only for $\mathbf{q}/M_A \ll 1$.

As we see from the previous discussion the cartesian expansion in invariant form factors is not too complicated for the special transitions treated. Unique relations of the cartesian form factors to multipole form factors can also easily be derived whereby those relations are then more or less simple expressions. This situation, however, will change if we go over to higher spins for the initial and final state nuclei. Then we have to carry out cumbersome calculations. In this case we have first to build up a suitable relativistic wave function basis. This basis can be obtained by parallel coupling of spin one wave functions or of spin one wave functions plus a Dirac spinor wave function (for half integer spins).

These wave functions have to obey not only the Dirac or Klein–Gordon equations but also the conditions of Rarita and Schwinger (1941).

The number of invariant cartesian form factors is then also increasing with $J_m = \min(J_i, J_f)$ (see Table 9.1).

The cartesian method will, therefore, for higher spins end up with complicated expressions and notations. For that reason we will not go into further details of that method but refer to the excellent discussion of that topic given by Delorme (1979). In that article further explicit relations between multipole and cartesian form factors can also be found.

Applications for beta-decay particularly have been described in the papers by Kim and Primakoff (1965), Armstrong and Kim (1972), Bottino and Ciochetti (1973), Kubodera and Kim (1973), Morita *et al.* (1976*a*), Hwang and Primakoff (1977), Goulard *et al.* (1977), Pascual (1978), Sato and Ohtsubo (1978), and Kim and Primakoff (1979).

As conclusion of this section we would state the main advantage of the cartesian method is its explicit covariance which might sometimes be useful for separating kinematic aspects or for a treatment of general properties of the hadron current. The main disadvantage of the cartesian method compared with the multipole method is its complexity, especially

TABLE 9.1 *Number of independent cartesian form factors (Delorme 1979)*

Spin	Parity change	Vector current	Axial vector current
Integer	$(-1)^{\Delta J}$	$4J_m+3(4J+2)$	$4J_m+1(4J)$
	$(-1)^{\Delta J+1}$	$4J_m+1(4J)$	$4J_m+3(4J+2)$
Half-integer	$(-1)^{\Delta J}$	$4J_m+2(4J+1)$	$4J_m+2(4J+1)$
	$(-1)^{\Delta J+1}$	$4J_m+2(4J+1)$	$4J_m+2(4J+1)$

($J_m = \min(J_i, J_f)$, in the case where the spins are equal, i.e. if $J_i = J_f = J$ the numbers are given in brackets.)

for higher spins of the nuclei taking part in the decay. It should in particular be noted that a unique and simple treatment of all types of beta-decay occurring in nature is not possible by applying the cartesian method of an invariant form factor expansion.

9.2. Holstein's method

In the last years another method of form factor decomposition which has been developed by Holstein (Holstein *et al.* 1972; Holstein 1974*a*, 1974*b*, 1977*a*, 1977*b*, 1977*c*; Holstein and Treiman 1977; Kleppinger *et al.* 1977; Firestone *et al.* 1978; Holstein 1979*a*, 1979b; for a review, see Holstein 1974*a*) has often been applied. This method lies somewhat in between the cartesian and the multipole expansion approach and is especially useful for a treatment of allowed beta-decays.

The advantage of this method is that under a Lorentz transformation the leading terms in the expansion of the T-matrix remain covariant on its own and have a direct relation to the cartesian nuclear form factors. In Holstein's method the treatment of the leading multipoles is carried out in the following way

$$T = \frac{G_\beta}{\sqrt{2}} \langle f | V_\mu + A_\mu | i \rangle L^\mu = \frac{G_\beta}{\sqrt{2}} \left\{ \delta_{J_i J_f} \delta_{M_i M_f} \left[a(q^2) \frac{PL}{2M_A} + e(q^2) \frac{qL}{2M_A} \right] + C(J_f \ 1 \ J_i; M_f \ k) \varepsilon_{ijk} \varepsilon_{ij\lambda\eta} \times \frac{1}{4M_A} [c(q^2) L^\lambda P^\eta - d(q^2) L^\lambda q^\eta] \right\}. \tag{9.49}$$

Here we have given the decomposition in the same form as listed by Holstein (1974). It should, however, be noted that the metric in Holstein's paper is different (Bjorken–Drell) from ours and that the four vectors P and q are in this case defined by $P = p_f + p_i$ and $q = p_i - p_f$ ($q_{\text{Holstein}} = -q_{\text{Behrens–Bühring}}$). M_A is the nuclear mass.

The ε_{ijk} and $\varepsilon_{ij\lambda\eta}$ are the Levi–Civita symbols, which are defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{even permutations of } 1, 2, 3 \\ 0 & \text{for 2 or more indices equal} \\ -1 & \text{odd permutations of } 1, 2, 3 \end{cases}$$

$$\varepsilon_{ij\lambda\eta} = \begin{cases} 1 & \text{even permutations of } 1, 2, 3, 4 \\ 0 & \text{for 2 or more indices equal} \\ -1 & \text{odd permutation of } 1, 2, 3, 4. \end{cases}$$

We have

$$\tfrac{1}{2} \varepsilon_{ijk} \varepsilon_{ij\lambda\eta} = -\{\delta_{\lambda 4} \delta_{k\eta} - \delta_{\eta 4} \delta_{\lambda k}\} \tag{9.50}$$

and therefore

$$T=\frac{G_\beta}{\sqrt{2}}\left\{\delta_{J_iJ_f}\delta_{M_iM_f}\left[a(q^2)\frac{P_0}{2M_A}+e(q^2)\frac{q_0}{2M_A}\right]L_0\right.$$
$$-\delta_{J_iJ_f}\delta_{M_iM_f}\left[a(q^2)\frac{\mathbf{P}}{2M_A}+e(q^2)\frac{\mathbf{q}}{2M_A}\right]\mathbf{L}$$
$$+C(J_f\ 1\ J_i;M_f\ k)\left(\left[c(q^2)\frac{P^4}{2M_A}-d(q^2)\frac{q^4}{2M_A}\right]L^k\right.$$
$$\left.\left.-\left[c(q^2)\frac{P^k}{2M_A}-d(q^2)\frac{q^k}{2M_A}\right]L^4\right)\right\}. \tag{9.51}$$

In Holstein's metric we have $L_0=L^4$, $V_0=V^4$ etc.

Going into the Breit system and taking into account that the sign of $\mathbf{q}$ changes in our definition we obtain in the expressions given above

$$\begin{aligned} &P_0=M_i+M_f=2M_A \qquad && \mathbf{P}=0\\ &q_0=\Delta && \mathbf{q}=-\{q^1,q^2,q^3\}. \end{aligned} \tag{9.52}$$

Then we get finally

$$T=\frac{G_\beta}{\sqrt{2}}\left\{\delta_{J_iJ_f}\delta_{M_iM_f}\left[a(q^2)+e(q^2)\frac{\Delta}{2M_A}\right]L_0\right.$$
$$-\delta_{J_iJ_f}\delta_{M_iM_f}e(q^2)\frac{\mathbf{q}}{2M_A}\mathbf{L}$$
$$+C(J_f\ 1\ J_i;M_f\ k)\left(\left[c(q^2)-d(q^2)\frac{\Delta}{2M_A}\right]L_k\right.$$
$$\left.\left.-d(q^2)\frac{\mathbf{q}}{2M_A}e_kL_0\right)\right\}. \tag{9.53}$$

In our method, however, the leading terms in the multipole expansion look like (see eqns. (6.24) and (6.25))

$$\langle f|\,V_0\,|i\rangle=\delta_{J_iJ_f}\delta_{M_iM_f}{}^{\mathrm{V}}F_{000}(\mathbf{q}^2) \tag{9.54a}$$

$$\langle f|\,\mathbf{V}\,|i\rangle=-\delta_{J_iJ_f}\delta_{M_iM_f}\frac{\mathbf{q}R}{3}{}^{\mathrm{V}}F_{011}(\mathbf{q}^2) \tag{9.54b}$$

$$\langle f|\,A_0\,|i\rangle=\sum_M(-1)^{J_f-M_f}\sqrt{(2J_i+1)}\begin{pmatrix}J_f & 1 & J_i\\ -M_f & M & M_i\end{pmatrix}$$
$$\times\frac{\mathbf{q}R}{\sqrt{3}}{}^{\mathrm{A}}F_{110}(\mathbf{q}^2)\mathbf{e}^*_M \tag{9.54c}$$

$$\langle f|\,\mathbf{A}\,|i\rangle=\sum_M(-1)^{J_f-M_f}\sqrt{(2J_i+1)}\begin{pmatrix}J_f & 1 & J_i\\ -M_f & M & M_i\end{pmatrix}{}^{\mathrm{A}}F_{101}(\mathbf{q}^2)\mathbf{e}^*_M. \tag{9.54d}$$

It is

$$\sqrt{(4\pi)}\overset{*}{Y}{}_1^M(\hat{q}) = \sqrt{3}\,\frac{\mathbf{q}}{|\mathbf{q}|}\,\mathbf{e}_M^* \tag{9.55}$$

and

$$\sqrt{(4\pi)}\overset{*}{Y}{}_{01}^0(\hat{q}) = -\frac{\mathbf{q}}{|\mathbf{q}|}. \tag{9.56}$$

For a one to one comparison of both methods, Holstein's and ours, we have to replace the $3j$-symbol in eqns (9.54c) and (9.54d) by Clebsch–Gordan coefficients, i.e.

$$(-1)^{J_f-M_f}\sqrt{(2J_i+1)}\begin{pmatrix} J_f & 1 & J_i \\ -M_f & M & M_i \end{pmatrix} = (-1)^{J_i-J_f+M} C(J_f\ 1\ J_i; M_f\ -M). \tag{9.57}$$

Then the expression in eqns (9.54) can be rewritten as

$$\begin{aligned} T = -\frac{G_\beta}{\sqrt{2}}\langle f|\, V_\mu + A_\mu\,|i\rangle\, L_\mu = \frac{G_\beta}{\sqrt{2}}\Big\{\delta_{J_iJ_f}\delta_{M_iM_f}\Big[{}^{\mathrm{V}}F_{000}(\mathbf{q}^2)L_0 \\ + \frac{R}{3}{}^{\mathrm{V}}F_{011}(\mathbf{q}^2)\mathbf{q}\mathbf{L}\Big] + (-1)^{J_i-J_f}C(J_f\ 1\ J_i; M_f\ k) \\ \times\Big[\frac{R}{\sqrt{3}}{}^{\mathrm{A}}F_{110}(\mathbf{q}^2)\mathbf{q}e_kL_0 - {}^{\mathrm{A}}F_{101}(\mathbf{q}^2)L_k\Big]\Big\} \end{aligned} \tag{9.58}$$

where $k = -M = M_i - M_f$.

As far as the factor $(-1)^{J_i-J_f}$ in eqn (9.58) is concerned, it came from a different definition of the reduced matrix elements in both cases. Holstein's definition

$$\langle J_fM_f|\, O_K^{M^*}\,|J_iM_i\rangle = C(J_f\ K\ J_i;\ M_f\ M)\langle J_f|\,|O_K|\,|J_i\rangle \tag{9.59}$$

differs from ours

$$\langle J_fM_f|\, O_K^M\,|J_iM_i\rangle = (-1)^{J_f-M_f}\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix}\langle J_f|\,|O_K|\,|J_i\rangle. \tag{9.60}$$

The factor $(-1)^{J_i-J_f}$ does not, however play any role since in the case of $J_i = J_f$ it is $+1$ and in the case $J_i \neq J_f$, where it is -1, the terms with ${}^{\mathrm{V}}F_{000}(\mathbf{q}^2)$ and ${}^{\mathrm{V}}F_{011}(\mathbf{q}^2)$ disappear. In the transition probability we always have T^+T (see eqn (5.54)), thus such a phase factor has no influence.

By a one to one comparison of eqns (9.53) and (9.58) we now obtain

$${}^{\mathrm{V}}F_{000}(\mathbf{q}^2) = a(q^2) + e(q^2)\frac{\Delta}{2M_{\mathrm{A}}} \tag{9.61a}$$

$${}^{\mathrm{V}}F_{011}(\mathbf{q}^2) = \frac{3}{2M_{\mathrm{A}}R}\,e(q^2) \tag{9.61b}$$

$$^{A}F_{101}(\mathbf{q}^2) = -\left\{c(q^2) - d(q^2)\frac{\Delta}{2M_A}\right\} \tag{9.61c}$$

$$^{A}F_{110}(\mathbf{q}^2) = -\frac{\sqrt{3}}{2M_A R}\, d(q^2). \tag{9.61d}$$

Writing now down the *higher order* terms in Holstein's form factor expansion method we have

$$\begin{aligned} T = \frac{G_\beta}{\sqrt{2}} \Big\{ &\ldots + \mathrm{i}\frac{b(q^2)}{2M_A} C(J_f\ 1\ J_i; M_f\ k)(\mathbf{q}\times\mathbf{L})_k \\ &+ C(J_f\ 2\ J_i; M_f\ k)\Big[\frac{f(q^2)}{2M_A} C(1\,1\,2; n\ n')L_n q_{n'} \\ &+ \frac{g(q^2)}{(2M_A)^3}(PL)\sqrt{\left(\frac{4\pi}{5}\right)} Y_2^k(\hat{q})\mathbf{q}^2\Big] + C(J_f\ 1\ J_i; M_f\ k)\varepsilon_{ijk} \\ &\times \varepsilon_{ij\lambda\eta}\frac{h(q^2)}{2(2M_A)^3} q^\lambda P^\eta(qL) + C(J_f\ 2\ J_i; M_f\ k) \\ &\times C(1\,2\,2; n\ n')L_n\sqrt{\left(\frac{4\pi}{5}\right)} Y_2^{n'}(\hat{q})\frac{\mathbf{q}^2}{(2M_A)^2} j_2(q^2) \\ &+ C(J_f\,3\,J_i; M_f\ k)C(1\ 2\ 3; n\ n')L_n\sqrt{\left(\frac{4\pi}{5}\right)} \\ &\times Y_2^{n'}(\hat{q})\frac{\mathbf{q}^2}{(2M_A)^2} j_3(q^2) + \ldots\Big\}. \end{aligned} \tag{9.62}$$

We see immediately that up to some trivial factors the higher multipole expansion coefficients are directly related to each other for both methods (with the exception of the pseudoscalar term connected with $h(q^2)$).

A complete comparision between the form factors used in the present book (Stech–Schülke method) and those introduced by Holstein for allowed transitions is now presented in the following (see also Behrens *et al.* 1978):†

$$^{V}F_{000}(\mathbf{q}^2) = a(q^2) + \frac{\Delta}{2M_A} e(q^2) \tag{9.63a}$$

† If we make the comparison of both methods not in the Breit reference system, but in the lab-system we would obtain slightly different results for $^{V}F_{011}(\mathbf{q}^2)$ and $^{A}F_{110}(\mathbf{q}^2)$, namely

$$^{V}F_{011}(\mathbf{q}^2) = -\frac{3}{2M_A R}[a(q^2) - e(q^2)]$$

$$^{A}F_{110}(\mathbf{q}^2) = -\frac{\sqrt{3}}{2M_A R}\left[c(q^2) + d(q^2) + \frac{\Delta}{2M_A} h(q^2)\right].$$

This result is quoted in Behrens *et al.* (1978). The difference between both methods is, however, negligible.

$$^{V}F_{011}(\mathbf{q}^2) = \frac{3}{2M_A R}\, e(q^2) \tag{9.63b}$$

$$^{A}F_{101}(\mathbf{q}^2) = -\left[c(q^2) - \frac{\Delta}{2M_A}\, d(q^2) - \tfrac{1}{3}h(q^2)\, \frac{\mathbf{q}^2}{(2M_A)^2}\right] \tag{9.63c}$$

$$^{V}F_{111}(\mathbf{q}^2) = -\sqrt{\left(\frac{3}{2}\right)}\, \frac{b(q^2)}{M_A R} \tag{9.63d}$$

$$^{A}F_{110}(\mathbf{q}^2) = -\frac{\sqrt{3}}{2M_A R}\left[d(q^2) + \frac{\Delta}{2M_A}\, h(q^2)\right] \tag{9.63e}$$

$$^{A}F_{121}(\mathbf{q}^2) = -\frac{5\sqrt{2}}{4}\, \frac{h(q^2)}{(M_A R)^2} \tag{9.63f}$$

$$^{V}F_{211}(\mathbf{q}^2) = \frac{\sqrt{3}}{2}\, \frac{f(q^2)}{M_A R} \tag{9.63g}$$

$$^{V}F_{220}(\mathbf{q}^2) = \frac{3\sqrt{5}}{4}\, \frac{g(q^2)}{(M_A R)^2} \tag{9.63h}$$

$$^{A}F_{221}(\mathbf{q}^2) = -\frac{3\sqrt{5}}{4}\, \frac{j_2(q^2)}{(M_A R)^2} \tag{9.63i}$$

$$^{A}F_{321}(\mathbf{q}^2) = -\frac{3\sqrt{5}}{4}\, \frac{j_3(q^2)}{(M_A R)^2}\,. \tag{9.63j}$$

Since $\Delta/2M_A \ll 1$ for nuclear beta-decay terms of this order have been neglected against 1 in the above formulae.

The axial vector form factors are the ones for β^--decay which change the sign for β^+-decay. In the method (Stech–Schülke formalism) used in our book the expansion into form factor coefficients is given by (see eqns (6.74) and (6.159)–(6.166))

$$F_{KLs}(\mathbf{q}^2) = F^{(0)}_{KLs} - \frac{(\mathbf{q}R)^2}{2(2L+3)}\, F^{(1)}_{KLs} + \ldots \tag{9.64}$$

where $\mathbf{q}^2$ stands for the square of the transferred three momentum. The analogous expansion of form factors in Holstein approach has the form

$$c(q^2) = c_1 + c_2\left(\frac{q}{M_A}\right)^2 + \ldots \tag{9.65}$$

and here $q^2 = \Delta^2 - \mathbf{q}^2$ is the square of the four momentum transferred.

If the above relations are inserted into the formulae for the observables (see Chapter 7) we find agreement between the two different approaches as far as the main terms are concerned. Very slight differences between the Holstein expression and our expression exist, however, in the order

$O(q/M_A)$. These small differences have their origin in the fact that Holstein has included an additional term† in the phase space factor which takes into account the kinetic recoil energy.

The advantage of Holstein's expansion, which is in a certain sense specialized to allowed transitions, is that it can easily be generalized into the cartesian method, at least as far as the leading terms are concerned. For the latter terms inherent symmetries can then clearly be shown up. Otherwise there are, however, only small differences to the Stech–Schülke multipole treatment.

9.3. Other multipole expansions

Since multipole expansions are frequently used in nuclear physics, other pure multipole expansion formalisms have been also developed for beta-decay where the expansion coefficients are more or less directly related to those utilized in our book.

For the convenience of the reader we will, however, also give these relations although they consist of trivial factors in most cases.

The first other formalism which should be mentioned in this context is that of Delorme (1979) to which we, by the way, refer to in many other places in this book. In the Breit reference system ($\mathbf{P}=0$) Delorme's expansion for the nuclear current reads as

$$\langle J_f M_f|\, V_0(0)+A_0(0)\,|J_i M_i\rangle = -\sum_{KM}\frac{(-1)^{K+M_f-M_i}}{\sqrt{(2J_f+1)}}\,C(J_i\ K\ J_f;\ M_i\ M)\times\sqrt{(4\pi)}\ Y_K^M(\hat{q})\,a^{(K)}(\mathbf{q}^2) \quad (9.66)$$

$$\langle J_f M_f|\, \mathbf{V}(0)+\mathbf{A}(0)\,|J_i M_i\rangle = \sum_{t=0,\pm1}(e_t)^+\sum_{KLM}\frac{(-1)^{L+M_f-M_i-t}}{\sqrt{(2J_f+1)}}\,C(J_i\ K\ J_f;\ M_i\ M)\times C(L\ 1\ K;\ M-t\ t)\sqrt{(4\pi)}\ Y_L^{-M+t}(\hat{q})\,a^{(L,K)}(\mathbf{q}^2). \quad (9.67)$$

After some simple transformations and a one to one comparison‡ with our expansion (see eqns (6.24), (6.25) and (6.28)) we obtain the following relations between the form factors $a^{(K)}(\mathbf{q}^2)$ or $a^{(L,K)}(\mathbf{q}^2)$ and our form

† The phase space factor of Holstein has the form

$$dR=\left(1+\frac{3W_e-W_0-3\mathbf{p}_e\hat{\mathbf{p}}_\nu}{M_A}\right)(W_0-W_e)^2 p_e W_e\,dW_e\,d\Omega_e\,d\Omega_\nu.$$

The difference to our dR lies in the first factor (for comparison see eqns (5.60) to (5.67) or eqn (7.21)).

‡ Note that

$$\mathbf{q}_{\text{Behrens-Bühring}}=-\mathbf{q}_{\text{Delorme}}$$

This fact has been taken into account in eqns (9.66) and (9.67).

factors $F_{KLs}(\mathbf{q}^2)$

$$F_{KK0}(\mathbf{q}^2) = (-1)^{K-1} \frac{(2K+1)!!}{(qR)^K} \frac{1}{\sqrt{(2J_i+1)}} a^{(K)}(\mathbf{q}^2) \tag{9.68}$$

$$F_{KL1}(\mathbf{q}^2) = (-1)^{L} \frac{(2L+1)!!}{(qR)^L} \frac{1}{\sqrt{(2J_i+1)}} a^{(L,K)}(\mathbf{q}^2). \tag{9.69}$$

The formulae given in Delorme's formalism can therefore very easily be converted into our formalism.

Another different multipole analysis which has its basis in the treatment of electromagnetic transitions like, for instance, electron scattering, has been extensively discussed for beta-decay and applied to many practical problems by Walecka *et al.* (see de Forest and Walecka 1966; Donnelly and Walecka 1972, 1973, 1975; O'Connell *et al.* 1972; Serot 1978; Donnelly and Peccei 1979). Walecka's multipole expansion can be written as (see, for example, Walecka 1975)

$$\langle J_f M_f | \, V_0 + A_0 \, | J_i M_i \rangle = \sum_K \sqrt{\{4\pi(2K+1)\}} (-1)^{J_f - M_f} \begin{pmatrix} J_f & K & J_i \\ -M_f & 0 & M_i \end{pmatrix} \langle J_f | \, | i^K \hat{M}_K | \, | J_i \rangle \tag{9.70}$$

$$\begin{aligned} \langle J_f M_f | \, \mathbf{V} + \mathbf{A} \, | J_i M_i \rangle = {} & \sum_K \sqrt{\{4\pi(2K+1)\}} (-1)^{J_f - M_f} \\ & \times \begin{pmatrix} J_f & K & J_i \\ -M_f & 0 & M_i \end{pmatrix} \langle J_f | \, | i^K \hat{L}_K | \, | J_i \rangle \, \mathbf{e}_0 + \sum_{M=\pm 1} \sum_{K \geq 1} \sqrt{\{2\pi(2K+1)\}} \\ & \times (-1)^{J_f - M_f} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} [-M \langle J_f | \, | i^K \hat{J}_K^{\text{mag}} | \, | J_i \rangle \\ & + \langle J_f | \, | i^K \hat{J}_K^{\text{el}} | \, | J_i \rangle] \mathbf{e}_M^*. \end{aligned} \tag{9.71}$$

In the above expansion† the z-axis is taken parallel to $\mathbf{q}$, or in the spherical basis $\mathbf{e}_0$ along $\mathbf{q}$. In addition it is assumed the initial and final nuclei are very heavy and recoil is neglected in the transition matrix elements. We can, therefore, also neglect the difference between the Breit and laboratory reference system when we are comparing the above expansion with our multipole decomposition. For a term by term comparison we have to specify our multipole expansion to the case $\mathbf{q}$ parallel to

† Note that

$$\mathbf{q}_{\text{Walecka}} = -\mathbf{q}_{\text{Behrens-Bühring}}.$$

This has been taken into account in eqns (9.70) and (9.71).

the z-axis, i.e. to $\hat{q} = (\Theta_q, \phi_q) = (0, \phi_q)$. Then we have

$$\overset{*}{Y}{}^{M}_{K}(0, \phi_q) = \sqrt{\left(\frac{2K+1}{4\pi}\right)}\delta_{M0} \tag{9.72}$$

and

$$\overset{*}{\mathbf{Y}}{}^{M}_{KL}(0, \phi_q) = \sqrt{\left(\frac{2L+1}{4\pi}\right)} \sum_{M} C(L\ 1\ K;\ 0\ M)\overset{*}{\mathbf{e}}_{M}. \tag{9.73}$$

From eqns (6.24) and (6.25) we then obtain the following relations† between both formalisms

$$\langle J_f| \, |\mathrm{i}^K \hat{M}_K| \, |J_i\rangle = \sqrt{\left(\frac{2J_i+1}{4\pi}\right)} \frac{(qR)^K}{(2K+1)!!} F_{KK0}(\mathbf{q}^2) \tag{9.74}$$

$$\langle J_f| \, |\mathrm{i}^K \hat{L}_K| \, |J_i\rangle = -\sqrt{\left(\frac{2J_i+1}{4\pi}\right)}\left\{\frac{(qR)^{K-1}}{(2K-1)!!}\sqrt{\left(\frac{K}{2K+1}\right)}\right.$$
$$\left. \times F_{KK-11}(\mathbf{q}^2) - \frac{(qR)^{K+1}}{(2K+3)!!}\sqrt{\left(\frac{K+1}{2K+1}\right)}F_{KK+11}(\mathbf{q}^2)\right\} \tag{9.75}$$

$$\langle J_f| \, |\mathrm{i}^K \hat{J}^{\mathrm{mag}}_K| \, |J_i\rangle = \sqrt{\left(\frac{2J_i+1}{4\pi}\right)} \frac{(qR)^K}{(2K+1)!!} F_{KK1}(\mathbf{q}^2) \tag{9.76}$$

$$\langle J_f| \, |\mathrm{i}^K \hat{J}^{\mathrm{el}}_K| \, |J_i\rangle = -\sqrt{\left(\frac{2J_i+1}{4\pi}\right)}\left\{\frac{(qR)^{K-1}}{(2K-1)!!}\sqrt{\left(\frac{K+1}{2K+1}\right)}F_{KK-11}(\mathbf{q}^2)\right.$$
$$\left. + \frac{(qR)^{K+1}}{(2K+3)!!}\sqrt{\left(\frac{K}{2K+1}\right)}F_{KK+11}(\mathbf{q}^2)\right\}. \tag{9.77}$$

The factor i^K (or i^{K-1}) on the left side of the above relations is necessary if single particle wave functions on both sides are used which are multiplied by i^l or $\mathrm{i}^{l'}$, respectively, as we use.

†We have:
for $M = 0$

$$C(L\ 1\ K;\ 0\ M) = \begin{cases} \sqrt{\left(\frac{L+1}{2L+1}\right)} & \text{if } L = K-1 \\ 0 & \text{if } L = K \\ -\sqrt{\left(\frac{L}{2L+1}\right)} & \text{if } L = K+1 \end{cases}$$

for $M = \pm 1$

$$C(L\ 1\ K;\ 0\ M) = \begin{cases} \sqrt{\left\{\frac{L+2}{2(2L+1)}\right\}} & \text{if } L = K-1 \\ -M\sqrt{\frac{1}{2}} & \text{if } L = K \\ \sqrt{\left\{\frac{L-1}{2(2L+1)}\right\}} & \text{if } L = K+1. \end{cases}$$

The multipole operators of Walecka are defined by

$$\hat{M}_K^M(|\mathbf{q}|) = {}^{\mathrm{V}}\hat{M}_K^M + {}^{\mathrm{A}}\hat{M}_K^M = \int \mathrm{d}^3x [j_K(|\mathbf{q}|\, x)\, Y_K^M(\hat{x})][V_0(\mathbf{x}) + A_0(\mathbf{x})] \quad (9.78)$$

$$\hat{L}_K^M(|\mathbf{q}|) = {}^{\mathrm{V}}\hat{L}_K^M + {}^{\mathrm{A}}\hat{L}_K^M = \frac{\mathrm{i}}{|\mathbf{q}|} \int \mathrm{d}^3x [\boldsymbol{\nabla}(j_K(|\mathbf{q}|\, x)\, Y_K^M(\hat{x}))][\mathbf{V}(\mathbf{x}) + \mathbf{A}(\mathbf{x})] \quad (9.79)$$

$$\hat{T}_K^{M\,\mathrm{el}}(|\mathbf{q}|) = {}^{\mathrm{V}}\hat{T}_K^{M\,\mathrm{el}} + {}^{\mathrm{A}}\hat{T}_K^{M\,\mathrm{el}} = \frac{1}{|\mathbf{q}|} \int \mathrm{d}^3x \times [\boldsymbol{\nabla} \times j_K(|\mathbf{q}|\, x) \mathbf{Y}_{KK}^M(\hat{x})][\mathbf{V}(\mathbf{x}) + \mathbf{A}(\mathbf{x})] \quad (9.80)$$

$$\hat{T}_K^{M\,\mathrm{mag}}(|\mathbf{q}|) = {}^{\mathrm{V}}\hat{T}_K^{M\,\mathrm{mag}} + {}^{\mathrm{A}}\hat{T}_K^{M\,\mathrm{mag}} = \int \mathrm{d}^3x \times [j_K(|\mathbf{q}|\, x) \mathbf{Y}_{KK}^M(\hat{x})][\mathbf{V}(\mathbf{x}) + \mathbf{A}(\mathbf{x})]. \quad (9.81)$$

These are the charge (or Coulomb), longitudinal and transverse electric and magnetic multipoles of the nuclear current, respectively, which are well-known from the multipole expansion of the electromagnetic current. $V_0(\mathbf{x})$, $A_0(\mathbf{x})$, $\mathbf{V}(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ are the corresponding current density operators for the vector and axial vector current.

Corresponding single particle matrix elements (harmonic oscillator) have been extensively tabulated by Donnelly and Haxton (1979) for this formalism.

As we will see in the next chapter we have an analogy between beta-decay and some electromagnetic processes, as, for example, electron scattering or gamma-transitions. The relations given above are therefore also of importance in that context.

10

SYMMETRIES

10.1. Time reversal invariance

CLASSICALLY, the operation of time reversal is defined by (Rose and Brink 1967; Frauenfelder and Henley 1975)

$$t \to -t, \qquad \mathbf{r} \to \mathbf{r} \tag{10.1}$$

with the consequence that the classical momentum $\mathbf{p}$ and the angular momentum $\mathbf{J}$ change also their sign

$$\mathbf{p} \to -\mathbf{p}, \qquad \mathbf{J} \to -\mathbf{J}. \tag{10.2}$$

If the laws of motion for classical systems are invariant with respect to the time reversal operation we are speaking of time reversal invariance.

Quantum mechanically, the time reversal transformation can be expressed as

$$T\mathbf{x}_{\text{op}}T^{-1} = \mathbf{x}_{\text{op}} \tag{10.3a}$$

$$T\mathbf{p}_{\text{op}}T^{-1} = -\mathbf{p}_{\text{op}} \tag{10.3b}$$

$$T\mathbf{J}_{\text{op}}T^{-1} = -\mathbf{J}_{\text{op}} \tag{10.3c}$$

$\mathbf{x}_{\text{op}}$, $\mathbf{p}_{\text{op}}$ and $\mathbf{J}_{\text{op}}$ are here operators and T is the operator of the time reversal. Time reversal invariance then holds if

$$TH(\mathbf{r}, t)T^{-1} = H(\mathbf{r}, -t) \tag{10.4}$$

or for an Hamiltonian without explicit time dependence if

$$[T, H] = 0. \tag{10.5}$$

Nuclear beta-decay is therefore T-conserving if its Hamiltonian density $H_\beta(x)$ (see eqn (6.1)) transforms under the time reversal operation as follows

$$TH_\beta(\mathbf{r}, t)T^{-1} = H_\beta(\mathbf{r}, -t). \tag{10.6}$$

The Hamiltonian density in the current interaction version introduced earlier has the form (see eqn (6.1))

$$H_\beta(\mathbf{r}, t) = \frac{G_\beta}{\sqrt{2}}\{J_\mu^+(\mathbf{r}, t)L_\mu(\mathbf{r}, t) + \text{h.c.}\}, \tag{10.7}$$

As already extensively discussed in Section 6.2 we do not know the exact

form of the hadron current (for that reason we expanded the current into multipole form factors). The lepton current, however, has a well-known explicit form given by (see eqn (6.3))

$$L_\mu(\mathbf{r}, t) = \mathrm{i}\bar{\psi}_{\nu_e}(\mathbf{r}, t)\gamma_\mu(1+\gamma_5)\psi_e(\mathbf{r}, t). \tag{10.8}$$

For the Dirac field operators (see eqns (6.6) and (6.7)) the time reversal operation is defined such that

$$T\psi(\mathbf{r}, t)T^{-1} = t_u\psi(\mathbf{r}, -t) \tag{10.9a}$$

and

$$T\bar{\psi}(\mathbf{r}, t)T^{-1} = \bar{\psi}(\mathbf{r}, -t)t_u^+ \tag{10.9b}$$

(see, for example, Bjorken and Drell 1964, 1965; Bernstein 1968) where t_u has in our notation the specific form

$$t_u = -\gamma_1\gamma_3 = \mathrm{i}\sigma_2 \tag{10.10}$$

where

$$\sigma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

is here a 4×4 matrix.

Since we have for any operators O

$$\begin{aligned} T[\bar{\psi}_{\nu_e}(\mathbf{r}, t)O\psi_e(\mathbf{r}, t)]T^{-1} &= T\bar{\psi}_{\nu_e}(\mathbf{r}, t)T^{-1}TOT^{-1}T\psi_e(\mathbf{r}, t)T^{-1} \\ &= \bar{\psi}_{\nu_e}(\mathbf{r}, t)t_u^+O^*t_u\psi_e(\mathbf{r}, t) \end{aligned} \tag{10.11}$$

we obtain for the time reversal transformation of the lepton current†

$$T\mathbf{L}(\mathbf{r}, t)T^{-1} = -\mathbf{L}(\mathbf{r}, -t) \tag{10.12a}$$

$$TL_0(\mathbf{r}, t)T^{-1} = L_0(\mathbf{r}, -t). \tag{10.12b}$$

Thus because of condition (10.6) time reversal invariance requires for the hadron current $J_\mu^+ = V_\mu + A_\mu$ that

$$T\mathbf{V}(\mathbf{r}, t)T^{-1} = -\mathbf{V}(\mathbf{r}, -t) \tag{10.13a}$$

$$T\mathbf{A}(\mathbf{r}, t)T^{-1} = -\mathbf{A}(\mathbf{r}, -t) \tag{10.13b}$$

$$TV_0(\mathbf{r}, t)T^{-1} = V_0(\mathbf{r}, -t) \tag{10.13c}$$

$$TA_0(\mathbf{r}, t)T^{-1} = A_0(\mathbf{r}, -t). \tag{10.13d}$$

Before proceeding further we now have to first discuss how the T-operation transforms a wave function. The application of the T-operators on the free particle Dirac solutions

$$\phi(\mathbf{r}, t) = u^m(p)\mathrm{e}^{\mathrm{i}(\mathbf{pr}-Wt)} \tag{10.14}$$

† Since in this case $O = \mathrm{i}\gamma_\mu(1+\gamma_5)$ we have

$$t_u^+O^*t_u = -\mathrm{i}t_u^+\gamma_\mu^*(1+\gamma_5^*)t_u = -\mathrm{i}\gamma_\mu(1+\gamma_5).$$

gives, for example,

$$T\phi(\mathbf{r}, t) = t_u \phi^*(\mathbf{r}, -t) = t_u u^{m*}(p) e^{-i(\mathbf{pr}+Wt)}$$
$$= (-1)^{\frac{1}{2}+m} u^{-m}(-p) e^{i(-\mathbf{pr}-Wt)} \quad (10.15)$$

i.e. T is an antilinear or antiunitary operator.

The last equation† describes a particle with reversed momentum and reversed spin. Thus the action of the T-operator on a spin $\frac{1}{2}$ state vector can be written as

$$T|\mathbf{p}, m\rangle = (-1)^{\frac{1}{2}+m} |-\mathbf{p}, -m\rangle. \quad (10.16)$$

For a system of nucleons the above expression can be generalized to‡ (see Rose and Brink 1967)

$$T\phi(\mathbf{r}, t) = e^{i\pi J_2} \phi^*(\mathbf{r}, -t) \quad (10.17)$$

with

$$J_2 = \sum_{n=1}^{A} \{\sigma_2\}_n.$$

For a nucleus of spin J and momentum $\mathbf{p}$ the T-operation then transforms the state as (see also Jacob and Wick 1959)

$$T|a, \mathbf{p}, J, M\rangle = (-1)^{J+m} |a_T, -\mathbf{p}, J, -M\rangle. \quad (10.18)$$

The symbol a denotes all the other quantum numbers and a_T the corresponding time reversed ones.

The behaviour of our nuclear current $\langle f| V_\mu(0) + A_\mu(0) |i\rangle$ under a T-transformation is then given by§

$$(-1)^{J_f+J_i+M_f+M_i} \langle -\mathbf{p}_f, J_f, -M_f| T[V_\mu(0) + A_\mu(0)]T^{-1} |-\mathbf{p}_i, J_i, -M_i\rangle^*$$
$$= \langle \mathbf{p}_f, J_f, M_f| V_\mu(0) + A_\mu(0) |\mathbf{p}_i, J_i, M_i\rangle. \quad (10.19)$$

† Note that there is a difference in the T-transformation of field operators and wave functions (see Bjorken and Drell 1964, 1965).

‡ For one nucleon, i.e. for a single nucleon wave function we have

$$e^{i\pi J_2} = e^{i(\pi/2)\sigma_2} = i\sigma_2$$

and

$$T\phi(\mathbf{r}, t) = i\sigma_2 \phi^*(\mathbf{r}, -t).$$

The time reversal operator T applied on our bound single particle wave functions given in eqn (8.16) leads then to the result

$$Tg(r, \kappa)\chi_\kappa^\mu(\hat{r}) = (-1)^{j+\mu} g(r, \kappa)\chi_\kappa^{-\mu}(\hat{r}).$$

§ $\langle T\phi_f(\mathbf{r}, t)| T O T^{-1} |T\phi_i(\mathbf{r}, t)\rangle = \langle t_u \phi_f^*(\mathbf{r}, -t)| t_u O^* t_u^+ |t_u \phi_i^*(\mathbf{r}, -t)\rangle = \langle \phi_f(\mathbf{r}, -t)| O |\phi_i(\mathbf{r}, -t)\rangle^*$.

Thus we can derive the following T-invariance condition for the nuclear current.

$$\langle \mathbf{p}_f J_f M_f | \mathbf{V}(0) + \mathbf{A}(0) | \mathbf{p}_i J_i M_i \rangle = (-1)^{J_f+J_i+M_f+M_i+1} \times \langle -\mathbf{p}_f, J_f, -M_f | \mathbf{V}(0) + \mathbf{A}(0) | -\mathbf{p}_i, J_i, -M_i \rangle^* \quad (10.20)$$

$$\langle \mathbf{p}_f J_f M_f | V_0(0) + A_0(0) | \mathbf{p}_i J_i M_i \rangle = (-1)^{J_f+J_i+M_f+M_i} \times \langle -\mathbf{p}_f, J_f, -M_f | V_0(0) + A_0(0) | -\mathbf{p}_i, J_i, -M_i \rangle^*. \quad (10.21)$$

Expanding both sides of the above equations into our multipoles (see eqns (6.24) and (6.25)) we obtain†

$$\sum_{KM} (-1)^{J_f-M_f} \sqrt{(2J_i+1)} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \sqrt{(4\pi)} \overset{*}{\mathbf{Y}}{}^M_{KL}(\hat{q}) \times \frac{(qR)^L}{(2L+1)!!} F_{KL1}(q^2) = (-1)^{J_f+J_i+M_f+M_i+1} \sum_{KLM} (-1)^{J_f+M_f} \times \sqrt{(2J_i+1)} \begin{pmatrix} J_f & K & J_i \\ M_f & M & -M_i \end{pmatrix} \sqrt{(4\pi)} \times (-1)^L \mathbf{Y}^M_{KL}(\hat{q}) \frac{(qR)^L}{(2L+1)!!} F^*_{KL1}(q^2) \quad (10.22)$$

$$= \sum_{KLM} (-1)^{J_f-M_f} \sqrt{(2J_i+1)} \begin{pmatrix} J_f & K & J_i \\ -M_f & -M & M_i \end{pmatrix} \sqrt{(4\pi)} \overset{*}{\mathbf{Y}}{}^{-M}_{KL}(\hat{q}) \frac{(qR)^L}{(2L+1)!!} \overset{*}{F}_{KL1}(q^2),$$

i.e. T-invariance requires that

$$F_{KL1}(q^2) = \overset{*}{F}_{KL1}(q^2). \quad (10.23)$$

By inserting the multipole decomposition also in eqn (10.21) we get similarly

$$F_{KK0}(q^2) = \overset{*}{F}_{KK0}(q^2). \quad (10.24)$$

Thus, the form factors are purely real. The last two relations hold, of course, also for the decay of the free neutron. Because of the relations given in eqns (9.29) the coupling constants f_M, f_S, λ, f_T and f_P are then also purely real for the neutron decay.

Beyond that the latter statement remains true if we treat the decaying nucleons inside the nucleus in impulse approximation. This follows from eqns (8.263a–h) since the nuclear matrix elements are defined such that they are real.

† Note that

$$\mathbf{Y}^M_{KL}(\hat{q}) = (-1)^{K+L+M+1} \overset{*}{\mathbf{Y}}{}^{-M}_{KL}(\hat{q})$$

and

$$Y^M_L(\hat{q}) = (-1)^L Y^M_L(-\hat{q}).$$

On the contrary, if time reversal invariance does not hold we would have imaginary contributions to the form factors. Possible tests which make use of this fact have already been discussed in Section 7.2.4.2.

It should, however, be remarked that up to now no evidence for a violation of time reversal invariance has been found in beta-decay.

A general and detailed discussion of that topic can be found in the book by Frauenfelder and Henley (1975), in the articles by Richter (1975) and by Wilkinson (1978) and in a summary given by Calaprice (1978).

10.2. Conserved vector current (CVC) theory

Classically, in the electromagnetic interaction the charge is conserved, the reason being that we have a continuity equation of the type

$$\nabla \mathbf{j}(\mathbf{x}, t) + \frac{\partial \rho(\mathbf{x}, t)}{\partial t} = 0 \tag{10.25}$$

which follows from Maxwell's equations. ρ is here the classical charge density and $\mathbf{j}$ the classical current density.

The corresponding classical interaction energy is then given by

$$H = \int \mathrm{d}^3x (\rho\phi - \mathbf{jA}) \tag{10.26}$$

where ϕ is the electrostatic and $\mathbf{A}$ the vector potential.

Quantum mechanically, the electromagnetic interaction density for a nucleus with an electromagnetic field reads as

$$H_{\mathrm{em}}(x) = -eJ_{\mu}^{\mathrm{em}}(x)A_{\mu}(x) \tag{10.27}$$

where $J_{\mu}^{\mathrm{em}}(x)$ denotes the hadron† (or here nuclear) current.

Now, in the electromagnetic interaction we did not observe a renormalization of the coupling constant, i.e. of the charge e. That means the charge e of a strong interacting (finite size) proton is always exactly the same as that of an electron, or the charge of a nucleus is always exactly equal to the sum of the charges e of the individual protons. The reason for that fact is that the nuclear current obeys the conservation law (continuity equation)

$$\frac{\partial J_{\mu}^{\mathrm{em}}(x)}{\partial x_{\mu}} = \nabla \mathbf{J}^{\mathrm{em}}(\mathbf{x}, t) + \frac{\partial J_0^{\mathrm{em}}(\mathbf{x}, t)}{\partial t} = 0. \tag{10.28}$$

† For a Dirac proton we would have

$$J_{\mu}(x) = \mathrm{i}\bar{\psi}_{\mathrm{p}}(x)\gamma_{\mu}\psi_{\mathrm{p}}(x)$$

where the $\psi_{\mathrm{p}}(x)$ are the field operators for the proton.

To see how such a conserved current leads to an unrenormalized charge, let us consider

$$\int_V d^3x \frac{\partial J_0^{em}(\mathbf{x}, t)}{\partial t} = -\int_V d^3x \nabla \mathbf{J}^{em}(\mathbf{x}, t) = -\int_S d\sigma \mathbf{J}^{em}(\mathbf{x}, t). \quad (10.29)$$

Here, S is the surface of the volume V. Since the strong interactions have finite range the last integral disappears if S is chosen infinitely away from the nucleus. Therefore, we have

$$\frac{\partial}{\partial t} \int_V d^3x J_0(x) = 0, \quad (10.30)$$

i.e. also

$$\frac{\partial}{\partial t} \langle f | \int_V d^3x J_0(x) | i \rangle = \frac{\partial}{\partial t} \langle f | Q_{op} | i \rangle = \frac{\partial}{\partial t} Q = 0. \quad (10.31)$$

If at $t = -\infty$ a nucleus consisting of Z Dirac protons and N Dirac neutrons is introduced into the vacuum and then the strong interaction is switched on, the total charge $Q = Ze$ does not change because of the above relation. Thus the strong interactions do not at all modify the total charge even if they modify many other properties.

The electromagnetic interaction of a nucleus can be expanded in powers of the momentum transfer to the field. In second order, we have just four terms whose values at $q^2 = 0$ correspond to the charge, electric dipole moment, magnetic dipole moment, and electric quadrupole moment, i.e.

$$H_{em} = Ze\phi_0(0) - \mathbf{D}\mathbf{E}(0) - \mathbf{M}\mathbf{H}(0) - \tfrac{1}{6} \sum_{ij} Q_{ij} \frac{\partial E_j(0)}{\partial x_i} \quad (10.32)$$

(see Jackson 1962).

Here, $\mathbf{D}$ is the electric dipole moment, $\mathbf{M}$ is the magnetic dipole moment and Q_{ij} the electric quadrupole tensor. $\phi_0(0)$ is the potential of the electromagnetic field evaluated at the nucleus, $\mathbf{H}(0)$ the magnetic field and $\mathbf{E}(0)$ the electric field. In the static case ($q^2 = 0$) the electric dipole term is generally zero. The first term can now also be written as†

$$Ze\phi_0 = e(\tfrac{1}{2}A - T_3)\phi_0 \quad (10.33)$$

where the first term is the isoscalar and the second the isovector part of the interaction.

Thus it follows that

$$\langle f | \int d^3x J_0^{em}(x) | i \rangle \phi_0 = 2\pi^3 \delta^3(\mathbf{p}_f - \mathbf{p}_i) \langle f | J_0^{em}(0) | i \rangle \phi_0$$
$$= 2\pi^3 \delta^3(\mathbf{p}_f - \mathbf{p}_i) Z\phi_0. \quad (10.34)$$

† $T_3 = \dfrac{N-Z}{2}$, $A = N + Z$.

For $q^2 = 0$ ($\mathbf{p}_i = \mathbf{p}_f$) we have therefore

$$\langle f| J_0^{\mathrm{em}}(0) |i\rangle = Z \tag{10.35}$$

and separately for the isoscalar and isovector part;
for $q^2 = 0$

$$\langle J_f M_f| J_0^{\mathrm{em}}(0) |J_i M_i\rangle_{\mathrm{isoscalar}} = \tfrac{1}{2} A \delta_{J_i J_f} \delta_{M_i M_f}; \tag{10.36}$$

for $q^2 = 0$

$$\langle J_f M_f| J_0^{\mathrm{em}}(0) |J_i M_i\rangle_{\mathrm{isovector}} = -\langle T_f T_{3f}| T_3 |T_i T_{3i}\rangle \times \delta_{J_i J_f} \delta_{M_i M_f} = -T_3 \delta_{J_i J_f} \delta_{M_i M_f} \delta_{T_f T_i} \delta_{T_{3f} T_{3i}}. \tag{10.37}$$

The conserved vector current hypothesis (Feynman and Gell-Mann 1958; Gell-Mann 1958; Gershtein and Zeldovich 1956) now suggests that for the weak vector interactions a similar principle holds, i.e. also a conservation law

$$\frac{\partial V_\mu(\mathbf{x}, t)}{\partial x_\mu} = 0. \tag{10.38}$$

This assumption then, of course, leads to the same consequences as discussed before for the electromagnetic interaction. That means; for $q^2 = 0$

$$\langle J_f M_f| V_0(0) |J_i M_i\rangle = {}^{\mathrm{V}}F_{000}(q^2) \delta_{J_i J_f} \delta_{M_i M_f} = \langle T_f T_{3f}| T_\mp |T_i T_{3i}\rangle \delta_{J_i J_f} \delta_{M_i M_f}. \tag{10.39}$$

Therefore we obtain

$${}^{\mathrm{V}}F_{000}(0) = {}^{\mathrm{V}}F_{000}^{(0)} = \langle T_f T_{3f}| T_\mp |T_i T_{3i}\rangle = \sqrt{\{(T_i \pm T_{3i})(T_i \mp T_{3i} + 1)\}} \tag{10.40}$$

(T_- for β^--decay, T_+ for β^+-decay and electron capture).

Thus if the CVC hypothesis is true ${}^{\mathrm{V}}F_{000}^{(0)}$ can be evaluated in a completely rigorous way.

In the impulse approximation treatment, on the other hand, we had derived the following formula for ${}^{\mathrm{V}}F_{000}^{(0)}$:

$${}^{\mathrm{V}}F_{000}^{(0)} = {}^{\mathrm{V}}\mathfrak{M}_{000}^{(0)} \mp \frac{f_{\mathrm{S}}}{R} \{W_0 R \pm \tfrac{6}{5} \alpha Z\} {}^{\mathrm{V}}\mathfrak{M}_{000}^{(0)} \tag{10.41}$$

(see eqns (8.263a) and (8.266a)), and for ${}^{\mathrm{V}}\mathfrak{M}_{000}^{(0)}$

$${}^{\mathrm{V}}\mathfrak{M}_{000}^{(0)} = \langle T_f T_{3f}| T_\mp |T_i T_{3i}\rangle = \sqrt{\{(T_i \pm T_{3i})(T_i \mp T_{3i} + 1)\}} \tag{10.42}$$

(see eqn (8.140)).

By a comparision of both results we can, therefore, conclude that the validity of the CVC theory requires that on the one side

$$f_{\mathrm{S}} = 0 \tag{10.43}$$

and that on the other side no exchange currents contribute to the vector form factor coefficient ${}^{V}F^{(0)}_{000}$. This latter statement follows from the fact that the simple impulse approximation calculation above suffices to obtain the exact result. In cases where ${}^{V}F^{(0)}_{000}$ gives the main contribution to the beta-decay, as for example in transitions of the type $J_i^\pi = J_f^\pi = 0^+$, we have, therefore, no renormalization of the weak vector coupling constant.

Besides, CVC gives us also the possibility to derive relations between certain vector form factor coefficients. The way of seeing this is to go back to eqn (6.16) and to note that

$$\langle f| \frac{\partial V_\mu(x)}{\partial x_\mu} |i\rangle = \frac{\partial}{\partial x_\mu} e^{-i(p_{f_\mu} - p_{i_\mu})x_\mu} \langle f| V_\mu(0) |i\rangle = 0. \tag{10.44}$$

After carrying out the differentiation we get

$$\mathbf{q} \langle f| \mathbf{V}(0) |i\rangle + \Delta \langle f| V_0(0) |i\rangle = 0. \tag{10.45}$$

Expanding $\mathbf{V}(0)$ and $V_0(0)$ into our multipole form factors (see eqns (6.24) and (6.25)) this equation reads as

$$-\sum_L \mathbf{q} \overset{*}{\mathbf{Y}}{}^M_{KL}(\hat{q}) \frac{(qR)^L}{(2L+1)!!} {}^{V}F_{KL1}(q^2) = \Delta \overset{*}{Y}{}^M_K(\hat{q}) \frac{(qR)^K}{(2K+1)!!} {}^{V}F_{KK0}(q^2). \tag{10.46}$$

This expression can easily be simplified if we choose $\mathbf{q}$ to be in the z-direction. Then we obtain† (see also Stech and Schülke 1964; Schopper 1966)

$$-(2K+1)\sqrt{\left(\frac{K}{2K+1}\right)} {}^{V}F_{KK-11}(q^2) + \frac{(qR)^2}{2K+3} \sqrt{\left(\frac{K+1}{(2K+1)}\right)} \times {}^{V}F_{KK+11}(q^2) = \Delta R {}^{V}F_{KK0}(q^2) \tag{10.47}$$

(parity selection rule $\pi_i \pi_f = (-1)^K$).

Up to now we have, however, not taken into account that the nucleus is charged, but the conservation of the weak vector current in the form of eqn (10.38) is exact only when the electromagnetic interaction is absent.

† Note that

$$\overset{*}{Y}{}^M_K(0, \phi_q) = \sqrt{\left(\frac{2K+1}{4\pi}\right)} \delta_{M0}$$

$$\overset{*}{\mathbf{Y}}{}^M_{KK-1}(0, \phi_q) = \sqrt{\left(\frac{K}{4\pi}\right)} e_0 \delta_{M0}$$

$$\overset{*}{\mathbf{Y}}{}^M_{KK+1}(0, \phi_q) = -\sqrt{\left(\frac{K+1}{4\pi}\right)} e_0 \delta_{M0}$$

$$\mathbf{Y}^M_{KK}(0, \phi_q) = 0$$

(see also Section 9.3).

In the presence of the minimal electromagnetic interaction, eqn (10.38) has to be replaced by (for β^--decay)

$$\left(\frac{\partial}{\partial x_\mu} - ieA_\mu\right) V_\mu(\mathbf{x}, t) = 0. \tag{10.48}$$

Then eqn (10.45) becomes

$$\mathbf{q}\,\langle f|\,\mathbf{V}(0)\,|i\rangle + \Delta\,\langle f|\,V_0(0)\,|i\rangle - e\,\langle f|\,\phi V_0(0)\,|i\rangle = 0. \tag{10.49}$$

$A_\mu = (\mathbf{0}, i\phi)$ is the potential built up by the charged protons† (and other charge dependent effects).

Now we can, as before $\mathbf{V}(0)$ and $V_0(0)$, also expand the last term into multipoles and get analogous to eqn (6.24)

$$\langle f|\,e\phi V_0(0)\,|i\rangle = -\sum_{KM} (-1)^{J_f - M_f} \sqrt{(2J_i+1)} \begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i \end{pmatrix} \times \sqrt{(4\pi)}\, \overset{*}{Y}{}^M_K(\hat{q})\, \frac{(qR)^K}{(2K+1)!!}\, C_K(q^2). \tag{10.50}$$

Then the relation (10.47) has also to be replaced by (see Stech and Schülke 1964)

$$-(2K+1)\sqrt{\left(\frac{K}{2K+1}\right)}^{\mathrm{V}} F_{KK-11}(q^2) + \frac{(qR)^2}{2K+3}\sqrt{\left(\frac{K+1}{2K+1}\right)}^{\mathrm{V}} F_{KK+11}(q^2) = \Delta R^{\mathrm{V}} F_{KK0}(q^2) \pm R C_K(q^2) \tag{10.51}$$

(the upper sign applies for β^--decay, the lower for β^+-decay and electron capture).

Without an explicit calculation of the quantity $C_K(q^2)$ the above relation is not of much worth. We have, therefore, to say something about $C_K(q^2)$. Let us first give a model independent order of magnitude estimation of that term. For that purpose we consider the two beta-transitions from both the outer members of an isobaric triplet to the same isobaric singlet, i.e. the transitions $T_i = 1$, $T_{3i} = 1$ to $T_f = 0$, $T_{3f} = 0$ (β^--decay) and $T_i = 1$, $T_{3i} = -1$ to $T_f = 0$, $T_{3f} = 0$ (β^+-decay).

Since the masses of the $2T+1$ members of an isobaric multiplet are known from a mass equation of the form

$$M(A, T_3) = a(A, T) + b(A, T)T_3 + c(A, T)T_3^2 \tag{10.52}$$

(see Jänecke 1969) in our example the beta-transition energy is determined to be

$$\Delta = a(A,1) - a(A,0) + c(A,1) \pm b(A,1) = \Delta' \mp [\mp c(A,1) - b(A,1)] = \Delta' \mp \Delta'' \tag{10.53}$$

(upper sign β^--decay, lower sign β^+-decay).

† Exactly, it is the potential difference between initial and final state.

The Coulomb (and other charge dependent) interaction part Δ'' is essentially determined by $b(A, T)$ and $c(A, T)$ which reads as

$$\begin{aligned} b(A, T) &= (M_N - M_p) - E_C^{(1)}(A, T) \\ c(A, T) &= 3E_C^{(2)}(A, T) \end{aligned} \tag{10.54}$$

where $E_C^{(1)}$ ($E_C^{(2)}$) is the vector (tensor) Coulomb energy. In our case $\Delta E_C = \Delta'' + (M_n - M_p)$ is equal to the Coulomb displacement energy. For the simple model of the homogeneous charged sphere we would have

$$\Delta'' = -(M_n - M_p) + \frac{6}{5}\frac{\alpha Z}{R}. \tag{10.55}$$

$M_n - M_p$ is the neutron–proton mass difference.

Because of isospin impurity effects the form factors $F_{KLs}(q^2)$ also differ for the β^--decay and β^+-decay in our example, i.e.

$$F_{KLs}(q^2) = F'_{KLs}(q^2)(1 \pm \delta). \tag{10.56}$$

δ however is very small and of second order in the isospin mass splitting, i.e. of order $(\alpha Z)^2$. This is a consequence of the Behrends Sirlin Ademollo Gatto theorem which says that corrections to the nuclear vector current matrix element $\langle f| V_\lambda(0) |i\rangle$ have the form $c(\delta M)^2$ if δM represent the multiplet mass splittings (Behrends and Sirlin 1960; Ademollo and Gatto 1964).

If we, for the moment, neglect the small effects of the δ we require that the right side of eqn (10.51) is equal for our β^-- and β^+-decay. That means

$$\begin{aligned} \Delta\, {}^{V}F_{KK0}(q^2) \pm C_K(q^2) = \Delta'\, {}^{V}F_{KK0}(q^2) \mp \Delta''\, {}^{V}F_{KK0}(q^2) \pm C_K(q^2) \\ = \Delta'\, {}^{V}F_{KK0}(q^2), \end{aligned} \tag{10.57}$$

i.e.

$$RC_K(q^2) = \Delta'' R\, {}^{V}F_{KK0}(q^2) = \left\{-(M_n - M_p) + \frac{6}{5}\frac{\alpha Z}{R}\right\} RF_{KK0}(q^2). \tag{10.58}$$

Since the above result does not contain any specific quantum numbers connected with our special example we can generalize it and obtain finally ($\Delta = W_0$) (see also Schopper 1966)

$$\begin{aligned} -(2K+1)\sqrt{\left(\frac{K}{2K+1}\right)}\, {}^{V}F_{KK-11}(q^2) + \frac{(qR)^2}{2K+3}\sqrt{\left(\frac{K+1}{2K+1}\right)}\, {}^{V}F_{KK+1}(q^2) \\ = \left(W_0 \mp (M_n - M_p) \pm \frac{6}{5}\frac{\alpha Z}{R}\right) R\, {}^{V}F_{KK0}(q^2) \end{aligned} \tag{10.59}$$

(upper sign β^--decay, lower sign β^+-decay and electron capture). A second possibility often applied to calculate $C_K(q^2)$ is the application of

the impulse approximation treatment. For that purpose we first approximate $e\phi$ by

$$-e\phi = V(r) = \frac{\alpha Z}{R} U(r) - (M_n - M_p), \tag{10.60}$$

i.e. by an average static potential difference acting on the nucleons. Then we obtain for $C_K(q^2)$ using the same methods as discussed in Sections 8.1.1.1. and 8.1.2:

$$C_K(q^2) = \left\{ -(M_n - M_p)\, {}^{V}F_{KK0}(q^2) + \frac{\alpha Z}{R} \frac{(2K+1)!!}{(qR)^K} \times \left\{ \int \left(\frac{r}{R}\right)^K U(r) j_K(qr) T_{KK0} j_0^V \right\} \right\} \tag{10.61}$$

(see, for example, Blin-Stoyle and Nair 1966; Behrens and Bühring 1971). Here use has been made of the more symbolic notation given in eqns (8.15*a* and *b*). The time component of the vector current j_0^V has to be taken from eqn (8.259*a*) (with $f_S = 0$) and $T_{KK0}(\hat{r})$ from eqns (6.27*a*) or (8.10*a*), respectively.

For practical applications some special cases of the form factor CVC relation are important. If $K = 0$, then only one term survives and eqn (10.51) becomes

$$\frac{(qR)^2}{3} {}^{V}F_{011}(q^2) = W_0 R\, {}^{V}F_{000}(q^2) \pm R C_0(q^2). \tag{10.62}$$

Besides, eqn (10.51) can also be used to derive similar relations for the form factor coefficients.

By inserting eqn (6.74) one obtains (see also Behrens and Bühring 1971)

$$-(2K+1+2N)\sqrt{\left(\frac{K}{2K+1}\right)} {}^{V}F^{(N)}_{KK-11} - 2N\sqrt{\left(\frac{K+1}{2K+1}\right)} {}^{V}F^{(N-1)}_{KK+11} = W_0 R\, {}^{V}F^{(N)}_{KK0} \pm R C^{(N)}_K \tag{10.63}$$

with either

$$C^{(N)}_K = \left\{ -(M_n - M_p) + \frac{6}{5}\frac{\alpha Z}{R} \right\} {}^{V}F^{(N)}_{KK0} \tag{10.64}$$

or

$$C^{(N)}_K = \left\{ -(M_n - M_p)\, {}^{V}F^{(N)}_{KK0} + \frac{\alpha Z}{R} \left(\int \left(\frac{r}{R}\right)^{K+2N} U(r) T_{KK0} j_0^V \right) \right\} \tag{10.65}$$

according to which of the approximations discussed above we use.

If $K=0$ this equation simplifies to

$$-2N\,{}^{V}F_{011}^{(N-1)} = W_0 R\,{}^{V}F_{000}^{(N)} \pm RC_0^{(N)} \tag{10.66}$$

and for $N=0$ to

$$-(2K+1)\sqrt{\left(\frac{K}{2K+1}\right)}\,{}^{V}F_{KK-11}^{(0)} = W_0 R\,{}^{V}F_{KK0}^{(0)} \pm RC_K^{(0)}. \tag{10.67}$$

As far as the form factor coefficients ${}^{V}F_{KLs}^{(N)}(k_e, m, n, \rho)$, which are related to the ${}^{V}F_{KLs}(q^2)$ by eqn (6.159), are concerned, similar model independent relations like eqns (10.63) to (10.67) cannot be derived for $\rho \neq 0$. Some relations which are, however, not very useful can be obtained if one either introduces impulse approximation at an earlier stage (see Behrens and Bühring 1971) or introduces additionally some new types of form factor coefficients.

In concluding this topic we should always keep in mind that the above CVC relations are exact and model independent,† i.e. they relate the vector form factors including exchange effects, induced interactions etc. to one another.

† It remains the question: can these relations be applied to the form factors calculated in impulse approximation. Let us consider a special example in that context, i.e. the form factor coefficients ${}^{V}F_{211}^{(0)}$ and ${}^{V}F_{220}^{(0)}$. They are related by (see eqn (10.67))

$${}^{V}F_{211}^{(0)} = -\frac{1}{\sqrt{(10)}}\{W_0 R \mp (M_n - M_p)R \pm \tfrac{6}{5}\alpha Z\}F_{220}^{(0)}.$$

In impulse approximation we have (see eqn (8.263a) and (8.263e)) if $f_S = 0$

$${}^{V}F_{211}^{(0)} = -\mathfrak{M}_{211}^{(0)} - \frac{f_M}{R}\{W_0 R \mp (M_n - M_p)R \pm \tfrac{6}{5}\alpha Z\}\,{}^{C}\mathfrak{N}_{211}^{(0)}$$

and

$${}^{V}F_{220}^{(0)} = \mathfrak{M}_{220}^{(0)} + \sqrt{(10)}\,\frac{f_M}{R}\,{}^{C}\mathfrak{N}_{211}^{(0)}.$$

The many particle $\mathfrak{M}_{KLs}^{(0)}$ in the independent particle model are always given by (see eqn (8.66))

$$\mathfrak{M}_{KLs}^{(0)} = C(K)\mathfrak{M}_{KLs_{\text{single particle}}}^{(0)}.$$

Therefore we obtain (see eqns (8.21a), (8.21d), (8.30a) and (8.30b))

$$\frac{\mathfrak{M}_{211}^{(0)}}{\mathfrak{M}_{220}^{(0)}} = \frac{R}{\sqrt{(10)}}\,\frac{\int_0^\infty g_f\{(E_i - E_f) - (V_i - V_f)\}g_i\left(\frac{r}{R}\right)^2 r^2\,dr}{\int_0^\infty g_f\left(\frac{r}{R}\right)^2 g_i r^2\,dr}$$

$$= \frac{R}{\sqrt{(10)}}\left\{E_i - E_f - \frac{\int_0^\infty g_f(V_i - V_f)\left(\frac{r}{R}\right)^2 g_i r^2\,dr}{\int_0^\infty g_f\left(\frac{r}{R}\right)^2 g_i r^2\,dr}\right\}.$$

For practical applications of these relations we have, however, to evaluate the quantity $C_K(q^2)$. Then approximations have to be introduced and the relations cannot be considered as rigorous any more.

As we have seen before, both the electromagnetic and weak currents are so far analogous to one another as their currents are both conserved. Now we can extend the parallelism between both currents assuming that the electromagnetic isovector and weak vector current are different components of one and the same current in isospin space (Gell-Mann 1958).

This latter assumption which represents the CVC theory in its strong version leads to the following operator equation†

$$V_\mu(0) = \mp[T_\mp, J_\mu^{em}(0)] \tag{10.68}$$

(Upper sign β^--decay, lower sign β^+-decay and electron capture).

Then we have, of course, also

$$\langle f|\, V_\mu(0)\,|i\rangle = \mp\langle f|[T_\mp, J_\mu^{em}(0)]\,|i\rangle. \tag{10.69}$$

†Note that in isospin space we have the following operators for a single neutron or proton

$t_\mp = t_1 \mp i t_2$	$\beta^\mp$-decay
t_3	electromagnetic interaction.

The corresponding spherical tensor operators are however

$$T_1^q = \frac{-q}{\sqrt{2}}(t_1 + iqt_2)$$

with $q = \mp 1$ for $\beta^\mp$-decay

and

$T_1^0 = t_3$ for the electromagnetic interaction

If the single particle potential difference is determined by the average Coulomb potential only we have

$$-\frac{\int_0^\infty g_f(V_i - V_f) g_i \left(\frac{r}{R}\right)^2 r^2\, dr}{\int_0^\infty g_f \left(\frac{r}{R}\right)^2 g_i r^2\, dr} \simeq \tfrac{6}{5}\alpha Z/R$$

and finally

$${}^{V}F_{211}^{(0)} = -\frac{1}{\sqrt{(10)}}\{W_0 R \mp (M_n - M_p)R \pm \tfrac{6}{5}\alpha Z\} F_{220}^{(0)}.$$

In this case impulse approximation treatment and exact calculation on the basis of CVC agrees. If, however, single particle potentials like the symmetry potential or r-dependent spin-orbit terms are included, both calculation disagree. If residual interactions are switched on the situation will be even less transparent. It should be stated that one has to be very careful applying CVC impulse approximation. An excellent discussion of that matter can be found in a paper by Hwang (1980).

If the final state $f=(T_f, T_{3f})$ and the initial state $i=(T_i, T_{3i})$ are members of the same isospin multiplet with isospin T we then obtain simply

$$\langle T, T_{3f}|\, V_\mu(0)\,|T, T_{3i}\rangle = \pm\sqrt{\{(T\pm T_{3i})(T\mp T_{3i}+1)\}} \times\{\langle T, T_{3f}|\, J_\mu^{\mathrm{em}}(0)\,|T, T_{3f}\rangle - \langle T, T_{3i}|\, J_\mu^{\mathrm{em}}(0)\,|T, T_{3i}\rangle\} \quad (10.70)$$

(upper sign β^--decay, lower sign β^+-decay).

If the final state has isospin $T-1$ and the intial state isospin T and their third components are maximal we get (for β^--decay)

$$\langle T-1, T-1|\, V_\mu(0)\,|T, T\rangle = \sqrt{(2T)}\langle T-1, T-1|\, J_\mu^{\mathrm{em}}(0)\,|T, T-1\rangle \quad (10.71)$$

where we used

$$T_-|T, T\rangle = \sqrt{(2T)}\,|T, T-1\rangle \qquad T_+|T-1, T-1\rangle = 0 \quad (10.72)$$

and (for β^+-decay)

$$\langle T-1, -T+1|\, V_\mu(0)\,|T, -T\rangle = -\sqrt{(2T)}\langle T-1, -T+1|\, J_\mu^{\mathrm{em}}(0)\,|T, -T+1\rangle. \quad (10.73)$$

A further consequence is the fact that the matrix elements reduced in the isospin space for both electromagnetic isovector and weak vector interaction are equal, and therefore for a transition $T, T_{3i} \to T, T_{3f}$ it is

$$\frac{\langle T, T_{3f}|\, V_\mu(0)\,|T, T_{3i}\rangle}{\langle T, T_{3f}|\, J_\mu^{\mathrm{em}}(0)\,|T, T_{3f}\rangle} = \mp\frac{\sqrt{2}\begin{pmatrix} T & 1 & T \\ -T_{3f} & \mp 1 & T_{3i}\end{pmatrix}}{\begin{pmatrix} T & 1 & T \\ -T_{3f} & 0 & T_{3f}\end{pmatrix}} = \mp\frac{\{(T\pm T_{3i})(T\mp T_{3i}+1)\}^{1/2}}{T_{3f}} \quad (10.74)$$

$$\frac{\langle T, T_{3f}|\, V_\mu(0)\,|T, T_{3i}\rangle}{\langle T, T_{3i}|\, J_\mu^{\mathrm{em}}(0)\,|T, T_{3i}\rangle} = \pm\frac{\sqrt{2}\begin{pmatrix} T & 1 & T \\ -T_{3f} & \mp 1 & T_{3i}\end{pmatrix}}{\begin{pmatrix} T & 1 & T \\ -T_{3i} & 0 & T_{3i}\end{pmatrix}} = \mp\frac{\{(T\pm T_{3i})(T\mp T_{3i}+1)\}^{1/2}}{T_{3i}}. \quad (10.75)$$

For the case $T, T_{3i} \to T-1, T_{3f}$ which we find very often realized in nature, we have on the other hand

$$\frac{\langle T-1, T_{3f}|\, V_\mu(0)\,|T, T_{3i}\rangle}{\langle T-1, T_{3f}|\, J_\mu^{\mathrm{em}}(0)\,|T, T_{3f}\rangle} = \mp\frac{\sqrt{2}\begin{pmatrix} T-1 & 1 & T \\ -T_{3f} & \mp 1 & T_{3i}\end{pmatrix}}{\begin{pmatrix} T-1 & 1 & T \\ -T_{3f} & 0 & T_{3f}\end{pmatrix}} = \pm\left\{\frac{T\pm T_{3f}+1}{T\mp T_{3f}}\right\}^{1/2}. \quad (10.76)$$

It should be noted that the above relations are exact only in the limit that we have pure isospin states in the initial and final nucleus. In reality that condition is, however, only approximately fulfilled.

The formulae presented above give us now the chance to relate electromagnetic to weak form factors or to relate electromagnetic observables to weak ones. Let us, therefore, now consider some of the most important electromagnetic observables.

The radiative widths Γ_γ for a γ-transition from $J_i \to J_f$ is given by (see de Forest and Walecka 1966; Überall 1971; Donnelly and Walecka 1975)†

$$\Gamma_\gamma = 8\pi\alpha \sum_{K=1}^{\infty} \left(\frac{K+1}{K}\right) \frac{q^{2K+1}}{[(2K+1)!!]^2} [B(EK, q) + B(MK, q)] \tag{10.77}$$

where we have here $\mathbf{q} = \mathbf{k}$ and $|\mathbf{q}| = \Delta$ (k = momentum of the photon).‡

Similarly we have for the photon absorption cross-section σ_γ (integrated over its widths Δk) for a transition $J_f \to J_i$

$$\int_{\Delta k} \sigma_\gamma(k)\, \mathrm{d}k = (2\pi)^3 \alpha \left(\frac{2J_i+1}{2J_f+1}\right) \sum_{K=1} \left(\frac{K+1}{K}\right) \frac{q^{2K-1}}{[(2K+1)!!]^2} [B(EK, q) + B(MK, q)]. \tag{10.78}$$

The differential cross-section for elastic and inelastic electron scattering for a transition $J_f \to J_i$ reads as (see, for example, Überall 1971)§

$$\frac{\mathrm{d}\sigma^{J_f \to J_i}}{\mathrm{d}\Omega} = 4\pi\alpha^2 f_{\mathrm{Rec}} \frac{1}{k_i^2} \frac{(2J_i+1)}{(2J_f+1)} \left\{ v_c(\theta) \sum_{K=0} \frac{q^{2K}}{[(2K+1)!!]^2} \times B(CK, q) + v_T(\theta) \sum_{K=1} \frac{K+1}{K} \frac{q^{2K}}{[(2K+1)!!]^2} [B(EK, q) + B(MK, q)] \right\} \tag{10.79}$$

with

$$v_c(\theta) = \frac{2k_i k_f}{q^4} (E_i E_f + \mathbf{k}_i \mathbf{k}_f + 1) \tag{10.80}$$

$$v_T(\theta) = \frac{2k_i k_f}{(q^2 - E^2)^2} (E_i E_f - (\mathbf{k}_i \mathbf{q})(\mathbf{k}_f \mathbf{q})/|\mathbf{q}^2| - 1) \tag{10.81}$$

† The radiative width Γ_γ is very often expressed in Weisskopf units which are defined as

$$\Gamma_W(EK) = \alpha \frac{2(K+1)}{K[(2K+1)!!]^2} \left[\frac{3}{K+3}\right]^2 q(qR)^{2K}$$

$$\Gamma_W(MK) = \alpha \frac{20(K+1)}{K[(2K+1)!!]^2} \left[\frac{3}{K+3}\right]^2 \frac{q}{(M_p R)^2} (qR)^{2K}$$

with M_p = mass of the proton.

‡ For a compilation of experimentally determined Γ_γ see Endt (1979).

§ The factor $(2J_i+1)/(2J_f+1)$ has to be introduced in eqns (10.78) and (10.79) because we consider in γ-transitions (as in β-decay) transitions of the type $J_i \to J_f$ but in photoabsorption and electron scattering transitions of the type $J_f \to J_i$.

where k_i and k_f are the electron three momenta before and after the scattering process ($\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$) and E_i and E_f the corresponding total energies ($E = E_i - E_f$). The recoil factor is given by

$$f_{\text{Rec}} = \left[1 + \frac{2k_i}{M_A} \sin^2 \tfrac{1}{2}\theta\right]^{-1} \tag{10.82}$$

(θ = angle between $\mathbf{k}_i$ and $\mathbf{k}_f$).

The $B(CK, q)$, $B(EK, q)$ and $B(MK, q)$ are the reduced transition strength ($B(\Lambda\downarrow)$) for longitudinal electric, transversal electric and transversal magnetic transitions (see, for example, Überall 1971).

These reduced transition strengths are connected with the corresponding electromagnetic form factors or matrix elements as follows:

$$B(CK, q) = \frac{1}{2J_i + 1} \frac{[(2K+1)!!]^2}{q^{2K}} \langle J_f || i^K \hat{M}_K || J_i \rangle^2_{\text{electromagnetic}} \tag{10.83}$$

$$B(EK, q) = \frac{1}{2J_i + 1} \frac{K}{K+1} \frac{[(2K+1)!!]^2}{q^{2K}} \langle J_f || i^K \hat{T}^{\text{el}}_K || J_i \rangle^2_{\text{electromagnetic}} \tag{10.84}$$

$$B(MK, q) = \frac{1}{2J_i + 1} \frac{K}{K+1} \frac{[(2K+1)!!]^2}{q^{2K}} \langle J_f || i^K \hat{T}^{\text{mag}}_K || J_i \rangle^2_{\text{electromagnetic}} \tag{10.85}$$

(for the definition of the above matrix elements see eqns (9.78) to (9.81)).

The parity selection rules belonging to these matrix elements are the following

$$\pi_i \pi_f = (-1)^K \qquad \text{(Coulomb, transverse electric)}$$
$$\pi_i \pi_f = (-1)^{K+1} \qquad \text{(transverse magnetic).}$$

By using eqns (10.70) to (10.76) we obtain relations between beta-decay form factors ${}^{\text{V}}F_{KLs}(q^2)$ and electromagnetic reduced transition strengths. For practical applications the most important cases in this context are the following:

(a) beta-transitions within an isospin multiplet, i.e. transitions of the type $T \to T$

Here we have $\pi_i \pi_f = +1$:

$$K = \text{even}$$
$$4\pi B_{i,f}(CK, q)_{\text{isovector}} = \frac{(T_{3i,f})^2}{(T \pm T_{3i})(T \mp T_{3i} + 1)} \cdot R^{2K} ({}^{\text{V}}F_{KK0}(q^2))^2 \tag{10.86}$$

$$K = \text{odd}$$
$$4\pi B_{i,f}(MK, q)_{\text{isovector}} = \frac{(T_{3i,f})^2}{(T \pm T_{3i})(T \mp T_{3i} + 1)} \frac{K}{K+1} \cdot R^{2K} ({}^{\text{V}}F_{KK1}(q^2))^2. \tag{10.87}$$

Thus, it follows that the beta-decay form factors ${}^{V}F_{KK0}(q^2)$ and ${}^{V}F_{KK1}(q^2)$ are related to the elastic electron (charge and magnetic) scattering reduced transition strengths.†

The index i or f denotes whether the beta-decay form factor is related to the corresponding $B(\Lambda\downarrow)$ of the initial or final nucleus, respectively. It should also be remarked that $B(EK, q) = 0$ for elastic electron scattering (see, for example, Überall 1971).

(b) beta-transitions of the type $T, T_{3i} \to T-1, T_{3f}$ electromagnetic transitions of the type $T, T_{3f} \to T-1, T_{3f}$.

For $\pi_i\pi_f = (-1)^K$

$$4\pi B(CK, q) = \frac{T \mp T_{3f}}{T \pm T_{3f}+1} R^{2K}\{{}^{V}F_{KK0}(q^2)\}^2 \tag{10.88}$$

$$4\pi B(EK, q) = \frac{T \mp T_{3f}}{T \pm T_{3f}+1} \frac{K}{K+1} \frac{2K+1}{(qR)^2} R^{2K} \times \left\{\sqrt{(K+1)}\,{}^{V}F_{KK-11}(q^2) + \frac{(qR)^2\sqrt{K}}{(2K+1)(2K+3)}\,{}^{V}F_{KK+11}(q^2)\right\}^2 \tag{10.89}$$

with (see eqns (10.51) and (10.59))

$$4\pi B(EK, q) \xrightarrow[q^2\to 0]{} \frac{T \mp T_{3f}}{T \pm T_3+1} \left(\frac{\Delta}{q}\right)^2 R^{2K}\{{}^{V}F_{KK0}(q^2)\}^2$$

where Δ is here the energy difference between the states T, T_{3f} and $T-1, T_{3f}$.

For $\pi_i\pi_f = (-1)^{K+1}$

$$4\pi B(MK, q) = \frac{T \mp T_{3f}}{T \pm T_{3f}+1} \frac{K}{K+1} R^{2K}\{{}^{V}F_{KK1}(q^2)\}^2. \tag{10.90}$$

The most important case of the above relations is, of course, that where $T = 1$, $T_{3i} = \pm 1$ and $T_{3f} = 0$, i.e. when we are comparing a beta-transition from both the outer members of an isospin triplet with a gamma-transition (or a photo- and/or electro excitation) from the middle member of this isospin triplet to an isospin singlet. Then the isospin factor in the above relations is equal to $\frac{1}{2}$.

In this case it is very useful that gamma-transition, photo absorption or electro excitation results can be related to beta-decay vector form factors. On the other hand these relations can, of course, also be used for a model-independent test of the CVC-hypothesis.

† Note that

$$4\pi B_{i,f}(C0, q)_{\text{isovector}} \xrightarrow[q^2\to 0]{} (T_{3i,f})^2.$$

Besides, the fact that the weak vector and the electromagnetic current are the result of a rotation of one and the same current in isospin space (if CVC is valid) has additional consequences for the coupling constants f_M and f_S. The most general weak vector current for one nucleon is (see eqn (8.236))

$$\langle N'|\, V_\mu(0)\,|N\rangle = \mathrm{i}\,\langle \bar{u}_{N'}|\,[f_V\gamma_\mu - f_M\sigma_{\mu\nu}q_\nu + \mathrm{i}f_S q_\mu]t_\mp\,|u_N\rangle. \qquad (10.91)$$

N and N' denote the nucleon states and the u's are, as before, the free Dirac spinors. In comparison with that the isovector part of the electromagnetic current has the form (see, for example, Bernstein 1968; Blin-Stoyle 1973)

$$\langle N'|\, J_\mu^{\mathrm{em}}(0)\,|N\rangle = \mathrm{i}\,\langle \bar{u}_{N'}|\,[F_1^V\gamma_\mu - F_2^V\sigma_{\mu\nu}q_\nu]t_3\,|u_N\rangle. \qquad (10.92)$$

In the limit $q^2 \to 0$ F_1^V and F_2^V have the values

$$F_1^V(0) = 1 \qquad F_2^V(0) = \frac{\kappa_p - \kappa_n}{2M_N} \qquad (10.93)$$

where κ_p and κ_n are the anomalous magnetic moments of neutron and proton, M_N is the nucleon mass.

Thus CVC (in its strong form) requires that in the limit $q^2 = 0$

$$f_V(0) = 1 \qquad f_M(0) = \frac{\kappa_p - \kappa_n}{2M_N} \qquad f_S = 0 \qquad (10.94)$$

with $\kappa_p = 1{\cdot}793$ and $\kappa_n = -1{\cdot}913$.

$f_M(0)$ has the numerical value

$$f_M = f_M(0) = 1{\cdot}008 \times 10^{-3}. \qquad (10.95)$$

The result $f_S = 0$ had already been obtained before from the simple current conservation, i.e. from the weak form of CVC (see eqn (10.43)).

Beyond that, CVC (in its strong form) also requires that for $q^2 > 0$

$$f_V(q^2) = F_1^V(q^2) \qquad (10.96)$$

and

$$f_M(q^2) = F_2^V(q^2). \qquad (10.97)$$

At this point it should be mentioned that the relation (10.95) particularly offers the best test possibility for the CVC theory in its strong form. We will discuss this point later on under the heading 'allowed transitions'.

Other articles or books where CVC theory in the context of beta-decay has been reviewed in detail have been published by Schopper (1966), Blin-Stoyle and Nair (1966), Blin-Stoyle (1973), Holstein (1974), Fujita and Une (1976), Wilkinson (1978), and Wu (1978).

As far as relations between matrix elements (previously the beta-decay

observables have exclusively been described in terms of matrix elements) on the basis of the CVC theory are concerned, the earlier papers by Fujita (1962), Eichler (1963) and Damgaard and Winther (1966) should also be mentioned.

Up to some years ago the CVC theory was nothing else than a pure but nevertheless nice theoretical hypothesis (but with a certain experimental evidence for its validity). Nowadays we have, however, a deeper understanding of a possible connection between electromagnetic and weak interactions. The reason is that now the gauge theories mentioned before (see, for example, Weinberg 1974; Beg and Sirlin 1974; Taylor 1976; Ryder 1977) offer us a scheme to unify electromagnetic and weak interaction in a more rigorous way. The CVC theory is, therefore, now one of the consequences which follow logically from the gauge theories, especially from the Weinberg–Salam model, favoured today.

10.3. Partially conserved axial current (PCAC)

The axial vector coupling constant $\lambda = 1{\cdot}25$ (see Kropf and Paul 1974) deviates remarkably from the value 1. We can, therefore, assume that renormalization effects due to strong interactions are present in that interaction. Thus it is probable that a strict conservation law of the type

$$\frac{\partial_\mu A_\mu(x)}{\partial x_\mu} = 0 \tag{10.98}$$

does not exist. In fact, it can be shown that a conserved axial vector current is in contradiction to some experimental results.

Firstly, exact axial vector current conservation would lead to a very large pseudoscalar coupling constant $f_P \simeq 10^3\,\lambda$, which has never been observed (Goldberger and Treiman 1958).

Secondly, exact axial vector current conservation would forbid a leptonic decay of the pion (Taylor 1958) which, however, is well known to occur in nature (a detailed discussion of these points can, for example, be found in the article by Blin-Stoyle and Nair (1966) or in the book by Bernstein (1968)).

A somewhat modified concept of axial vector conservation has, therefore, been introduced (Nambu 1960; Gell-Mann and Lévy 1960; Bernstein *et al.* 1960) where the axial current conservation is only valid in the limit of $m_\pi = 0$ (m_π = pion mass), i.e.

$$\frac{\partial A_\mu(x)}{\partial x_\mu} = m_\pi^2 f_\pi \phi_{\pi^\mp}. \tag{10.99}$$

f_π is the charged pion decay constant. The $\phi_{\pi^\mp}$ are the field operators† for the pion (π^- or π^+).

Analogously to our treatment in the case of the vector current we now have to consider

$$\langle f|\frac{\partial A_\mu(x)}{\partial x_\mu}-m_\pi^2 f_\pi \phi_{\pi^\mp}|i\rangle. \tag{10.100}$$

Since we have

$$(\Box - m_\pi^2)\phi_{\pi^\mp}(x) = -j_{\pi^\mp}(x) \tag{10.101}$$

($j_{\pi^\mp}$ is the current generating the pion field) we can further write‡

$$\langle f|\, m_\pi^2 f_\pi \phi_{\pi^\mp}|i\rangle = \frac{m_\pi^2 f_\pi}{q^2-\Delta^2+m_\pi^2}\langle f|\, j_{\pi^\mp}|i\rangle. \tag{10.102}$$

Similarly, as we have done in eqn (10.50), we now expand the pion nucleus vertex into multipoles in the Breit system and obtain

$$\langle f|\, j_{\pi^\mp}(0)\,|i\rangle = \mathrm{i}\sum_{KM}(-1)^{J_f-M_f}\sqrt{(2J_i+1)}\begin{pmatrix} J_f & K & J_i \\ -M_f & M & M_i\end{pmatrix} \times\sqrt{(4\pi)}\overset{*}{Y}{}_K^M(\hat{q})\frac{(qR)^K}{(2K+1)!!}A_K^{\pi NN'}(q^2). \tag{10.103}$$

Following the same lines as in the foregoing chapter (see eqn (10.51)) we can derive a relation between different axial vector form factors which reads as §

$$-(2K+1)\sqrt{\left(\frac{K}{2K+1}\right)}{}^{\mathrm{A}}F_{KK-11}(q^2)+\frac{(qR)^2}{(2K+3)}\sqrt{\left(\frac{K+1}{2K+1}\right)}{}^{\mathrm{A}}F_{KK+11}(q^2) = \Delta R\,{}^{\mathrm{A}}F_{KKO}(q^2)-\frac{m_\pi^2 f_\pi}{q^2-\Delta^2+m_\pi^2}RA_K^{\pi NN'}(q^2). \tag{10.104}$$

For $q^2=0$ the above equation takes the form of a generalized Goldberger–Treiman relation‖

$$-(2K+1)\sqrt{\left(\frac{K}{K+1}\right)}{}^{\mathrm{A}}F^{(0)}_{KK-11} = \Delta R\,{}^{\mathrm{A}}F^{(0)}_{KKO}-f_\pi RA_K^{\pi NN'}(0). \tag{10.105}$$

† $$\phi=\frac{1}{\sqrt{V}}\sum_p[\mathrm{e}^{\mathrm{i}px}a(p)+a^+(p)\mathrm{e}^{-\mathrm{i}px}]$$

where $a^+(p)$ and $a(p)$ are the corresponding creation and annihilation operators.

‡ Here, $q^2=|\mathbf{q}^2|$ is the square of the three momentum transfer.

§ As in the case of the corresponding CVC relation one additional term on the right side has to be added if the Coulomb interaction (gauge invariance) of the charged nucleus is taken into account.

‖ Since $\Delta^2 \ll m_\pi^2$, we have replaced $m_\pi^2-\Delta^2$ by m_π^2.

Since the $A_K^{\pi NN'}(q^2)$ are not known in general the above relation is not of much value with the exception of the decay of a single nucleon.

Then we have

$$\langle p|\, j_\pi\, |n\rangle = \mathrm{i}\sqrt{2}\; g_r K_{\pi NN}(q^2-\Delta^2)\,\langle \bar{u}_p|\, \gamma_5\, |u_n\rangle \tag{10.106}$$

and therefore

$$A_1^{\pi NN'}(q^2) = -\frac{3\sqrt{2}}{2M_N R}\, g_r K_{\pi NN}(q^2-\Delta^2) \tag{10.107}$$

where g_r is the rationalized, renormalized pion-nucleon coupling constant $(g_r^2/4\pi) = 14{\cdot}4$ and $K_{\pi NN}(q^2-\Delta^2)$ the invariant pion nucleon vertex function normalized so that $K_{\pi NN}(-m_\pi^2) = 1$.

By applying eqns (9.29*a*) to (9.29*f*) we then can relate the coupling constants λ and f_P to the form factor ${}^A F_{101}(q^2)$, ${}^A F_{110}(q^2)$ and ${}^A F_{121}(q^2)$, and obtain

$$2M_N\lambda(q^2) + (q^2-\Delta^2) f_P(q^2) = \frac{m_\pi^2 f_\pi}{q^2-\Delta^2+m_\pi^2}\sqrt{2}\; g_r K_{\pi NN}(q^2-\Delta^2). \tag{10.108}$$

We have

$$f_\pi = \frac{\sqrt{2}\, M_N \lambda}{g_r K_{\pi NN}(0)}. \tag{10.109}$$

Neglecting the q^2 dependence of $K_{\pi NN}$ we get finally for $q^2 = 0$

$$f_P = -\frac{2M_N\lambda}{q^2-\Delta^2+m_\pi^2} \simeq -\frac{2M_N\lambda}{m_\pi^2}. \tag{10.110}$$

For beta-decay we have therefore

$$f_P \simeq -0{\cdot}0492\,\lambda \simeq -\tfrac{1}{20}\lambda. \tag{10.111}$$

10.4. G-parity transformation and second class currents

Weinberg has introduced a current classification according to their behaviour under a G-parity transformation (Weinberg 1958) which is defined as

$$G = C\mathrm{e}^{\mathrm{i}\pi T_2}. \tag{10.112}$$

C represents, here, the charge conjugation operation (see also eqn (6.118)) and $\mathrm{e}^{\mathrm{i}\pi T_2}$ the charge symmetry operation.†

† It should be noted that the strong interaction is invariant under a G-parity transformation.

If our V–A current is a first class current then we have

$$GV_\mu^{(\mathrm{I})}G^{-1} = V_\mu^{(\mathrm{I})} \tag{10.113a}$$

$$GA_\mu^{(\mathrm{I})}G^{-1} = -A_\mu^{(\mathrm{I})}. \tag{10.113b}$$

If it is of the second class type it holds

$$GV_\mu^{(\mathrm{II})}G^{-1} = -V_\mu^{(\mathrm{II})} \tag{10.114a}$$

$$GA_\mu^{(\mathrm{II})}G^{-1} = A_\mu^{(\mathrm{II})}, \tag{10.114b}$$

i.e. any current can be decomposed into first and second class currents

$$V_\mu^{(\mathrm{I})} = \tfrac{1}{2}(V_\mu + GV_\mu G^{-1}) \tag{10.115a}$$

$$V_\mu^{(\mathrm{II})} = \tfrac{1}{2}(V_\mu - GV_\mu G^{-1}) \tag{10.115b}$$

and

$$A_\mu^{(\mathrm{I})} = \tfrac{1}{2}(A_\mu - GA_\mu G^{-1}) \tag{10.116a}$$

$$A_\mu^{(\mathrm{II})} = \tfrac{1}{2}(A_\mu + GA_\mu G^{-1}). \tag{10.116b}$$

Let us first consider the behaviour of the vector and axial vector current under the charge conjugation transformation. From CPT invariance we know that for the Hamiltonian density (see eqn (6.1)) we have (see, for example, Frauenfelder and Henley 1975)

$$CPTH_\beta(0)(CPT)^{-1} = H_\beta(0). \tag{10.117}$$

Since we are interested in the transformation properties of $V_\mu(0)$ and $A_\mu(0)$ we now study the corresponding properties of the lepton current $L_\mu(0)$ whose functional form can be explicitly given (see eqn (6.3)). If we apply the parity transformation on a Dirac field operator we get

$$P\psi(\mathbf{r}, t)P^{-1} = \gamma_4\psi(-\mathbf{r}, t) \tag{10.118a}$$

$$P\bar{\psi}(\mathbf{r}, t)P^{-1} = \bar{\psi}(-\mathbf{r}, t)\gamma_4 \tag{10.118b}$$

and for the charge conjugation transformation

$$C\psi(\mathbf{r}, t)C^{-1} = c\bar{\psi}(\mathbf{r}, t) \tag{10.119a}$$

$$C\bar{\psi}(\mathbf{r}, t)C^{-1} = -\psi(\mathbf{r}, t)c^+. \tag{10.119b}$$

In our representation of the γ's we have

$$c = \gamma_4\gamma_2. \tag{10.120}$$

Together with the time reversal transformation properties discussed before (see Section 10.1) we then obtain

$$CPT\mathbf{L}(0)(CPT)^{-1} = -\mathbf{L}^+(0) \tag{10.121a}$$

$$CPTL_0(0)(CPT)^{-1} = -L_0^+(0). \tag{10.121b}$$

CPT invariance, therefore, requires that

$$CPT(\mathbf{V}(0)+\mathbf{A}(0))(CPT)^{-1}=-(\mathbf{V}(0)+\mathbf{A}(0))^{+} \quad (10.122a)$$

$$CPT(V_0(0)+A_0(0))(CPT)^{-1}=-(V_0(0)+A_0(0))^{+}. \quad (10.122b)$$

By definition we have, on the other hand,

$$P\mathbf{V}(\mathbf{r},t)P^{-1}=-\mathbf{V}(-\mathbf{r},t) \quad (10.123a)$$

$$PV_0(\mathbf{r},t)P^{-1}=V_0(-\mathbf{r},t) \quad (10.123b)$$

$$P\mathbf{A}(\mathbf{r},t)P^{-1}=\mathbf{A}(-\mathbf{r},t) \quad (10.123c)$$

$$PA_0(\mathbf{r},t)P^{-1}=-A_0(-\mathbf{r},t) \quad (10.123d)$$

and then for parity and time reversal transformation together (assuming time reversal invariance to be valid)

$$PT\mathbf{V}(0)(PT)^{-1}=\mathbf{V}(0) \quad (10.124a)$$

$$PTV_0(0)(PT)^{-1}=V_0(0) \quad (10.124b)$$

$$PT\mathbf{A}(0)(PT)^{-1}=-\mathbf{A}(0) \quad (10.124c)$$

$$PTA_0(0)(PT)^{-1}=-A_0(0). \quad (10.124d)$$

From these equations it then follows finally that always

$$C\mathbf{V}(0)C^{-1}=-\mathbf{V}^{+}(0) \quad (10.125a)$$

$$CV_0(0)C^{-1}=-V_0^{+}(0) \quad (10.125b)$$

$$C\mathbf{A}(0)C^{-1}=\mathbf{A}^{+}(0) \quad (10.125c)$$

$$CA_0(0)C^{-1}=A_0^{+}(0). \quad (10.125d)$$

At this point it should be kept in mind that the behaviour of the weak interaction currents under charge conjugation is unique. Thus, the behaviour under the charge conjugation transformation cannot be responsible for a mixture of first and second class in the weak current (if time reversal and CPT invariance holds).

Now we have to go over to the charge symmetry transformation. From the CVC theory (see Section 10.2) it follows that $V_\mu(0)$ is a member of an isospin triplet. Analogously the PCAC theory requires that $A_\mu(0)$ also transforms as a member of an isospin triplet (since $\partial_\mu A_\mu(0)/\partial x_\mu$ is proportional to the pion field which is an isospin triplet).

Then, we can conclude that under charge symmetry our current transforms as

$$e^{i\pi T_2}(\mathbf{V}(0)+\mathbf{A}(0))e^{-i\pi T_2}=-(\mathbf{V}(0)+\mathbf{A}(0))^{+} \quad (10.126a)$$

$$e^{i\pi T_2}(V_0(0)+A_0(0))e^{-i\pi T_2}=-(V_0(0)+A_0(0))^{+} \quad (10.126b)$$

Any current that transforms like this equation is called charge symmetric. Thus, in this case we would obtain

$$Ce^{i\pi T_2}(\mathbf{V}(0)+\mathbf{A}(0))(Ce^{i\pi T_2})^{-1}=\mathbf{V}(0)-\mathbf{A}(0) \qquad (10.127a)$$

and

$$Ce^{i\pi T_2}(V_0(0)+A_0(0))(Ce^{i\pi T_2})^{-1}=V_0(0)-A_0(0), \qquad (10.127b)$$

i.e. the currents would purely be of the first class type.

Second class currents can, therefore, only be occur if the behaviour of the weak current under charge symmetry differs in the sign from the charge symmetric behaviour. That means for a second class current we would have

$$e^{i\pi T_2}\mathbf{J}^{II}(0)e^{-i\pi T_2}=\mathbf{J}^{II+}(0) \qquad (10.128a)$$

$$e^{i\pi T_2}J_0^{II}(0)e^{-i\pi T_2}=J_0^{II+}(0). \qquad (10.128b)$$

Since the charge symmetry operation applied on a wave function gives

$$e^{i\pi T_2}|T, T_3\rangle=(-1)^{T+T_3}|T,-T_3\rangle \qquad (10.129)$$

we obtain the relation

$$\begin{aligned}\langle T_f T_{3f}|\, V_\mu(0)+A_\mu(0)\,|T_i T_{3i}\rangle &= (-1)^{T_f+T_i+T_{3f}+T_{3i}}\\ &\times\langle T_f,-T_{3f}|\, e^{i\pi T_2}(V_\mu(0)+A_\mu(0))e^{-i\pi T_2}\,|T_i,-T_{3i}\rangle\\ &=(-1)^{T_f+T_i+T_{3f}+T_{3i}}\varepsilon\,\langle T_f,-T_{3f}|\,(V_\mu(0)+A_\mu(0))^+\,|T_i,-T_{3i}\rangle\\ &=(-1)^{T_f+T_i+T_{3f}+T_{3i}}\varepsilon\,\langle T_i,-T_{3i}|\, V_\mu(0)+A_\mu(0)\,|T_f,-T_{3f}\rangle^* \qquad (10.130)\end{aligned}$$

with $\varepsilon=-1$ for first class currents and $\varepsilon=+1$ for second class currents. Expanding now both sides of the above equation into multipoles (analogous with what we have done in Section 10.1) we get the following conditions for our form factors (assuming time reversal to be valid)

$$F_{KLs}(q^2)^{i\to f}_{T_{3i}\to T_{3f}}=(-1)^{K-s+J_f-J_i+T_f-T_i+1}\varepsilon\sqrt{\left(\frac{2J_f+1}{2J_i+1}\right)}F_{KLs}(q^2)^{f\to i}_{-T_{3f}\to -T_{3i}} \qquad (10.131)$$

and also

$$F_{KLs}(q^2)_{T_{3i}\to T_{3f}}=(-1)^{T_f-T_i+1}\varepsilon\,F_{KLs}(q^2)_{-T_{3i}\to -T_{3f}}. \qquad (10.132)$$

Making use of the relations between β^- and β^+ decay observables discussed in Section 7.4.1 we can explictly write (see eqns (7.163a) and (7.163b))

$${}^{V}F_{KLs}(q^2)^{\beta^-}_{T_{3i}\to T_{3f}}=(-1)^{T_f-T_i+1}\varepsilon\,{}^{V}F_{KLs}(q^2)^{\beta^+}_{-T_{3i}\to -T_{3f}} \qquad (10.133a)$$

$${}^{A}F_{KLs}(q^2)^{\beta^-}_{T_{3i}\to T_{3f}}=(-1)^{T_f-T_i}\varepsilon\,{}^{A}F_{KLs}(q^2)^{\beta^+}_{-T_{3i}\to -T_{3f}}. \qquad (10.133b)$$

Thus we have obtained a relation between the form factors of a transition to those of its mirror transition. Each and every form factor can consist of first and second class current parts:

$$F_{KLs}(q^2) = F^{\mathrm{I}}_{KLs}(q^2) + F^{\mathrm{II}}_{KLs}(q^2). \tag{10.134}$$

For mirror decays we see, however, that if the first class (second class) contributions to the form factors change the sign, the second class (first class) parts of the form factors do not. In the case where no second class currents exists, observables which depend on the squares of the form factors only (like, for example, simply the ft-values) then must be identical for both transitions.†

For other observables the difference between β^-- and β^+-decay can definitely be predicted.

Thus the study of β-transitions offers, in a certain sense, a model independent possibility to search for a possible existence of second class current contributions. This test method has therefore been applied very often in the past, but with almost a negative result, i.e. no finite second class current contributions have been found (for reviews of the experiments see Wilkinson 1978; Calaprice 1978; Behrens *et al.* 1978; Oka and Kubodera 1980).

Going back to the impulse approximation treatment of the induced interactions outlined in Section 8.1.2 we find for the transformation properties of the different nuclear matrix elements

$$\mathfrak{M}_{KL\bar{s}}(q^2)^{\,i\to f}_{T_{3i}\to T_{3f}} = (-1)^{K-\bar{s}+J_f-J_i+T_f-T_i}\,\mathrm{s}\,\mathfrak{M}_{KL\bar{s}}(q^2)^{f\to i}_{-T_{3f}\to -T_{3i}} \tag{10.135}$$

$${}^{\mathrm{C}}\mathfrak{M}_{KK0}(q^2)^{\,i\to f}_{T_{3i}\to T_{3f}} = (-1)^{K+J_f-J_i+T_f-T_i}\,\mathrm{s}\,{}^{\mathrm{C}}\mathfrak{M}_{KK0}(q^2)^{f\to i}_{-T_{3f}\to -T_{3i}} \tag{10.136}$$

$${}^{\mathrm{D}}\mathfrak{M}_{KL1}(q^2)^{\,i\to f}_{T_{3i}\to T_{3f}} = (-1)^{K-1+J_f-J_i+T_f-T_i}\,\mathrm{s}\,{}^{\mathrm{D}}\mathfrak{M}_{KL1}(q^2)^{f\to i}_{-T_{3f}\to -T_{3i}} \tag{10.137}$$

$${}^{\mathrm{D}}\mathfrak{M}_{KK0}(q^2)^{\,i\to f}_{T_{3i}\to T_{3f}} = (-1)^{K+J_f-J_i+T_f-T_i+1}\,\mathrm{s}\,{}^{\mathrm{D}}\mathfrak{M}_{KK0}(q^2)^{f\to i}_{-T_{3f}\to -T_{3i}}$$

$${}^{\mathrm{C}}\mathfrak{M}_{KL1}(q^2)^{\,i\to f}_{T_{3i}\to T_{3f}} = (-1)^{K+J_f-J_i+T_f-T_i}\,\mathrm{s}\,{}^{\mathrm{C}}\mathfrak{M}_{KL1}(q^2)^{f\to i}_{-T_{3f}\to -T_{3i}}. \tag{10.139}$$

with $\mathrm{s} = \{(2J_f+1)/(2J_i+1)\}^{\frac{1}{2}}$.

By inspection of eqns (8.263a) to (8.263h) and by comparing the above transformation properties with those of eqn (10.131) we see immediately that for form factors based on pure first class currents it should hold ‡

$$f_{\mathrm{S}} = 0 \quad \text{and} \quad f_{\mathrm{T}} = 0, \tag{10.140}$$

† If isospin impurity effects caused by the Coulomb interaction are neglected.

‡ Of course, eqns (10.130) can also be applied to the decay of a single neutron. Then we come directly to the same result.

i.e. a finite second class contribution would be introduced by finite values for the induced scalar and for the induced tensor coupling constants.† The above consequence for f_S and f_T offers a second test possibility for a possible existence of second class currents (by comparision of theory and experiment) which is, however, model dependent since it depends on the validity of the impulse approximation treatment and on the quality of the nuclear models used (for an example, see Behrens *et al.* 1978).

From a theoretical point of view detailed discussions of the second class current problem can also be found in the book by Blin-Stoyle (1973) and in the articles by Weinberg (1958), Blin-Stoyle and Nair (1966), Delorme and Rho (1971), Kubodera *et al.* (1973), Holstein (1974), and Wilkinson (1978).

†It should be kept in mind that $f_S = 0$ follows also from the validity of the CVC theory (see Section 10.2).

11

INVERSE BETA-DECAY

THE INVERSE beta-decay manifests itself in the charge changing neutrino reactions†

$$\nu_e + (Z, A) \rightarrow (Z+1, A) + e^- \tag{11.1}$$

$$\overline{\nu_e} + (Z, A) \rightarrow (Z-1, A) + e^+. \tag{11.2}$$

In the following we summarize the basic expressions for the above reactions. Specifically, we shall try to elucidate the analogies between the beta-decay and its inverse process. Analogous with the approach of Sections 6.1 and 6.2 we first consider the underlying elementary processes and their corresponding S- or T-matrices. Similarly as given in eqns (6.9a–c) we then obtain for the S-matrix

$$\nu_e + n \rightarrow p + e^- \qquad S_{fi} = -i \langle pe^- | \int H_\beta(x)\, d^4x \, | n\nu_e \rangle \tag{11.3}$$

$$\overline{\nu_e} + p \rightarrow n + e^+ \qquad S_{fi} = -i \langle ne^+ | \int H_\beta(x)\, d^4x \, | p\bar{\nu}_e \rangle. \tag{11.4}$$

By inserting the Hamiltonian density of eqn (6.1) with the field operators of the eqns (6.6) and (6.7) and choosing the initial and final states to

$$|\nu_e n\rangle = a^+_{\nu_e} a^+_n |0\rangle \tag{11.5a}$$

$$|pe^-\rangle = a^+_p a^+_e |0\rangle \tag{11.5b}$$

$$|\overline{\nu_e} p\rangle = b^+_{\nu_e} a^+_p |0\rangle \tag{11.5c}$$

$$|ne^+\rangle = a^+_n b^+_e |0\rangle \tag{11.5d}$$

we then get for the T-matrix (see eqns 6.13a–c) in terms of free particle Dirac spinors

$$\nu_e + n \rightarrow p + e^-$$

$$T = \frac{G_\beta}{\sqrt{2}} [\bar{u}_p \gamma_\mu (1 + \lambda\gamma_5) u_n][\bar{u}_e \gamma_\mu (1 + \gamma_5) u_\nu] \tag{11.6}$$

$$\overline{\nu_e} + p \rightarrow n + e^+$$

$$T = \frac{G_\beta}{\sqrt{2}} [\bar{u}_n \gamma_\mu (1 + \lambda\gamma_5) u_p][\bar{v}_\nu \gamma_\mu (1 + \gamma_5) v_e]. \tag{11.7}$$

† Equation (11.1) is to be considered as the inverse process of $(Z+1, A) \rightarrow (Z, A) + e^+ + \nu_e$ and eqn (11.2) as the one of $(Z-1, A) \rightarrow (Z, A) + e^- + \bar{\nu}_e$.

Note that in the process which is analogous† to the β^--decay v_ν has to be replaced by u_ν and in the process which is analogous to β^+-decay $\overline{u_\nu}$ by $\overline{v_\nu}$. That is what we expect since we are now dealing with the corresponding neutrino particles (antiparticles) compared to antiparticles (particles) in the β^- (β^+)-decay. It should, however, be mentioned that the use of the correct formalism, as we have done above, should be preferred‡ because it always automatically leads to the correct particle or antiparticle wave functions with their corresponding correct momenta. In the more general case of a complex nucleus we then get for the T-matrix (see eqns (6.17a–c)

$$\nu_e + (Z, A) \rightarrow (Z+1, A) + e^-$$

$$T = -\frac{G_\beta}{\sqrt{2}} \langle f | \, V_\mu(0) + A_\mu(0) \, | i \rangle \, \mathrm{i}\overline{u_e} \gamma_\mu (1+\gamma_5) u_\nu \tag{11.8}$$

$$\overline{\nu_e} + (Z, A) \rightarrow (Z-1, A) + e^+$$

$$T = -\frac{G_\beta}{\sqrt{2}} \langle f | \, V_\mu(0) + A_\mu(0) \, | i \rangle \, \mathrm{i}\overline{v_\nu} \gamma_\mu (1+\gamma_5) v_e. \tag{11.9}$$

In contrast to beta-decay (see eqns (5.14) and (5.7)) the recoil three momentum is now given by

$$\mathbf{q} = \mathbf{p}_\nu - \mathbf{p}_e \tag{11.10}$$

and the electron energy (if kinetic nuclear recoil is neglected) by

$$W_e = W_\nu - \Delta. \tag{11.11}$$

Δ is defined in the same way as before for the inverse β-decay, i.e.

$$\Delta = (M_i - M_f)_{\beta\text{-decay}} = (M_f - M_i)_{\text{neutrino reactions}}. \tag{11.12}$$

Equation (11.11) has the consequence that the above reactions can only take place if the neutrino (antineutrino) energy is higher than the threshold energy

$$W_{\nu_{\text{threshold}}} = \Delta. \tag{11.13}$$

Next we expand the nuclear current matrix element in terms of form

† Note that a difference is made between the analogous and the inverse beta-decay. The analogous β^--decay corresponds to the process given in eqn (11.1) and the analogous β^+-decay to that of eqn (11.2). The word analogous should be understood with respect to the Coulomb functions.

‡ In older publications it is very often directly argued in terms of wave functions whereby this approach is then used as the basis for further calculations, but, in many cases this simplified method was the source of mistakes, i.e. of a use of wrong lepton wave functions (particle instead of antiparticle and vice versa).

factor coefficients in the same way as shown in Section 6.2 and the lepton matrix element into spherical waves as shown in Section 6.4. Then, we obtain for the (ν, e^-) reaction

$$T=\frac{G_\beta}{\sqrt{2}}\frac{1}{\pi^{3/2}}\sum_{KLsM}\sum_{\substack{\kappa_e\mu_e\\ \kappa_\nu\mu_\nu}}(-1)^{J_f-M_f+j_e-\mu_e+L+M}$$

$$\times\sqrt{(2J_i+1)}\begin{pmatrix} J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix} j_e & K & j_\nu\\ -\mu_e & -M & \mu_\nu\end{pmatrix}$$

$$\times a^*_{\kappa_e\mu_e}b_{\kappa_\nu\mu_\nu}\int_0^\infty q^2\,dq\int_0^\infty r^2\,dr\frac{(qR)^L}{(2L+1)!!}j_L(qr)F_{KLs}(q^2)$$

$$\times\langle\phi_{\kappa_{e_{\text{electron}}}}||T_{KLs}(1+\gamma_5)||\phi_{\kappa_{\nu_{\text{neutrino}}}}\rangle \qquad (11.14)$$

(for comparison with β^--decay see eqn (6.132)) or in terms of the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$

$$T=\frac{G_\beta}{4\pi}\sum_{KM}\sum_{\substack{\kappa_e\mu_e\\ \kappa_\nu\mu_\nu}}(-1)^{J_f-M_f+K+M}(-1)^{j_e-\mu_e}\sqrt{\{(2J_i+1)(2K+1)\}}$$

$$\times\begin{pmatrix} J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix} j_e & K & j_\nu\\ -\mu_e & -M & \mu_\nu\end{pmatrix}a^*_{\kappa_e\mu_e}b_{\kappa_\nu\mu_\nu}\alpha_{\kappa_e}$$

$$\times\{M_K(k_e, k_\nu)+\text{sign}(\kappa_e)m_K(k_e, k_\nu)\}. \qquad (11.15)$$

The quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ can be obtained from eqn (6.151) by replacing

$$-\text{sign}(\kappa_\nu)\to 1$$

$$j_{\bar{1}}(p_\nu r)\to -j_{\bar{1}}(p_\nu r)$$

For the $(\bar{\nu}, e^+)$ reaction we obtain

$$T=\frac{G_\beta}{\sqrt{2}}\frac{1}{\pi^{3/2}}\sum_{KLsM}\sum_{\substack{\kappa_e\mu_e\\ \kappa_\nu\mu_\nu}}(-1)^{J_f-M_f+j_\nu-\mu_\nu+L+M}\sqrt{(2J_i+1)}$$

$$\times\begin{pmatrix} J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix} j_\nu & K & j_e\\ \mu_\nu & -M & -\mu_e\end{pmatrix}$$

$$\times(-1)^{j_e+\mu_e+j_\nu+\mu_\nu}a^*_{\kappa_e\mu_e}b_{\kappa_\nu\mu_\nu}\int_0^\infty q^2\,dq\int_0^\infty r^2\,dr\frac{(qR)^L}{(2L+1)!!}j_L(qr)$$

$$\times F_{KLs}(q^2)\langle\phi_{\kappa_{\nu_{\text{antineutrino}}}}||T_{KLs}(1+\gamma_5)||\phi_{\kappa_{e_{\text{positron}}}}\rangle \qquad (11.16)$$

(for comparison with β^+-decay see eqn (6.145)) or

$$T=\frac{G_\beta}{4\pi}\sum_{KM}\sum_{\substack{\kappa_e\mu_e\\ \kappa_\nu\mu_\nu}}(-1)^{J_f-M_f+K+M}(-1)^{j_e-\mu_e}\sqrt{\{(2J_i+1)(2K+1)\}}$$
$$\times\begin{pmatrix} J_f & K & J_i\\ -M_f & M & M_i\end{pmatrix}\begin{pmatrix} j_e & K & j_\nu\\ -\mu_e & -M & \mu_\nu\end{pmatrix}$$
$$\times a^*_{\kappa_e\mu_e}b_{\kappa_\nu\mu_\nu}\,\mathrm{sign}(\kappa_e)\mathrm{sign}(\kappa_\nu)$$
$$\times\alpha_{\kappa_e}\{M_K(k_e,k_\nu)+\mathrm{sign}(\kappa_e)m_K(k_e,k_\nu)\} \tag{11.17}$$

whereby the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ can be calculated from eqn (7.153) by adding a factor $-\mathrm{sign}(\kappa_\nu)$ and by replacing

$$j_T(p_\nu r)\to -j_T(p_\nu r).$$

Before proceeding further we have first to discuss the kinematics of the neutrino reactions and to derive the necessary formulae for cross-sections. Because here we only have two particles in the final state (recoil nucleus and electron or positron) we get for the transition probability per unit time (for comparison see eqn (5.54))

$$\frac{\mathrm{d}W}{\mathrm{d}t}=\sum_f\sum_s\int\frac{(2\pi)^4\delta^4(p_f-p_i)V}{V^4}T^+_{if}T_{fi}\frac{\mathrm{d}^3p_fV}{(2\pi)^3}\frac{\mathrm{d}^3p_eV}{(2\pi)^3}. \tag{11.18}$$

Dividing this by the initial flux of neutrinos v_ν/V, where $v_\nu=p_\nu/W_\nu$ is the velocity of the neutrinos, we then obtain for the cross-sections, if we assume the target to be at rest and if the kinetic recoil energy is neglected,

$$\mathrm{d}\sigma=\frac{1}{(2\pi)^2}\frac{W_\nu}{p_\nu}\sum_f\sum_s\int|T|^2\,\delta^3(\mathbf{p}_f+\mathbf{p}_e-\mathbf{p}_\nu)$$
$$\times\delta(W_f+W_e-W_\nu-M_i)\,\mathrm{d}^3p_f\,\mathrm{d}^3p_e. \tag{11.19}$$

The integrations over the δ-functions can partly be carried out (see the discussion in Section 5.3) with the result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_e}=\frac{1}{(2\pi)^2}\frac{W_\nu}{p_\nu}p_eW_e\sum_f\sum_s|T|^2. \tag{11.20}$$

If we assume further the rest mass of the neutrino to be zero we have simply (since $W_\nu/p_\nu=1$)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_e}=\frac{p_eW_e}{(2\pi)^2}\sum_f\sum_s|T|^2. \tag{11.21}$$

Introducing the density matrix of eqn (7.7) we obtain more explicitly

$$\frac{d\sigma}{d\Omega_e}=\frac{p_e W_e}{(2\pi)^2}\frac{1}{2J_i+1}\sum_{M_iM_i'}\sum_{M_fM_f'}\sum_{m_em_e'}\sum_{m_\nu m_\nu'}\rho_\beta(M_iM_i'M_fM_f'm_em_e'm_\nu m_\nu')$$
$$\times\delta_{M_iM_i'}\delta_{M_fM_f'}\delta_{m_em_e'}\delta_{m_\nu m_\nu'} \tag{11.22}$$

when we are not observing any polarizations in initial and final states.† In beta-decay a summation of the above kind has, however, just been carried out for electron–neutrino correlations of unoriented nuclei (see Section 7.2.4.1). Thus we have only to look for the difference between the quantity

$$\sum_{M_iM_i'}\sum_{M_fM_f'}\sum_{m_em_e'}\sum_{m_\nu m_\nu'}\rho_\beta(M_iM_i'M_fM_f'm_em_e'm_\nu m_\nu')$$
$$\times\delta_{M_iM_i'}\delta_{M_fM_f'}\delta_{m_em_e'}\delta_{m_\nu m_\nu'} \tag{11.23}$$

for electron-neutrino correlations in beta-decay and that for neutrino reactions.

Let us compare the T-matrix for beta-decay with that for neutrino reactions (see, for example, eqn (6.174) and eqn (11.15)) for this purpose. We then see that the main difference between beta-decay and neutrino reactions for $\rho_\beta(M_iM_i'M_fM_f'm_em_e'm_\nu m_\nu')=T(M_iM_fm_em_\nu)T^*(M_i'M_f'm_e'm_\nu')$ lies in the following factor:

$$\begin{aligned}&\text{beta-decay} &&\rightarrow \quad \text{neutrino reactions}\\ &b^*_{\kappa_\nu\mu_\nu}b_{\kappa_\nu'\mu_\nu'} &&\rightarrow \quad (-1)^{j_\nu+j_\nu'+\mu_\nu+\mu_\nu'}\,\mathrm{sign}(\kappa_\nu)\\ &&&\quad\times\mathrm{sign}(\kappa_\nu')b_{\kappa_\nu\mu_\nu}b^*_{\kappa_\nu'\mu_\nu'}\end{aligned} \tag{11.24}$$

and in the following replacement with the $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$:

$$\begin{aligned}&\text{beta-decay} &&\rightarrow \quad \text{neutrino reactions}\\ &j_T(p_\nu r) &&\rightarrow \quad -j_T(p_\nu r).\end{aligned} \tag{11.25}$$

Since

$$b_{\kappa_\nu\mu_\nu}b^*_{\kappa_\nu'\mu_\nu'}=(-1)^{-\mu_\nu-\mu_\nu'+m_\nu+m_\nu'}\,\mathrm{sign}(\kappa_\nu)\mathrm{sign}(\kappa_\nu')$$
$$\times b^*_{\kappa_\nu-\mu_\nu}b_{\kappa_\nu'-\mu_\nu'} \tag{11.26}$$

the above relation corresponds to:

$$\begin{aligned}&\text{beta-decay} &&\rightarrow \quad \text{neutrino reactions}\\ &b^*_{\kappa_\nu\mu_\nu}b_{\kappa_\nu'\mu_\nu'} &&\rightarrow \quad (-1)^{k_\nu+k_\nu'-1+m_\nu+m_\nu'}b^*_{\kappa_\nu-\mu_\nu}b_{\kappa_\nu'-\mu_\nu'}.\end{aligned} \tag{11.27}$$

Since we have to sum over $m_\nu=m_\nu'$ and over all possible μ_ν and μ_ν', nothing changes if we sum over all $-m_\nu=-m_\nu'$ and over all $-\mu_\nu$ and $-\mu_\nu'$.

† The neutrino or antineutrino is always completely polarized. Therefore, in principle a summation over m_ν is superfluous. Our formalism, however, guarantees automatically that the results remain correct even if this redundant summation is carried out for symmetry reasons.

Therefore we end up with the simple prescription that the following replacements have to be made going from electron–neutrino correlations to neutrino cross-sections:

$$\begin{array}{ll} \text{beta-decay} & \rightarrow \quad \text{neutrino reactions} \\ M_K(k_e, k_\nu) & (-1)^k M_K(k_e, k_\nu) \\ m_K(k_e, k_\nu) & (-1)^k m_K(k_e, k_\nu) \\ j_T(p_\nu r) & -j_T(p_\nu r) \end{array} \tag{11.28}$$

which is equivalent to the simpler replacement

$$p_\nu \rightarrow -p_\nu. \tag{11.29}$$

This result can always be intuitively understood since we replace an outgoing particle by an ingoing without observing any polarizations. Nevertheless, we have derived the above result exactly because intuitive arguments very often lead to wrong conclusions in this subject.

Finally the neutrino reaction cross-section is obtained as

$$\frac{d\sigma}{d\Omega_e} = \frac{G_\beta^2}{4\pi^2} p_e W_e F_0 \sum_k D^{(k)} P_k(\cos\theta_{e\nu}) \tag{11.30}$$

where the electron neutrino correlation coefficients $D^{(k)}$ have to be taken from eqns (7.96) and (7.98) but with the replacement

$$D^{(k)}(p_\nu) \rightarrow D^{(k)}(-p_\nu).$$

$\theta_{e\nu}$ is the angle between electron and neutrino.

For the (ν_e, e^-) reaction one has to start from the $D^{(k)}$ for β^--decay and for the $(\overline{\nu_e}, e^+)$ reaction from those for β^+-decay. At this point it should be mentioned that corresponding formulae for neutrino reaction cross-sections, which, however, do not include Coulomb effects, have also been derived and discussed in detail by Walecka (1975), Donnelly and Walecka (1975), Serot (1978), and Donnelly and Peccei (1979). These authors give additionally a number of applications for special nuclei where they have calculated explicit numerical values by using different nuclear models.

For neutrino energies near the threshold (long wavelength limit $p_\nu R \ll 1$) the neutrino cross-sections can be directly related to some beta-decay observables. In this case we have

$$\frac{d\sigma}{d\Omega_e} = \frac{G_\beta^2}{4\pi^2} p_e W_e F_0 L_0 C(W_e)_\nu \left[1 + \frac{D^{(1)}}{L_0 C(W_e)_\nu} P_1(\cos\theta_{e\nu})\right] \tag{11.31}$$

since $D^{(0)} = L_0 C(W_e)_\nu$. Here, $C(W_e)_\nu$ is analogous to the shape factor in beta-decay (see eqn (7.56)), or, exactly, it is the shape factor $C(W_e)_\beta$ of the analogous beta-transition but with the replacement $p_\nu \rightarrow -p_\nu$.

It is interesting to see that the Coulomb interaction influences the neutrino reactions in the same way as the beta-decay, i.e. through the Fermi function.

Going back to eqn (7.60) we obtained, on the other hand, for the ft-value of the corresponding inverse beta-transition

$$ft = \frac{2\pi^3 \ln 2}{G_\beta^2 \overline{C(W_e)_\beta}}. \tag{11.32}$$

We can now express the coupling constant G_β by this ft-value and arrive at

$$\frac{d\sigma}{d\Omega_e} = \frac{\pi \ln 2}{2} p_e W_e \frac{F_0 L_0 C(W_e)_\nu}{ft \overline{C(W_e)_\beta}} [1 + d_1 P_1 (\cos \theta_{e\nu})] \tag{11.33}$$

for the differential cross-section, and at

$$\sigma = 2\pi^2 \ln 2 \, p_e W_e \frac{F_0 L_0 C(W_e)_\nu}{ft \overline{C(W_e)_\beta}} \tag{11.34}$$

for the integrated cross-section.†

d_1 is the asymmetry coefficient of the electron–neutrino correlation (in the relevant formulae for d_1, however, p_ν has to be replaced by $-p_\nu$). It should be noticed that the above formula is valid for every type of beta-transition, i.e. for every degree of forbiddenness.

For allowed transitions (see Sections 14.2.1.3.1 and 14.2.1.3.4) we have simply

$$\frac{d\sigma}{d\Omega_e} = \frac{\pi \ln 2}{2} S p_e W_e \frac{F_0 L_0}{ft} [1 + d_1 P_1 (\cos \theta_{e\nu})] \tag{11.35}$$

with (see eqns (14.198) and (14.199))

$$d_1 = \frac{A_0^2 - \frac{1}{3} C_0^2}{A_0^2 + C_0^2} \frac{p_e}{W_e} \Lambda_1 \qquad S = \frac{2J_f + 1}{2J_i + 1} \tag{11.36}$$

and

$$\sigma = 2\pi^2 \ln 2 \, p_e W_e \frac{F_0 L_0}{ft} S. \tag{11.37}$$

Here, d_1 is exactly equal to the electron–neutrino correlation coefficient if only dominant terms are taken into account.

By applying the formulae for the cross-sections just derived we obtain values in natural units. Usually cross-sections are, however, given in cm^2.

† As a consequence of the gauge theories (see Chapter 12) neutral current neutrino reaction processes of the type $\nu_e + N \rightarrow \nu_e + N^*$ (or $\overline{\nu_e} + N \rightarrow \overline{\nu_e} + N^*$) also exist. The total cross-section for this type of reaction is about half as large as that of the charge changing reactions quoted in eqn (11.34) (see the comprehensive treatment of Donnelly and Peccei 1979).

For convenience of the reader we therefore add the corresponding equation which allows the calculation of cross-sections in units of cm^2. It is

$$\sigma = 2{\cdot}629 \times 10^{-41} p_e W_e \frac{F_0 L_0 C(W_e)_\nu}{ft\, \overline{C(W_e)_\beta}} \text{ cm}^2. \tag{11.38}$$

Here the ft-values have to be inserted in seconds, as they are usually given. The other quantities like F_0L_0, p_e, W_e etc. have to be taken, as before, in natural units (i.e. p_e in units of m_ec and W_e in units of m_ec^2).

Up to now we have tried to show up the analogies between the charged neutrino reactions and the nuclear beta-decay. We therefore have treated the long wavelength limit, i.e. the case where $p_\nu R \ll 1$, because in this limit the analogy between both processes will be especially evident. In the long wavelength limit for neutrino reactions we have always $p_eR \ll 1$ and $W_eR \ll 1$ as in beta-decay. Thus we were able to expand the electron radial wave functions in the way discussed in Section 4.3 (see especially eqns (4.88) to (4.91)). However, as shown in eqn (4.83*b*), this expansion is justified only if

$$(W_\nu + W_e)R < 1 \tag{11.39}$$

or

$$(2W_\nu - \Delta)R < 1. \tag{11.40}$$

Since otherwise the convergence of the whole transition amplitude is not guaranteed (see the detailed discussion in Section 4.3). Thus for higher neutrino energies we have either to solve the whole problem numerically by calculating the electron radial wave functions numerically and integrating eqns (6.155) and (6.156) numerically or to neglect the Coulomb interaction. The latter method should be preferred since for higher neutrino energies the loss of accuracy is small but the advantage of an analytical solution is retained.

If the Coulomb interaction is neglected it is, of course, not necessary to expand electron and neutrino wave functions into spherical waves.

The summations over the different lepton spin directions of the lepton matrix element i.e. the lepton traces, can simply be carried out directly (see, for example, Walecka 1975; Donnelly and Walecka 1975).

For completeness we have, however, applied our formalism to neutrino cross-sections for higher neutrino energies where $p_\nu R < 1$, although it may be not so suitable for that purpose. Near the threshold, on the other hand, the strong analogy between beta-decay and its analogous process allows us to take additionally into account the final state Coulomb interaction in the same way as in beta-decay. In this latter case we are, therefore, easily able to derive neutrino reaction cross-sections which include Coulomb effects. For this reason they can be more properly

related to beta-decay observables than cross-sections which do not take care of the Coulomb interaction. Very often those cross-sections without final state Coulomb interaction are, however, compared with beta-decay results which include Coulomb effects, but near the neutrino reaction threshold the errors of this simplified procedure are of the same order of magnitude as they are if the beta-decay is treated without Coulomb interaction.

Low energy antineutrinos are produced by nuclear reactors. These antineutrinos have an energy spectrum peaked at energies $\simeq 0{\cdot}3$ MeV with an upper limit of $\simeq 10$ MeV. Their energy distribution is an average over a hundred of fission product beta-decays. These antineutrino spectra have, however, been calculated by some authors (Davis *et al.* 1979; Avignone *et al.* 1979) since their knowledge is needed for the interpretation of reactor antineutrino experiments. A survey over such neutrino experiments at nuclear reactors has been given by Borovoi (1980). Solar neutrino experiments have been reviewed by Kuchowicz (1976), Bilenky and Pontecorvo (1978), and Bahcall (1978).

12

SUBMICROSCOPIC DESCRIPTION OF NEUTRON DECAY AND GAUGE THEORIES

THE CONVENTIONAL Fermi-(V–A)-Cabibbo theory that we have introduced in Section 6.1 worked perfectly well for the description of the nuclear beta-decay and the low energy phenomena of its inverse process, but it should be kept in mind that this theory is not a fundamental one, the reason being that it is not renormalizable.

This means a perturbation expansion of the usual method diverges. Each order beyond the first one in such an expansion contains infinities which cannot be removed by a redefinition of masses, coupling constants and wave functions as in quantum electrodynamics. This situation has the consequence that when one tries to calculate anything other than the low energy transition amplitudes one gets either infinities or unphysical results. Two examples should be mentioned in that context.

First, if the (electromagnetic) radiative corrections to nuclear beta-decay, i.e. all contributions of the electromagnetic interaction beyond the electrostatic interaction are considered one finds that even the terms of order α are infinite (see the next chapter).

Second, the cross-sections for the processes $\nu_\mu + e^- \to \mu^- + \nu_e$ and $\nu_e + e^- \to \nu_e + e^-$, in the Fermi theory, increase with the square of the c.m. energy $\sigma \sim G^2 W_\nu^2$. On the other hand, unitary predicts these cross sections should be bounded ($\sigma < 4\pi\lambda^2$). At energies of about 300 GeV and 450 GeV, respectively, the results of the conventional theory are therefore in contradiction to unitarity (see, for example, Llewellyn Smith 1972).

For completeness we now give a brief description of the modern gauge theories although they are not of much relevance for nuclear beta-decay. We will therefore not go into much details and only discuss the nowadays preferred Weinberg–Salam or standard model (see, for example, Lee 1973; Bernstein 1974; Weinberg 1974; Beg and Sirlin 1974; Taylor 1976; Ryder 1977; Harari 1978; and Fadeev and Slavnov 1980). We will restrict ourselves to those parts of the theory which are of importance for a submicroscopic description of the neutron decay.

In that context we are confronted with the fact that neutrons and protons are not elementary particles. It is today assumed that they are built up of three quarks which are at the moment believed to be elementary. Since a fundamental weak interaction theory should deal with elementary particles we will, therefore, describe the neutron and

proton in terms of their submicroscopic constituents in the following treatment (see, for example, Kokkedee 1969; Greenberg 1978).

The u, d, and s fractionally charged quarks (Gell-Mann–Zweig model) are sufficient to construct the nucleons and these three quarks exist in three different colours (red, white, and blue). Usually the other degrees of freedom u, d, and s are denoted as flavour.† Just two of these quarks, the u and the d quark, are necessary to explain the structure of the neutron and proton. The u quark is assigned a charge of $+\frac{2}{3}$ and the d quark a charge of $-\frac{1}{3}$. The u quark and the d quark have isospin $\frac{1}{2}$ with the third components $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. The baryon number of both quarks is $+\frac{1}{3}$ and their strangeness is 0.

The neutron is then described by assembling two d quarks and one u quark, and the proton by assembling two u quarks and one d quark.‡ Then, the neutron beta-decay can be written as

$$\begin{matrix} d & d \\ u & \end{matrix} \rightarrow \begin{matrix} & d \\ u & u \end{matrix} + e^- + \bar{\nu}_e \tag{12.1}$$

with the elementary process

$$d \rightarrow u + e^- + \bar{\nu}_e. \tag{12.2}$$

The corresponding elementary hadron current is then (see eqn (6.4))

$$J^+_{\mu_{had}} = \mathrm{i} \cos\theta\, \bar{\psi}_u \gamma_\mu (1+\gamma_5)\psi_d \tag{12.3}$$

where θ is the Cabibbo angle (see eqn (6.2)).

Now the weak Hamiltonian density responsible for beta-decay is

$$H_\beta = \frac{G}{\sqrt{2}} \{J^+_{\mu_{had}} L_\mu + h.c.\}. \tag{12.4}$$

If we go one stage further and write this Hamiltonian in terms of self-interacting currents we obtain

$$H_\beta = \frac{G}{\sqrt{2}} J^+_\mu J_\mu \tag{12.5}$$

† There also exists another important variant of the original quark model (Han–Nambe model) where three non-identical triplets of quarks are introduced. All charges are integral in this case.

‡ In our definition the third isospin component of the neutron is

$$t_3(n) = -\{t_3(d) + t_3(d) + t_3(u)\}$$

and of the proton

$$t_3(p) = -\{t_3(d) + t_3(u) + t_3(u)\}.$$

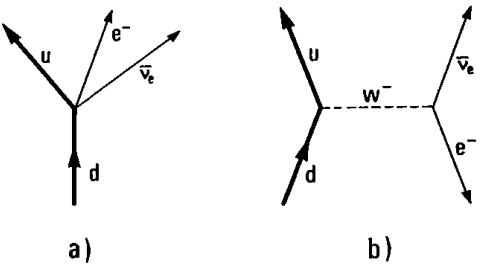


FIG. 12.1. Diagrams for the quark beta-decay (a) four point interaction; (b) intermediate vector boson exchange.

where now†

$$J_\mu = J_{\mu_{\text{had}}} + L_\mu. \tag{12.6}$$

Equation (12.5) is usually called the current–current interaction and represents the highest generalization of the conventional (four point) weak interaction formalism. If we now assume that the basic weak interaction will be governed by the exchange of a weak intermediate boson W (see Fig. 12.1), then the Hamiltonian will be of the form

$$H_\beta = g\{J_\mu^+ W_\mu + h.c.\} \tag{12.7}$$

where W is the intermediate boson field operator. Thus beta-decay is a process of second order in g ($G \propto g^2$), i.e.

$$\begin{array}{l} d \to u + W^- \\ \qquad\quad \hookrightarrow e^- + \bar{\nu}_e. \end{array} \tag{12.8}$$

Equation (12.7) is very similar to the form of the electromagnetic interaction given by

$$H_{\text{em}} = -eJ_\mu^{\text{em}} A_\mu \tag{12.9}$$

where J_μ^{em} is the electromagnetic current of the quarks and leptons, A_μ is the photonic field, and e is the electric charge.

The introduction of the intermediate vector boson hypothesis allows us

† Generally the lepton current L_μ has to be written as

$$L_\lambda^+ = i\bar{\psi}_{\nu_e}\gamma_\lambda(1+\gamma_5)\psi_e + \psi_{\nu_\mu}\gamma_\lambda(1+\gamma_5)\psi_\mu + \ldots$$

and the hadron current as

$$J_\lambda^+ = i\cos\theta\,\bar{\psi}_d\gamma_\lambda(1+\gamma_5)\psi_u + i\sin\theta\bar{\psi}_d\gamma_\lambda(1+\gamma_5)\psi_s + \ldots$$

Since we are, however, only interested in processes where u and d quarks, electrons and neutrinos together with their antiparticles are involved, the other terms are not considered here.

to express the universal weak interaction coupling constant in terms of g, such that‡

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{12.10}$$

where M_W is the mass of the charged intermediate boson. The intermediate vector boson theory does not, however, improve the overall situation very much since this theory is also not renormalizable.

The problems by which we are confronted are now, however, different from those in the Fermi theory.

If now the radiative corrections of order α to nuclear beta-decay are calculated it can be shown that the divergences cancel in the final answer if terms of order $\alpha q^2/M_W$ (q is the momentum transfer) are neglected. The latter terms are, however, divergent. The trouble with unitarity (in order g^2) although it exists, does not look so strange as before, because the unitarity bound is now only violated logarithmically. By considering the difficulties of the intermediate vector boson theory in more detail the finite mass of W is identified as the source why the theory is not renormalizable, i.e. the term $q_\mu q_\nu$ in the propagator.

For a massless W the theory would be renormalizable without problems. On the other hand, the intermediate vector boson W must certainly be massive (short range of the weak interaction). However, Higgs (1964) discovered that a certain class of theories (gauge theories) force massless and massive particles into a single family whereby the 'bare' masses of all particles are zero, but the finite masses may be generated by a so-called 'spontaneous breakdown of symmetry'. On the basis of these new gauge theories Weinberg and Salam (Weinberg 1967; Salam 1968) developed a model which unifies weak and electromagnetic interaction. In this model the 'bare' mass of the intermediate vector boson is zero, but it picks up a large mass from the symmetry breaking. Later on it was indeed shown by t'Hooft (1971) that this model is renormalizable (for reviews see Weinberg 1974; Beg and Sirlin 1974; Taylor 1976; Bailin 1977; Harari 1978; Faddeev and Slavnov 1980).

Let us now begin with a discussion of how the lepton part of this theory is constructed. The treatment of the hadron part will follow later on.

Weinberg and Salam assumed that the left handed (negative helicity) component of the electron and its neutrino transform as a SU(2) doublet (weak isospin $I^W = \frac{1}{2}$) while the right handed (positive helicity)

‡ The vector propagator for a boson of mass M_W has the form

$$\Delta_{\mu\nu} = \frac{\delta_{\mu\nu} + q_\mu q_\nu / M_W^2}{q_\mu^2 + M_W^2}.$$

component of the electron was chosen as a singlet,† i.e.

$$I^W = \tfrac{1}{2} \qquad L = \left(\frac{1+\gamma_5}{2}\right)\begin{pmatrix}\nu_e\\ e\end{pmatrix} = \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L \tag{12.11a}$$

$$I^W = 0 \qquad R = \left(\frac{1-\gamma_5}{2}\right)e = e_R \tag{12.11b}$$

For reasons of simplicity a shorthand notation for the field operators of the electron, neutrino, and quarks is now introduced which will be used in the following (the imaginary unit i is also omitted). The lepton current (electron neutrino part) then reads as

$$\begin{aligned} L_\mu &= \bar{L}\gamma_\mu t_- L + \bar{L}\gamma_\mu t_+ L \\ &= \bar{e}_L \gamma_\mu \nu_e + \bar{\nu}_e \gamma_\mu e_L \\ &= J_{\mu^-} + J_{\mu^+}. \end{aligned} \tag{12.12}$$

The first part corresponds here to the charge lowering current $\nu \rightarrow e^-$ and the second part to the charge raising current $\bar{\nu}_e \rightarrow e^+$. The $t_\mp = t_1 \mp i t_2$ are the usual charge lowering and raising isospin operators introduced before. Since

$$[t_+, t_-] = 2t_3 \tag{12.13}$$

we have

$$[J_+, J_-] = 2J_3 = \bar{\nu}_e \gamma_\mu \nu_e - \bar{e}_L \gamma_\mu e_L. \tag{12.14}$$

The condition for renormalizability, that the physical current generate a symmetry group, requires that J_3 is a physical current. It is a neutral current and represents the physical processes $\nu_e \rightarrow \nu_e$ (elastic neutrino scattering) and $e_L \rightarrow e_L$ (electron scattering of left handed electrons).

Thus neutral current reactions like the process

$$\nu_e + A \rightarrow \nu_e + A^*$$

are predicted. The electromagnetic current J^{em} as a whole is then given by

$$\begin{aligned} -J^{em} = \bar{e}\gamma_\mu e &= \bar{e}\gamma_\mu \left(\frac{1+\gamma_5}{2}\right)e + \bar{e}\gamma_\mu \left(\frac{1-\gamma_5}{2}\right)e \\ &= \bar{e}_L \gamma_\mu e_L + \bar{e}_R \gamma_\mu e_R \\ &= \bar{L}\gamma_\mu (\tfrac{1}{2} - t_3) L + \bar{R}\gamma_\mu R \\ &= -J_3 + \tfrac{1}{2}(\bar{L}\gamma_\mu L + 2\bar{R}\gamma_\mu R). \end{aligned} \tag{12.15}$$

Notice that we have now four different currents: three triplet currents

$$J_+, J_-, \text{ and } J_3$$

† A similar choice is made for the other leptons.

and the singlet current

$$J_Y = \tfrac{1}{2}(\bar{L}\gamma_\mu L + 2\bar{R}\gamma_\mu R). \tag{12.16}$$

The electromagnetic current J^{em} is just given by a linear combination of J_3 and J_y. For the construction of the interaction Hamiltonian we need four different vector bosons. This interaction density with symmetry SU(2)×U(1) can then be written as†

$$H_{int} = g\left\{\frac{1}{\sqrt{2}} J^+_{\mu_+} W_{\mu_+} + \frac{1}{\sqrt{2}} J^+_{\mu_-} W_{\mu_-} + J_{\mu_3} W_{\mu_3}\right\} + g' J_{\mu_Y} Y_\mu. \tag{12.17}$$

It is practical to eliminate J_Y in favour of J_3 and J^{em}. Thus

$$H_{int} = g\left\{\frac{1}{\sqrt{2}} J^+_{\mu_+} W_{\mu_+} + \frac{1}{\sqrt{2}} J^+_{\mu_-} W_{\mu_-} + J_{\mu_3} W_{\mu_3}\right\} + g'(J_{\mu_3} - J^{em}_\mu) Y_\mu. \tag{12.18}$$

This interaction Hamiltonian must contain the usual electromagnetic interaction of eqn (12.9). Such a term can be extracted by writing W_3 and Y as an appropriate linear combination of the two neutral boson fields Z_μ and A_μ. One takes

$$W_{\mu_3} = \cos\theta_W Z_\mu - \sin\theta_W A_\mu \tag{12.19a}$$

$$Y_\mu = \sin\theta_W Z_\mu + \cos\theta_W A_\mu \tag{12.19b}$$

and obtains

$$\begin{aligned} H_{int} = {} & g\left\{\frac{1}{\sqrt{2}} J^+_{\mu_+} W_{\mu_+} + \frac{1}{\sqrt{2}} J^+_{\mu_-} W_{\mu_-}\right\} \\ & + \{g\cos\theta_W J_{\mu_3} + g'\sin\theta_W J_{\mu_3} - g'\sin\theta_W J^{em}_\mu\} Z_\mu \\ & - \{g\sin\theta_W J_{\mu_3} - g'\cos\theta_W J_{\mu_3} + g'\cos\theta_W J^{em}_\mu\} A_\mu \end{aligned} \tag{12.20}$$

where θ_W is the Weinberg angle.

If we require

$$g\sin\theta_W = g'\cos\theta_W = e \tag{12.21}$$

the electromagnetic part is of the standard form. Then, we get finally

$$\begin{aligned} H_{int} = {} & -eJ^{em}_\mu A_\mu + \frac{e}{\sin\theta_W\cos\theta_W}(J_{\mu_3} - \sin^2\theta_W J^{em}_\mu) Z_\mu \\ & + \frac{e}{\sin\theta_W\sqrt{2}}(J^+_{\mu_+} W_{\mu_+} + J^+_{\mu_-} W_{\mu_-}). \end{aligned} \tag{12.22}$$

† $W_{\mu_\pm} = \frac{1}{\sqrt{2}}(W_{\mu_1} \pm iW_{\mu_2})$.

The first part of the above interaction represents the usual electromagnetic interaction, the second part the coupling of the Z-vector boson to the neutral current

$$J_\mu^N = J_{\mu_3} - \sin^2\theta_W J_\mu^{em} \tag{12.23}$$

and the last part the coupling of the usual weak interaction with a charged intermediate boson. The universal weak interaction coupling constant G is now, however, related to the elementary charge e by (see eqns (12.10) and (12.21))

$$G = \frac{e^2}{4\sqrt{2}\, M_W^2 \sin^2\theta_W} \tag{12.24}$$

if the mass of the $W^\pm$-fields is such that the Hamiltonian density obtained by exchanging W's among charged currents is reduced to the phenomenological current–current form at low momentum transfer.

The above equation holds in natural units. It can, therefore, also be written as

$$M_W^2 = \frac{\pi\alpha}{\sqrt{2}\, G} \frac{1}{\sin^2\theta_W}. \tag{12.25}$$

By inserting the values for the fine structure constant α ($\alpha = 1/137{\cdot}036$) and for the universal weak vector coupling constant G this gives for M_W, in GeV

$$M_W = \frac{37{\cdot}3}{\sin\theta_W}\ \text{GeV}. \tag{12.26}$$

Hence the Weinberg–Salam theory sets a lower limit on the mass of the charged intermediate vector boson of 37 GeV.

As mentioned earlier, the relevant mechanism for the generation of finite masses is that of spontaneous symmetry breaking (Higgs mechanism). In the Weinberg–Salam model we have (see Weinberg 1967)

$$M_W = \eta g \tag{12.27}$$

$$M_Z = \eta\sqrt{(g^2 + g'^2)} \tag{12.28}$$

and therefore

$$M_Z = \frac{M_W}{\cos\theta_W} \tag{12.29}$$

or

$$M_Z = \frac{74{\cdot}6}{\sin(2\theta_W)}\ \text{GeV}. \tag{12.30}$$

It should now be noticed that the weak interactions are weak because the

intermediate vector bosons are heavy. The coupling constants themselves are of the same order of magnitude as those of electromagnetism. Intrinsically the weak interactions are, therefore, not weak.

The important parameter θ_W, the Weinberg angle, cannot be derived from the theory itself. It has to be determined experimentally, for instance, from processes of the type

$$\nu_\mu + e^- \rightarrow \nu_\mu + e^- \tag{12.31a}$$

$$\bar{\nu}_\mu + e^- \rightarrow \bar{\nu}_\mu + e^- \tag{12.31b}$$

$$\bar{\nu}_e + e^- \rightarrow \bar{\nu}_e + e^- \tag{12.31c}$$

$$\nu + N \rightarrow \nu + N. \tag{12.31d}$$

A value of

$$\sin^2\theta_W = 0{\cdot}228 \pm 0{\cdot}010 \tag{12.32}$$

is now available from the different studies† (see Bricman *et al.* 1980 and the references quoted therein).

It remains to extend the gauge formalism to hadrons. Following a suggestion made first by Glashow, Iliopoulos, and Maiani (1970) (four quarks were, however, only included at that time) three left handed doublets

$$L_1 = \frac{1+\gamma_5}{2}\begin{pmatrix} u \\ d_c \end{pmatrix} = \begin{pmatrix} u \\ d_c \end{pmatrix}_L \tag{12.33}$$

$$I^W = \tfrac{1}{2} \qquad L_2 = \frac{1+\gamma_5}{2}\begin{pmatrix} c \\ s_c \end{pmatrix} = \begin{pmatrix} c \\ s_c \end{pmatrix}_L \tag{12.34}$$

$$L_3 = \frac{1+\gamma_5}{2}\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{12.35}$$

and six right handed singlets

$$I_W = 0 \qquad R_n = \frac{1-\gamma_5}{2}\, n \tag{12.36}$$

are introduced where n sums over all six quarks u, c, t, d_c, s_c, and b. The quarks d or s are not coupled directly to the quarks u or c, but rather their Cabibbo rotated combinations

$$d_c = d\cos\theta + s\sin\theta \tag{12.37}$$

$$s_c = -d\sin\theta + s\cos\theta. \tag{12.38}$$

† By using this value we obtain for the masses M_W and M_Z from the eqns (12.26) and (12.30)

$$M_W = 78{\cdot}1 \pm 1{\cdot}7 \text{ GeV}$$
$$M_Z = 88{\cdot}9 \pm 1{\cdot}4 \text{ GeV}.$$

s denotes the strange quark, c the charmed quark, b the bottom quark, and t the top quark.

The three pairs of quarks exist also in three different colours. In our context of nuclear beta-decay the left handed doublet L_1 and the right handed singlet u_R and d_{c_R} are of most importance. We will therefore not consider the other parts further even if they are, for example, a condition *sine qua non* for the exclusion of $\Delta S = 1$ neutral currents. Analogous to eqn (12.12) we then have

$$\begin{aligned} J_{\mu_{-\text{had}}} &= \sum_{n=1}^{3} \bar{L}_n \gamma_\mu t_- L_n \\ &= \bar{d}_{c_L} \gamma_\mu u_L + \dots \\ &= \tfrac{1}{2} \cos\theta\, \bar{d} \gamma_\mu (1+\gamma_5) u + \tfrac{1}{2} \sin\theta\, \bar{s} \gamma_\mu (1+\gamma_5) u + \dots \end{aligned} \tag{12.39}$$

and

$$\begin{aligned} J_{\mu_{+\text{had}}} &= \sum_{n=1}^{3} \bar{L}_n \gamma_\mu t_+ L_n \\ &= \bar{u}_L \gamma_\mu d_{c_L} + \dots \\ &= \tfrac{1}{2} \cos\theta\, \bar{u} \gamma_\mu (1+\gamma_5) d + \tfrac{1}{2} \sin\theta\, \bar{u} \gamma_\mu (1+\gamma_5) s + \dots \end{aligned} \tag{12.40}$$

As before (see eqns (12.13) and (12.14)) we can form the commutator

$$\begin{aligned} [J_+, J_-] &= 2J_3 \\ &= 2 \sum_{n=1}^{3} \bar{L}_n \gamma_\mu t_3 L_n \end{aligned} \tag{12.41}$$

and the neutral hadronic current J_3.

Next, let us consider the electromagnetic current. It can be written as

$$J^{\text{em}}_{\mu_{\text{had}}} = \tfrac{2}{3}(\bar{u}\gamma_\mu u + \bar{c}\gamma_\mu c + \bar{t}\gamma_\mu t) - \tfrac{1}{3}(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu s + \bar{b}\gamma_\mu b) \tag{12.42}$$

or in terms of left and right handed projections

$$J^{\text{em}}_{\mu_{\text{had}}} = J_3 + \tfrac{1}{6} \sum_{n=1}^{3} \bar{L}_n \gamma_\mu L_n + \tfrac{2}{3}(\bar{R}_u \gamma_\mu R_u + \dots) - \tfrac{1}{3}(\bar{R}_d \gamma_\mu R_d + \dots) \tag{12.43}$$

i.e. the singlet current J_Y is now defined as

$$-J_{Y_{\text{had}}} = \tfrac{1}{6} \sum_{n=1}^{3} \bar{L}_n \gamma_\mu L_n + \tfrac{2}{3}(\bar{R}_u \gamma_\mu R_u + \dots) - \tfrac{1}{3}(\bar{R}_d \gamma_\mu R_d + \dots). \tag{12.44}$$

The quark–vector boson interaction can now be expressed in straight

analogy with the lepton–vector boson interaction. It then reads as†

$$H_{\text{int}} = -eJ^{\text{em}}_{\mu_{\text{had}}}A_\mu + \frac{e}{\sin\theta_W \cos\theta_W}(J_{\mu_{3_{\text{had}}}} - \sin^2\theta_W J^{\text{em}}_{\mu_{\text{had}}})Z_\mu$$

$$+ \frac{e}{\sin\theta_W\sqrt{2}}(J^+_{\mu_{+_{\text{had}}}}W_{\mu_+} + J^+_{\mu_{-_{\text{had}}}}W_{\mu_-}). \tag{12.45}$$

The weak neutral current term

$$J^N_{\mu_{\text{had}}} = J_{\mu_{3_{\text{had}}}} - \sin^2\theta_W J^{\text{em}}_{\mu_{\text{had}}}$$

has now the explicit form

$$J^N_{\mu_{\text{had}}} = \tfrac{1}{2}(\bar{u}_L\gamma_\mu u_L - \bar{d}_L\gamma_\mu d_L) + (\bar{c}_L\gamma_\mu c_L - \bar{s}_L\gamma_\mu s_L) + \ldots - \sin^2\theta_W J^{\text{em}}_\mu. \tag{12.46}$$

The structure of our unified model for the weak and electromagnetic interaction discussed above can be summarized as follows. The gauge group is SU(2)×U(1) with the left handed doublets and right handed singlets as given by:

leptons:
$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L$$
$$e^-_R \quad \mu^-_R \quad \tau^-_R \tag{12.47}$$

quarks:
$$\begin{pmatrix}u\\ d_c\end{pmatrix}_L \quad \begin{pmatrix}c\\ s_c\end{pmatrix}_L \quad \begin{pmatrix}t\\ b\end{pmatrix}_L$$
$$u_R \quad d_{c_R} \quad c_R \quad s_{c_R} \quad t_R \quad b_R \tag{12.48}$$

τ is the τ-lepton and ν_τ the associated neutrino.

For the renormalizability of gauge theories one additional condition exists (condition under which the gauge model is free from triangle anomalies) which can simply be cast in the form (see Bardeen 1969; Wess and Zumino 1971)

$$\sum_i (T^i_3)^2 Q^i = 0 \tag{12.49}$$

where T_3 denotes the third component of weak isospin and Q^i is the

† Because of the gauge symmetry SU(2)×U(1) the corresponding Lagrangian is invariant under a gauge transformation

$$\psi \to e^{i\frac{1}{2}g\omega\tau + g'\omega'}\psi$$

where the ψ are the left handed doublets L and right handed singlets R, $\tau = \{\tau_1, \tau_2, \tau_3\}$ the Pauli matrices, ω an arbitrary vector, ω' an arbitrary scalar, g and g′ coupling constants. The *conserved* currents which belong to that gauge invariance are

$$j^\nu_\mu = \bar{L}\gamma_\mu \mathbb{t} L \qquad j^\nu_\mu = \bar{R}\gamma_\mu R.$$

charge. The sum runs over all fundamental fields involved in the model. In our case it sufficies to state that these difficulties are avoided if the total charge of all fermions is zero. It is easily checked that the above relation is fulfilled if the fundamental fermions are the six quarks with colour and the six leptons. Their existence is, therefore, nowadays assumed even if they are not all fully established at the moment.

The mixing between the d and s quark introduced in eqns (12.37) and (12.38) in our case of six quarks is not the most general one. The mixing can occur more generally between all quarks of charge $-\frac{1}{3}$ and can then be expressed by (see, for example, Harari 1978)

$$\begin{pmatrix} d_c \\ s_c \\ b_c \end{pmatrix} = A \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{12.50}$$

where

$$A = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 + s_2 s_3 e^{i\delta} & c_1 c_2 s_3 - s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 - c_2 s_3 e^{i\delta} & c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix} \tag{12.51}$$

is the Kobayashi–Maskawa mixing matrix (Kobayashi and Maskawa 1973).†

Here we have $c_i = \cos\theta_i$, $s_i = \sin\theta_i$ for $i = 1, 2, 3$. CP violation of the $K^0 \to \pi\pi$ decay is incorporated which enters via the phase δ.

In the limit $\theta_2 = \theta_3 = \delta = 0$ this matrix reduces to

$$A = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12.52}$$

† It cannot be excluded that neutrinos have a finite rest mass. For neutrinos with non-zero masses a mismatch between mass eigenstates and weak eigenstates can on the other hand occur. Under these circumstances neutrino oscillations are to be expected analogous to the K^0 and $\overline{K^0}$ system (here K_L and K_S have definite masses while for K^0 and $\overline{K^0}$ that is not the case).

The charged-current neutrino eigenstates ν_α ($\alpha = e, \mu, \tau$) are then related to the mass eigenstates ν_i with masses m_i ($i = 1, 2, 3$) by a Kobayashi–Maskawa matrix A as given in eqn (12.51)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = A \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ are the mixing angles and δ is the CP non-conservation phase (see Bilenky and Pontecorvo 1978; Wolfenstein 1978). It is amusing that we then would have a situation for the neutrinos which is completely analogous to that for the d, s, and b quarks. Recently workers have tried to determine experimentally the different mixing angles θ_i and the CPN phase δ (see de Rujula *et al.* 1980; Reines *et al.* 1980; Barger *et al.* 1980). A definite conclusion cannot, however, be drawn up to now although some evidence for such kind of neutrino oscillations has been found.

i.e. to the usual Cabibbo mixing (see eqns (12.37) and (12.38)) where θ_1 denotes the Cabibbo angle.

The following values for the parameters θ_1, θ_2, and θ_3 are reported at the moment (see Bricman *et al.* 1980)

$$|\cos\theta_1| = 0{\cdot}9737 \pm 0{\cdot}0025 \tag{12.53a}$$

$$|\sin\theta_1 \cos\theta_3| = 0{\cdot}219 \pm 0{\cdot}011 \tag{12.53b}$$

$$|\sin\theta_1 \sin\theta_3| = 0{\cdot}06 \pm 0{\cdot}06. \tag{12.53c}$$

θ_2 and δ cannot be determined without additional assumptions. Here, the charged quark current can be written as

$$J_{\mu_{+\text{had}}} = (\bar{u}\bar{c}\bar{t})\gamma_\mu \frac{1+\gamma_5}{2} A \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{12.54}$$

As it has been demonstrated above the gauge formalism unifies weak and electromagnetic interaction. Thus, this model should inherently also contain the conserved vector current theory. In the gauge theories the strangeness conserving part of the hadronic charged weak current is identified with the operator (see eqn (12.40))

$$O = \bar{u}\gamma_\mu(1+\gamma_5)d \tag{12.55}$$

whereby the vector part is represented by the operator

$${}^{V}O = \bar{u}\gamma_\mu d. \tag{12.56}$$

Since the isospin wave functions of u and d quark are just given by

$$\Phi_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Phi_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{12.57}$$

which satisfy the relations

$$\Phi_u = t_+\Phi_d \qquad t_-\Phi_d = 0 \tag{12.58a}$$

$$\Phi_d = t_-\Phi_u \qquad t_+\Phi_u = 0 \tag{12.58b}$$

where $t_\pm$ are the usual charge raising and lowering isospin operators. Thus eqn (12.56) can be written in the isospin formalism† as follows

$${}^{V}O_\mu^+ = \Phi_u\gamma_\mu t_+\Phi_d \tag{12.59}$$

or more generally as

$${}^{V}O_\mu^+ = \sum_{n=1}^{2}\sum_{n'=1}^{2} \bar{q}_{n'}\gamma_\mu t_+ q_n \tag{12.60}$$

† Here we are dealing with the ordinary strong isospin which is different from the weak isospin introduced before. These two should not be confused.

where the n represent the different isospin components $\pm\frac{1}{2}$. Analogously, we have in addition

$$^{V}O_{\mu}^{-} = \sum_{n=1}^{2}\sum_{n'=1}^{2} \bar{q}_{n'}\gamma_{\mu}t_{-}q_{n} \tag{12.61}$$

The operator of the electromagnetic current is, on the other hand, given by (see eqn (12.42))

$$O_{\mu}^{em} = \tfrac{2}{3}\bar{u}\gamma_{\mu}u - \tfrac{1}{3}\bar{d}\gamma_{\mu}d. \tag{12.62}$$

By making use of the Gell-Mann–Nishijima relation

$$Q = t_3 + \tfrac{1}{2}Y \tag{12.63}$$

where Q denotes the charge and Y ($Y = \frac{1}{3}$) the hypercharge of the quarks this relation can be expressed as

$$O_{\mu}^{em} = \tfrac{1}{6}(\bar{u}\gamma_{\mu}u + \bar{d}\gamma_{\mu}d) + \bar{u}\gamma_{\mu}t_3u + \bar{d}\gamma_{\mu}t_3d \tag{12.64}$$

or more generally as

$$O_{\mu}^{em} = \sum_{n=1}^{2}\{\tfrac{1}{6}\bar{q}_n\gamma_{\mu}q_n + \bar{q}_n\gamma_{\mu}t_3q_n\}. \tag{12.65}$$

From eqns (12.60), (12.61), and (12.65) it then follows immediately that the electromagnetic isovector and the weak vector current are different components of one and the same current in isospin space. This is, however, just the requirement of the CVC theory in its strong form. If flavor-SU_3 symmetry holds they are automatically conserved. We recognize that these currents transform, therefore, under charge symmetry (see eqn (10.126)), as charge symmetric. Because time reversal and CPT invariance is not violated by the formalism outlined above, we also find, following the discussion in Section 10.4, that second class vector currents are absent.

We can now repeat the whole discussion for the axial vector current† which is represented by the operator

$$^{A}O_{\mu} = \bar{u}\gamma_{\mu}\gamma_5 d$$

and end up with

$$^{A}O_{\mu}^{\mp} = \sum_{n=1}^{2}\sum_{n'=1}^{2} \bar{q}_{n'}\gamma_{\mu}\gamma_5 t_{\mp}q_{n}.$$

† In this model the axial vector coupling constant λ, i.e. the ratio f_A/f_V, can easily be calculated if each quark is assumed to move independently in a scalar potential $U(r)$. Then, a value of

$$\lambda = \tfrac{5}{3}(1-2\delta)$$

is obtained with δ being a relativistic correction (see for example, Kokkedee 1969).

This operator, which is one of the approximately conserved currents of chiral $SU(2) \times SU(2)$ symmetry, transforms also under charge symmetry (see eqn (10.126)) as charge symmetric. Thus, the axial vector current is also of purely first class type.

13
RADIATIVE CORRECTIONS

THE PRIMARY purpose of this book is to describe how the electrostatic part of the Coulomb interaction influences the beta-decay observables via the distorted electron radial wave functions. No attempt will be made, however, to discuss other aspects of the topic 'Coulomb interaction and nuclear beta-decay' which have to be treated by applying the techniques of 'Feynman diagrams' in detail. Coulomb corrections which are not of a static nature are called radiative corrections (if we, for the moment, exclude atomic cloud effects like autoionization, autoexcitation etc.). They have usually to be handled by using the Feynman diagram approach and are therefore far outside the scope of this text. On the other hand, for a detailed comparison of experimental and theoretical beta-decay observables these radiative corrections are sometimes needed and cannot be neglected in some cases. Thus, for completeness we nevertheless will present the most important results (final formulae) and give a short survey over the whole field without going into details and without giving any derivations.

Of course, the radiative corrections depend on the form of the underlying weak and strong interaction models. It has, however, been shown (Sirlin 1967) that these corrections can be split off into two parts: one part which is independent of the details of the strong and weak interaction and usually called 'outer radiative corrections', and another part which depends on the special form of the weak and strong interaction theory and is usually called 'inner radiative corrections'.

The former type of corrections (outer radiative corrections) affects the electron energy dependence of the observables, e.g. the form of the beta-spectrum, and the latter (inner radiative corrections) renormalize the coupling constants but do not otherwise change the specific form of the formulae for the observables.

Let us now start with the outer radiative corrections. Since they are independent of any strong and weak interaction details it is obvious to compute them by applying the simplest possible model. Thus, impulse approximation (see Section 8.1) and point like nucleons (nucleons having no anomalous magnetic moment) are assumed. The lowest order (order α, $Z\alpha^2$ and $Z^2\alpha^3$) of outer radiative corrections to the nuclear beta-decay are based on contributions from the Feynman diagrams shown in Fig. 13.1.

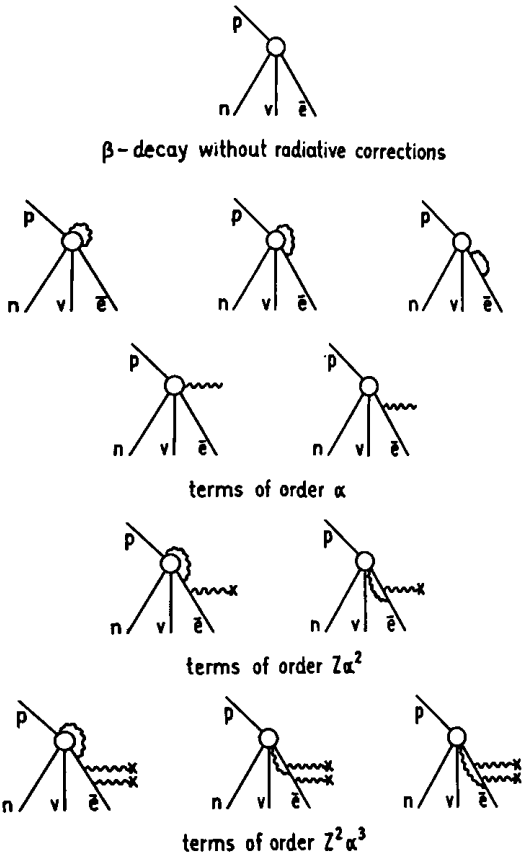


FIG. 13.1. Feynman diagrams for the radiative corrections.

Then, the following outer radiative corrections† for different observables are obtained.

(i) *Shape factor*

The radiative correction $\delta_R(W_e, Z)$ for the shape factor can be written in the form

$$C_R(W_e) = C(W_e)[1 + \delta_R(W_e, Z)]. \tag{13.1}$$

† After introducing the Coulomb interaction into the transition matrix T we had made the approximation to set the Coulomb potential $V_C^l(r)$ equal to $V_C^{l'}(r)$ (see Section 6.3, eqns (6.79) and (6.80)). That means, in Section 6.3 we had neglected the charge change in nuclear beta-decay. This effect of charge change is, however, included in the radiative corrections discussed in this chapter (see Berman 1958).

$\delta_R(W_e, Z)$ is a sum of terms of order α, $Z\alpha^2$ and $Z^2\alpha^3$ which are denoted by δ_1, δ_2 and δ_3, respectively,

$$\delta_R(W_e, Z) = \delta_1(W_e) + \delta_2(W_e, Z) + \delta_3(W_e, Z). \tag{13.2}$$

The term of order α can be written as (see Berman 1958; Kinoshita and Sirlin 1959; Berman and Sirlin 1962; Sirlin 1967)

$$\delta_1(W_e) = \frac{\alpha}{2\pi} g(W_e, W_0) \tag{13.3}$$

where (Sirlin (1967)

$$\begin{aligned} g(W_e, W_0) = {} & 3 \ln M_N - \frac{3}{4} + 4\left(\frac{\operatorname{arctanh} \beta}{\beta} - 1\right) \\ & \times \left\{\frac{W_0 - W_e}{3W_e} - \frac{3}{2} + \ln[2(W_0 - W_e)]\right\} + \frac{4}{\beta} L\left(\frac{2\beta}{1+\beta}\right) \\ & + \frac{1}{\beta}(\operatorname{arctanh} \beta)\left[2(1+\beta^2) + \frac{(W_0 - W_e)^2}{6W_e^2} - 4 \operatorname{arctanh} \beta\right]. \end{aligned} \tag{13.4}$$

In this equation W_0 is the end point energy of the electron, $\beta = p_e/W_e$ is the ratio of the electron velocity to the velocity of light and $L(x)$ is the Spence function defined by

$$L(x) = \int_0^x \frac{dt}{t} \ln(1-t). \tag{13.5}$$

The terms of order $Z\alpha^2$ and $Z^2\alpha^3$ read as (Jaus and Rasche 1970; Jaus 1972)

$$\delta_2(W_e, Z) = |Z|\, \alpha^2 \ln M_N + \ldots \tag{13.6}$$

$$\delta_3(W_e, Z) = Z^2\alpha^3\pi^{-1}(3 \ln 2 - \tfrac{3}{2} + \tfrac{1}{3}\pi^2) \ln M_N + \ldots \tag{13.6}$$

In these expressions the dots stand for energy dependent terms. They have been calculated for $\delta_2(W_e, Z)$ and came out to be some 10% of the first term (Jaus and Rasche 1970). It is believed that the terms of order $Z\alpha^2$ and $Z^2\alpha^3$ have no remarkable influence on the spectral shape (at a 0·1% level), but they are of importance for the *ft*-value. The above formulae demonstrate also that $\delta_3(W_e, Z)$ is smaller than $\delta_2(W_e, Z)$ by a factor of about αZ. Thus, corrections of the form $Z^n\alpha^{n+1}$ with $n > 2$ may be neglected.†

† It has been shown by Beg *et al.* (1972) that the radiative corrections for superallowed Fermi transitions (see Section 14.2.2.1) can generally be written in the form

$$\delta(W_e, Z) = \Delta_R^{(V)} = \sum_{\substack{m=1 \\ n=0}}^{\infty} a_{mn}\alpha^m(\alpha Z)^n$$

i.e. no term exists where the power of Z is higher than the power of α.

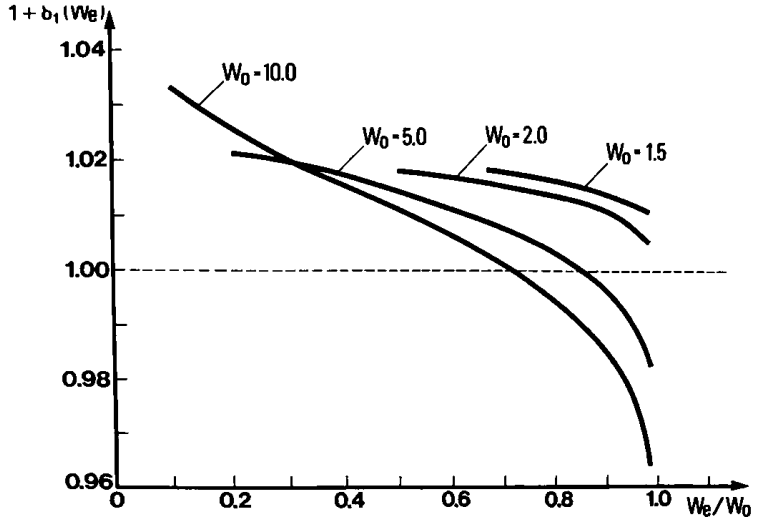


FIG. 13.2. Outer radiative correction $1+\delta_1(W_e)$ to the beta-spectrum (see eqn (13.3)).

Here, it should be pointed out that the outer radiative corrections, especially the energy dependent term $(\alpha/2\pi)\, g(W_0, W_e)$, are of the order of some percent. In the case of precise measurements of the shape factor they have therefore always to be taken into account (see Fig. 13.2).

(ii) *ft-value*

In principle, these outer radiative corrections are determined by those of the shape factor. They follow from eqn (13.1) to

$$(ft)_R = ft[1+\overline{\delta(W_e, Z)}] \tag{13.8}$$

whereby $\overline{\delta_R(W_e, Z)}$ means $\delta_R(W_e, Z)$ averaged over the beta-spectrum. As shown before, the terms of order $Z\alpha^2$ and $Z^2\alpha^3$ are essentially energy independent. The function which has therefore to be averaged over the beta-spectrum is $g(W_0, W_e)$. Numerical values for $g(W_0, W_e)$ have been evaluated by Wilkinson and Macefield (1970). In many cases a good approximation for $\overline{g(W_0, W_e)}$ is, however, given by Wilkinson and Macefield 1970) for $W_0 \to 1$

$$\overline{g(W_0, W_e)} = 3 \ln M_N - \tfrac{27}{4} + \tfrac{8}{9} p_0^2 [\ln 2p_0 - \tfrac{668}{315}] \tag{13.9a}$$

for $W_0 \to \infty$

$$\overline{g(W_0, W_e)} = 3 \ln \frac{M_N}{2W_0} + \frac{81}{10} - \frac{4\pi^2}{3} \tag{13.9b}$$

which is valid asymptotically for smaller and larger W_0, respectively.

TABLE 13.1 *Radiative corrections* $\overline{\delta(W_e, Z)}$ *for the* $0^+ - 0^+$ *superallowed transitions*

Parent nucleus	$\overline{\delta_1(W_e)}$ (%)	$\overline{\delta_2(W_e, Z)}$ (%)	$\delta_3(W_e, Z)$ (%)
^{10}C	1·48	0·18	0·01
^{14}O	1·30	0·26	0·02
^{18}Ne	1·22	0·33	0·03
^{22}Mg	1·13	0·41	0·04
^{26m}Al	1·12	0·44	0·05
^{26}Si	1·07	0·49	0·06
^{30}S	1·02	0·56	0·08
^{34}Cl	1·01	0·60	0·09
^{34}Ar	0·97	0·63	0·10
^{38m}K	0·98	0·67	0·12
^{38}Ca	0·94	0·71	0·13
^{42}Se	0·95	0·75	0·14
^{42}Ti	0·92	0·78	0·16
^{46}V	0·92	0·82	0·17
^{46}Cr	0·89	0·86	0·19
^{50}Mn	0·88	0·90	0·21
^{54}Co	0·85	0·97	0·24
^{62}Ga	0·83	1·11	0·32

In order to give an indication of how important this type of corrections for the *ft*-value is, a compilation of numerical values is given in Table 13.1 for the very important superallowed $0^+ - 0^+$ transitions (see Section 14.2.2.1).

(iii) *Asymmetry of electrons emitted from oriented nuclei*

The outer radiative correction of order α for the asymmetry coefficient a_1 of electrons emitted from oriented nuclei (see eqns (7.31) and (14.147)) is given by (Yokoo *et al.* 1973; Yokoo and Morita 1976; Garcia and Maya 1978)

$$a_{1_R} = a_1\left[1+\frac{\alpha}{2\pi}h(W_e, W_0)\right] \tag{13.10}$$

where

$$h(W_e, W_0) = 4\left(\frac{\operatorname{arctanh}\beta}{\beta}-1\right)\left[\frac{W_0-W_e}{3p_e^2W_e}+\frac{(W_0-W_e)^2}{24p_e^2}\right]$$

$$+\frac{1}{\beta}\operatorname{arctanh}\beta\left[\frac{2}{W_e^2}-\frac{(W_0-W_e)^2}{6W_e^2}\right]. \tag{13.11}$$

In order to get a feeling for the order of magnitude of this correction we

can consider the limits $\beta \to 0$ and $\beta \to 1$. We then obtain

$$\beta \to 0 \qquad h(W_e, W_0) = 2 + \tfrac{4}{9}(W_0 - 1) - \tfrac{1}{9}(W_0 - 1)^2 \tag{13.12}$$

$$\beta \to 1 \qquad h(W_e, W_0) = -\frac{1}{6}\left(\frac{W_0 - W_e}{W_e}\right)^2 \tag{13.13}$$

(iv) *Electron–neutrino angular correlation*

In this case the outer radiative correction of order α to the asymmetry coefficient d_1 (see eqns (7.97) and (14.198)) has the same form as before, i.e. (Yokoo and Morita 1976; see also Garcia and Maya 1978)

$$d_{1_R} = d_1\left[1 + \frac{\alpha}{2\pi} h(W_e, W_0)\right]. \tag{13.14}$$

As has been mentioned in Section 7.2.4.1 the electron–neutrino angular correlation determines also the recoil momentum spectrum (see eqns (7.104) and (14.211)). Therefore we have obtained the radiative correction for that observable in addition if we insert the above expression into eqn (7.104) or into eqn (14.211), respectively (see also Christian and Kühnelt 1978).

(v) *Electron longitudinal polarization*

Here, the outer radiative correction to the longitudinal polarization P_e of the electron can be expressed as (Sirlin 1967)

$$P_{e_R} = P_e\left[1 + \frac{\alpha}{\pi}\frac{1-\beta^2}{\beta}\operatorname{arctanh}\beta\right]. \tag{13.15}$$

For the definition of P_e see eqns (7.151) or (14.239), respectively. In this case, however, inner radiative corrections do not contribute. Thus the above formula represents the total radiative correction and holds rigorously.

In the limits $\beta \to 0$ and $\beta \to 1$ we have

$$\text{for } \beta \to 0 \qquad P_{e_R} = P_e\left[1 + \frac{\alpha}{\pi}\right] \tag{13.16}$$

$$\text{for } \beta \to 1 \qquad P_{e_R} = P_e \tag{13.17}$$

(vi) *Electron capture to positron decay ratio*

For the electron capture to positron decay ratio $\lambda_c/\lambda_{\beta^+}$ (see eqns (14.144) and (14.145)) the radiative corrections of order α have been calculated to (Holstein 1979*b*)

$$\left(\frac{\lambda_c}{\lambda_{\beta^+}}\right)_R = \frac{\lambda_c}{\lambda_{\beta^+}}\left\{1 - \frac{\alpha}{2\pi}\left[\overline{g(W_e, W_0)} - \lim_{\substack{\beta \to 0 \\ W_e \to -1}} g(W_e, W_0)\right]\right\}. \tag{13.18}$$

$\overline{g(W_e, W_0)}$ denotes $g(W_e, W_0)$ averaged over the positron energy. Corresponding numerical values can be found in the paper of Wilkinson and

Macefield (1970). It should, however, be noted that this correction is insignificant at the present level of experimental precision in most cases.

In the context of all the outer radiative corrections listed above it should, however, be remarked that they are calculated for allowed transitions (see Section 14.2). Therefore, they rigorously hold only for allowed transitions. Nevertheless, it could be expected that they can be applied as a first approximation to forbidden transitions too.

It should also be noticed that the inner radiative corrections, which can be absorbed into the vector and axial vector coupling constants as a renormalization, influence some of the above mentioned observables (asymmetry of electrons emitted from oriented nuclei, electron neutrino angular correlation) in addition if this renormalization is different for vector and axial vector coupling constant. The main importance of the inner radiative corrections lies, however, in the fact that the total transition rate, i.e. the ft-value is changed if they come into the game. For an isolated consideration of beta-decay alone they would, of course, not be worth a further discussion since they are contained in the coupling constants experimentally determined by beta-decay experiments. This can be expressed as

$$f_V^R = f_V[1+\tfrac{1}{2}\Delta_R^{(V)}] \tag{13.19}$$

$$f_A^R = f_A^R[1+\tfrac{1}{2}\Delta_R^{(A)}] \tag{13.20}$$

where $\Delta_R^{(V)}$ and $\Delta_R^{(A)}$ represent the inner radiative corrections for the vector and axial vector interaction. Thus if f_V^R and f_A^R are known all beta-decay observables can be described and we can finish the discussion at this stage.

However, these corrections will became immediately important if we go one stage beyond and consider the beta-decay as one process under many others which are all governed by one and the same universal weak interaction. Then, the coupling constant G of the weak interaction should be the same for all types of weak interaction processes, i.e. for the muon decay, pion decay, Kaon decay etc. (see Chapter 12). The inner radiative corrections are, however, not expected to be the same for all these processes. For a universal consideration we have therefore to try, at least, to calculate the differences of the inner radiative corrections between these different processes. As a normalization basis the coupling constant

$$g_\mu^R = g_\mu(1+\tfrac{1}{2}\Delta_\mu) \tag{13.21}$$

for the purely leptonic muon decay is usually chosen. Then, the difference of

$$\Delta_R^{(V)} - \Delta_\mu \quad \text{and} \quad \Delta_R^{(A)} - \Delta_\mu \tag{13.22}$$

is only considered.

However, it should be mentioned that from an universality point of view the inner radiative corrections to the axial vector coupling constant are not of such importance since in this case the main renormalization effect is caused by the strong interactions (a factor 10 greater than $\Delta_R^{(A)}$) in a way specific for nuclear beta-decay. This renormalization seems even to be different from nucleus to nucleus (see Sections 8.2 and 14.2.2.2). Because the renormalization of λ through the strong interaction cannot be calculated with the required precision for universality considerations it is not of much worth to evaluate and discuss the additional renormalization through the Coulomb interaction. In this case one is now restricted to take the value for λ from beta-decay experiments (see Kropf and Paul 1974).

For the weak vector coupling constant, renormalization effects of the strong interaction are, on the other hand, excluded by CVC (see Sections 10.2 and 14.2.2.1). Here, of course a universal consideration of beta-decay makes sense. In the following we will restrict ourselves therefore to the discussion of the inner radiative correction $\Delta_R^{(V)}$ for the vector interaction only.

Starting once again with the simplest approach (impulse approximation, point like nucleons, point 4-fermion interaction) one obtains for the inner radiative correction of order α by taking into account the Feynman diagrams shown in Fig. 13.1 (Berman 1958; Kinoshita and Sirlin 1959; Berman and Sirlin 1962)

$$\Delta_R^{(V)} = \frac{\alpha}{2\pi}\left[6 \ln \frac{\Lambda}{M_N} + \frac{9}{4}\right] \tag{13.23}$$

where Λ is an arbitrary cut-off parameter to limit the divergence.† Δ_μ for the muon is zero in this case.

As we have discussed in Chapter 12 the introduction of the gauge theories offers the possibility to really overcome the divergences in the

† In this case we also get

$$\Delta_R^{(A)} = \frac{\alpha}{2\pi}\left[6 \ln \frac{\Lambda}{M_N} + \frac{9}{4}\right].$$

If we go, however, a small step beyond the assumption of a point like nucleon by writing the hadron part as (see eqn 6.4)

$$J_\mu^+ = i\bar{\psi}_p \gamma_\mu (1 + \lambda\gamma_5)\psi_n$$

we obtain (Yokoo and Morita 1976)

$$\Delta_R^{(V)} = \frac{\alpha}{2\pi}\left[(3+3\lambda) \ln \frac{\Lambda}{M_N} + \frac{9\lambda}{4}\right]$$

$$\Delta_R^{(A)} = \frac{\alpha}{2\pi}\left[(3+3/\lambda) \ln \frac{\Lambda}{M_N} + \frac{4+5/\lambda}{4}\right].$$

radiative corrections indicated by the cut-off Λ in the local theory because these theories are renormalizable. For point like unstructured nucleons in the absence of strong interactions Sirlin obtained in the SU(2)×U(1) Weinberg–Salam gauge model described in Chapter 12 (Sirlin 1974)

$$\Delta_R^{(V)} - \Delta_\mu = \frac{\alpha}{2\pi}\left[6 \ln \frac{M_Z}{M_N} + \frac{9}{4}\right]. \tag{13.24}$$

Comparison with eqn (13.23) shows that now the cut-off Λ of the local theory is replaced by the Z-vector boson mass M_Z. It is rather remarkable that nothing other than this replacement has been changed. It is also surprising that it is just the mass of the Z-vector boson which enters.†

For the next step nucleon structure has to be introduced. That means effects caused by the strong interactions and the fact that the nucleons are not elementary particles but are built up by quarks, should now be evaluated. In the Weinberg–Salam model the radiative correction $\Delta_R^{(V)}$ is then modified to‡ (Sirlin 1974)

$$\Delta_R^{(V)} - \Delta_\mu = \frac{\alpha}{2\pi}\left[3 \ln \frac{M_Z}{M_N} + 6\bar{Q} \ln \frac{M_Z}{M_A} + 2C\right]. \tag{13.25}$$

Here $\bar{Q}$ is the mean charge of the fields that carry the axial current in the nucleon. In the Gell-Mann–Zweig quark model these are the u and d quark. Thus we have§

$$\bar{Q} = \tfrac{1}{2}(\tfrac{2}{3} - \tfrac{1}{3}) = \tfrac{1}{6} \tag{13.26}$$

The first term in eqn (13.25) comes from the vector current and is independent of the nucleon model. The second and third terms represent the asymptotic and non-asymptotic contributions of the axial current. M_A in the second term is the mass of some 'axial' structure within the nucleon. M_A will not be less than the mass of the A_1 meson and is usually identified with M_{A_1}. The nonasymptotic part C is probably of the order 1 (for a discussion of that term see Abers *et al.* 1968; Sirlin 1974).

Before closing this section we would like to mention that the above results are not (essentially) restricted to the simple Weinberg–Salam gauge model. Hence, we have an infinite number of other more or less probable gauge models. The question, therefore, arises, are the expressions specifically connected with the special gauge model chosen above or are they more general.

†This result is also very similar to that obtained in the simple intermediate boson (not gauge) theory, with the difference that M_Z rather than M_W plays the role of the cut-off Λ (see Lee 1962).

‡An expression similar to eqn (13.25) was first derived by Abers *et al.* (1968) making use of the old intermediate boson hypothesis. M_Z was, however, replaced by M_W.

§If the fundamental fields are the nucleons themselves we would have $\bar{Q} = \frac{1}{2}$.

A first intuitive indication for the more general validity of the above expressions is already given by the fact that their form was very stable compared with the drastic changes of the underlying theories. Sirlin has, in fact, demonstrated that the form of the inner radiative corrections does not change very much if the treatment is generalized to a wider range of quark and gauge models. He gets (Sirlin 1975; Sirlin 1978; Shei *et al.* 1979; Sirlin 1980)

$$\Delta_R^{(V)} - \Delta_\mu = \frac{\alpha}{2\pi}\left[3\ln\frac{M_W}{M_N} + 6\bar{Q}\ln\frac{M_W}{M_A} + 2C + \frac{3}{2}\frac{R\ln R}{R-1} \times \tan^2\theta_W(1+2\bar{Q})\right] \quad (13.27)$$

where $R = (M_W/M_Z)^2$ and θ_W is the weak interaction angle.

In the Weinberg–Salam model we have

$$\frac{R}{R-1}\tan^2\theta_W = -1 \quad (13.28)$$

and eqn (13.27) reduces to eqn (13.25). Thus, in the case of general symmetry breaking the inner radiative corrections depend on the two additional independent parameters R and $\tan\theta_W$.

14

SPECIAL FORMULAE

14.1. General considerations to the classification of different β-transitions

IN CHAPTER 7 we had shown before that all observables can be represented in terms of the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ which have to be summed up over K, k_e, and k_ν. The sum over the tensor rank K of the form factors has its source in our multipole decomposition of the nuclear current and the sum over k_e, k_ν in the expansion of electron and neutrino wave functions into spherical waves.

We have seen in Section 7.2.2.1 that the most important terms are always those with the smallest possible K given by

$$K = K_{\min}, K_{\min}+1 \quad \text{with} \quad K_{\min} = |J_i - J_f| \tag{14.1}$$

and the smallest possible k_e, k_ν given by

$$k_e + k_\nu = K+1 \tag{14.2}$$

and

$$k_e + k_\nu = K+2. \tag{14.3}$$

In order to obtain these terms we now have to evaluate $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$, i.e. the eqns (6.171) and (6.172), for these values. Then the coefficients $G_{KLs}(\kappa_e, \kappa_\nu)$ are needed for $k_e + k_\nu = K+1$ and $k_e + k_\nu = K+2$.

The following explicit expressions (Behrens and Bühring 1971) are valid, if $k_e + k_\nu = K+1$:

$$G_{KK-11}(-k_e, -k_\nu) = C_K \frac{2K+1}{\sqrt{K}} \tag{14.4a}$$

$$G_{KK-11}(+k_e, +k_\nu) = -C_K \frac{1}{\sqrt{K}} \tag{14.4b}$$

$$G_{KK+11}(-k_e, -k_\nu) = 0 \tag{14.4c}$$

$$G_{KK+11}(+k_e, +k_\nu) = 2C_K\sqrt{(K+1)} \tag{14.4d}$$

$$G_{KK0}(\mp k_e, \pm k_\nu) = \pm C_K\sqrt{(2K+1)} \tag{14.4e}$$

$$G_{KK1}(\mp k_e, \pm k_\nu) = -C_K\sqrt{\left\{\frac{(K+1)(2K+1)}{K}\right\}} \tag{14.4f}$$

$$G_{KK-11}(\pm k_e, \mp k_\nu) = G_{KK+11}(\pm k_e, \mp k_\nu) = 0 \tag{14.4g}$$

$$G_{KK0}(\mp k_e, \mp k_\nu) = G_{KK1}(\mp k_e, \mp k_\nu) = 0 \tag{14.4h}$$

$$C_K = \sqrt{\left\{\frac{(2k_\nu - 1)!!\,(2k_e - 1)!!K!}{(k_\nu - 1)!\,(k_e - 1)!\,(2K+1)!!}\right\}} \tag{14.4i}$$

and if $k_e + k_\nu = K + 2$:

$$G_{KK0}(\mp k_e, \mp k_\nu) = \tilde{C}_K \sqrt{(2K+1)} \tag{14.4j}$$

$$G_{KK1}(\mp k_e, \mp k_\nu) = \pm \tilde{C}_K (k_\nu - k_e) \sqrt{\left\{\frac{2K+1}{K(K+1)}\right\}} \tag{14.4k}$$

$$G_{KK-11}(+k_e, -k_\nu) = \tilde{C}_K \frac{2(k_\nu - 1)}{\sqrt{K}} \tag{14.4l}$$

$$G_{KK-11}(-k_e, +k_\nu) = -\tilde{C}_K \frac{2(k_e - 1)}{\sqrt{K}} \tag{14.4m}$$

$$G_{KK+11}(+k_e, -k_\nu) = \tilde{C}_K \frac{2k_e - 1}{\sqrt{(K+1)}} \tag{14.4n}$$

$$G_{KK+11}(-k_e, +k_\nu) = -\tilde{C}_K \frac{2k_\nu - 1}{\sqrt{(K+1)}} \tag{14.4o}$$

$$G_{KK0}(\mp k_e, \pm k_\nu) = G_{KK1}(\mp k_e, \pm k_\nu) = 0 \tag{14.4p}$$

$$G_{KK-11}(\mp k_e, \mp k_\nu) = G_{KK+11}(\mp k_e, \mp k_\nu) = 0 \tag{14.4r}$$

$$\tilde{C}_K = \sqrt{\left\{\frac{(2k_\nu - 3)!!\,(2k_e - 3)!!\,(K+1)!}{(k_\nu - 1)!\,(k_e - 1)!\,(2K+1)!!}\right\}} \tag{14.4s}$$

where $(-1)!! = 1$.

For a discussion of the order of the different terms within the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ we also have additionally to take into account that we have form factors connected with non-relativistic nuclear matrix elements and other ones connected with relativistic nuclear matrix elements (see Section 8.1.1.2). The latter are smaller by a factor of about $1/MR \simeq \frac{1}{20}$ than the first ones. Of what type the form factors are depends on the parities of initial and final nuclear states (see also Section 7.4.1), i.e. if $\pi_i \pi_f = (-1)^K$

$${}^{V}F_{KK0}(q^2) \simeq d(q^2) \tag{14.5a}$$

$${}^{V}F_{KK-11}(q^2) \simeq \frac{d(q^2)}{MR} \tag{14.5b}$$

$${}^{V}F_{KK+11}(q^2) \simeq \frac{d(q^2)}{MR} \tag{14.5c}$$

$${}^{A}F_{KK1}(q^2) \simeq d(q^2) \tag{14.5d}$$

if $\pi_i\pi_f = (-1)^{K+1}$

$$^{A}F_{KK0}(q^2) \simeq \frac{d(q^2)}{MR} \tag{14.5e}$$

$$^{A}F_{KK-11}(q^2) \simeq d(q^2) \tag{14.5f}$$

$$^{A}F_{KK+11}(q^2) \simeq d(q^2). \tag{14.5g}$$

$$^{V}F_{KK1}(q^2) \simeq \frac{d(q^2)}{MR}. \tag{14.5h}$$

Here, $d(q^2)$ denotes only the order of magnitude of the form factors but not any relation between them.

In beta-decay and electron capture the momenta of the participating particles are always small (see Section 5.2). We have

$$0 \leqslant qR < W_0R \ll 1$$

$$0 \leqslant p_eR < W_0R \ll 1$$

$$0 \leqslant p_\nu R < W_0R \ll 1.$$

If $\alpha Z = 0$, the form factors are only needed for small values of q. Thus it is possible to expand the $F_{KLs}(q^2)$ as shown in eqn (6.74) and take into account the first two terms of that expansion only (see eqn (6.75)). Then the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ are linear combinations of form factor coefficients which can be derived from the eqns (6.167) and (6.168). Even if $\alpha Z \neq 0$ it is, however, suitable to use the more general expansions of the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ into form factor coefficients as given in eqns (6.171) and (6.172) because the first two (or three) terms of this expansion are also of relevance only. In the following, therefore, we present the formulae for the beta-decay observables in terms of form factor coefficients.

Now we are in a position to write down the dominant terms in the linear combinations $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ of form factor coefficients (see also Behrens and Bühring 1971);
for $\pi_i\pi_f = +1$

$$M_0(1, 1) = {}^{V}F^{(0)}_{000}, \tag{14.6}$$

for $\pi_i\pi_f = (-1)^{K+1}$ and $K \geqslant 1$

$(k_e + k_\nu = K + 1)$

$$M_K(k_e, k_\nu) = -D_K(p_eR)^{k_e-1}(p_\nu R)^{k_\nu-1}\sqrt{\left(\frac{2K+1}{K}\right)}\,{}^{A}F^{(0)}_{KK-11} \tag{14.7}$$

with

$$M_1(1, 1) = -{}^{A}F^{(0)}_{101} \quad \text{for} \quad K = 1, \tag{14.8}$$

for $\pi_i\pi_f = -1$

$$M_0(1,1) = {}^{A}F^{(0)}_{000} - \tfrac{1}{3}\alpha Z\, {}^{A}F^{(0)}_{011}(1,1,1,1) - \tfrac{1}{3}W_0 R\, {}^{A}F^{(0)}_{011} \tag{14.9}$$

$$m_0(1,1) = -\tfrac{1}{3}m_e R\, {}^{A}F^{(0)}_{011} \tag{14.10}$$

for $\pi_i\pi_f = (-1)^K$ and $K \geq 1$

$(k_e + k_\nu = K+1)$

$$\begin{aligned} M_K(k_e, k_\nu) = D_K (p_e R)^{k_e - 1} (p_\nu R)^{k_\nu - 1} \Bigg\{ & -\sqrt{\left(\frac{2K+1}{K}\right)}\, {}^{V}F^{(0)}_{KK-11} \\ & - \frac{\alpha Z}{2k_e+1}\, {}^{V}F^{(0)}_{KK0}(k_e,1,1,1) - \left(\frac{W_e R}{2k_e+1} + \frac{p_\nu R}{2k_\nu+1}\right) {}^{V}F^{(0)}_{KK0} \\ & - \frac{\alpha Z}{2k_e+1}\sqrt{\left(\frac{K+1}{K}\right)}\, {}^{A}F^{(0)}_{KK1}(k_e,1,1,1) - \left(\frac{W_e R}{2k_e+1} - \frac{p_\nu R}{2k_\nu+1}\right) \\ & \times \sqrt{\left(\frac{K+1}{K}\right)}\, {}^{A}F^{(0)}_{KK1} \Bigg\} \end{aligned} \tag{14.11}$$

$$\begin{aligned} m_K(k_e, k_\nu) = D_K (p_e R)^{k_e - 1} (p_\nu R)^{k_\nu - 1} \\ \times \frac{m_e R}{2k_e+1}\left[-{}^{V}F^{(0)}_{KK0} - \sqrt{\left(\frac{K+1}{K}\right)}\, {}^{A}F^{(0)}_{KK1} \right], \end{aligned} \tag{14.12}$$

$(k_e + k_\nu = K+2)$

$$\begin{aligned} M_K(k_e, k_\nu) = \tilde{D}_K (p_e R)^{k_e - 1} (p_\nu R)^{k_\nu - 1} \sqrt{\left\{\frac{K+1}{(2k_e-1)(2k_\nu-1)}\right\}} \\ \times \left\{ {}^{V}F^{(0)}_{KK0} - \frac{k_e - k_\nu}{K+1}\sqrt{\left(\frac{K+1}{K}\right)}\, {}^{A}F^{(0)}_{KK1} \right\} \end{aligned} \tag{14.13}$$

$$M_{K+1}(k_e, k_\nu) = -\tilde{D}_K (p_e R)^{k_e - 1} (p_\nu R)^{k_\nu - 1}\, {}^{A}F^{(0)}_{K+1\,K\,1}. \tag{14.14}$$

The factors D_K and $\tilde{D}_K$ have the form

$$D_K = \sqrt{\tfrac{1}{2}}\sqrt{\left\{\frac{(2K)!!}{(2K+1)!!}\right\}}\sqrt{\left\{\frac{1}{(2k_e-1)!\,(2k_\nu-1)!}\right\}} \tag{14.15}$$

with $k_e + k_\nu = K+1$,

$$\tilde{D}_K = \sqrt{\left\{\frac{(2K)!!}{(2K+1)!!}\right\}}\sqrt{\left\{\frac{1}{(2k_e-1)!\,(2k_\nu-1)!}\right\}} \tag{14.16}$$

with $k_e + k_\nu = K+2$.

The two quantities D_K and $\tilde{D}_K$ are related by

$$\tilde{D}_{K-1} = \sqrt{\left(\frac{2K+1}{K}\right)}\, D_K. \tag{14.17}$$

Let us now consider as a criterion for a classification the half-life or ft-value (reduced half-life), respectively (see Section 7.2.2.2). From eqn (7.64) we have

$$ft = \frac{2\pi^3 \ln 2}{G_\beta^2 \overline{C(W_e)}} \tag{14.18}$$

where $\overline{C(W_e)}$ is the shape factor averaged over the beta-spectrum (see eqn (7.62)).

For an order of magnitude estimation the shape factor $C(W_e)$ can be approximated to (see eqn (7.56))

$$C(W_e) \simeq \sum_{k_e, k_\nu} \left\{ M^2_{K_{\min}}(k_e, k_\nu) + M^2_{K_{\min}+1}(k_e, k_\nu) \right\}. \tag{14.19}$$

Furthermore we assume for the moment† that $\overline{C(W_e)} \simeq C(W_e^*)$ with $W_e^* = (W_0^2 - 1)/2W_0 \simeq W_0/2$ (W_e^* corresponds to that electron energy where $p_e = p_\nu$).

We then obtain for the ft-value

if $\pi_i \pi_f = +1$ and $K = 0, 1$

$$ft = \frac{2\pi^3 \ln 2}{G_\beta^2 \{ ({}^{\mathrm{V}}F^{(0)}_{000})^2 + ({}^{\mathrm{A}}F^{(0)}_{101})^2 \}}, \tag{14.20}$$

if $\pi_i \pi_f = -1$ and $K = 0, 1$

$$ft = \frac{2\pi^3 \ln 2}{G_\beta^2 \{ A_0^2 + C_0^2 \}} \tag{14.21}$$

with

$$A_0 = {}^{\mathrm{A}}F_{000} - \tfrac{1}{3}\alpha Z \, {}^{\mathrm{A}}F^{(0)}_{011}(1, 1, 1, 1) - \tfrac{1}{3} W_0 R \, {}^{\mathrm{A}}F_{011} \tag{14.22}$$

$$C_0 = {}^{\mathrm{V}}F^{(0)}_{101} + \frac{\alpha Z}{3}\sqrt{\tfrac{1}{3}} \, {}^{\mathrm{V}}F^{(0)}_{110}(1, 1, 1, 1) + \frac{W_0 R}{3}\sqrt{\tfrac{1}{3}} \, {}^{\mathrm{V}}F^{(0)}_{110}$$

$$+ \frac{\alpha Z}{3}\sqrt{\tfrac{2}{3}} \, {}^{\mathrm{A}}F^{(0)}_{111}(1, 1, 1, 1), \tag{14.23}$$

if $\pi_i \pi_f = (-1)^{K+1}$ and $K \geqslant 1$

$$ft \simeq \frac{2\pi^3 \ln 2 \, \{(2K-1)!!\}^2 K}{G_\beta^2 (W_0 R)^{2K-2} ({}^{\mathrm{A}}F^{(0)}_{KK-11})^2} \tag{14.24}$$

which for $K = 1$ reduces to

$$ft = \frac{2\pi^3 \ln 2}{G_\beta^2 ({}^{\mathrm{A}}F^{(0)}_{101})^2}, \tag{14.25}$$

† That is, of course, in no way exact. This assumption can only be justified at this place because it is our intention to derive nothing else than simple order of magnitude formulae.

if $\pi_i\pi_f=(-1)^K$ and $K\geqslant 1$

$$ft\simeq\frac{2\pi^3\ln 2\,\{(2K-1)!!\}^2K}{G_\beta^2(W_0R)^{2K-2}C_{K-1}^2} \tag{14.26}$$

with

$$C_{K-1}={}^{\mathrm{v}}F_{KK-11}^{(0)}+\frac{\alpha Z}{2k_e^*+1}\sqrt{\left(\frac{K}{2K+1}\right)}\,{}^{\mathrm{v}}F_{KK0}^{(0)}(k_e^*,1,1,1)+\frac{W_0R}{2k_e^*+1}$$
$$\times\sqrt{\left(\frac{K}{2K+1}\right)}\,{}^{\mathrm{v}}F_{KK0}^{(0)}+\frac{\alpha Z}{2k_e^*+1}\sqrt{\left(\frac{K+1}{2K+1}\right)}\,{}^{\mathrm{A}}F_{KK1}^{(0)}(k_e^*,1,1,1) \tag{14.27}$$

with

$$k_e^*=\frac{K+1}{2}\quad\text{for } K=\text{odd}$$

$$k_e^*=\frac{K}{2}\qquad\text{for } K=\text{even}.$$

For $K=1$ eqn (14.26) reduces to

$$ft\simeq\frac{2\pi^3\ln 2}{G_\beta^2C_0^2} \tag{14.28}$$

which should be compared with eqn (14.21).

The above approximate expressions for the ft-values show that the degree of retardation for a beta-transition is determined (note that $W_0R\lesssim 0{\cdot}2$) by K and the corresponding parity selection rule.

The fastest transitions are those where $\pi_i\pi_f=+1$ and $K=0,1$, since here no retardation factor $[\{(2K-1)!!\}^2K]/(W_0R)^{2K-2}$ occurs and the form factor coefficients are of the non-relativistic type. They are therefore usually denoted as allowed transitions.

Then we have two types of retarded transitions which are usually denoted as forbidden ones: one type, where the retardation is only determined by the retardation factor $\{(2K-1)!!\}^2K/(W_0R)^{2K-2}$ and another type, where the retardation is caused (not) by the retardation factor $\{(2K-1)!!\}^2K/(W_0R)^{2K-2}$ and (but) by the fact that the leading form factor coefficients are of the relativistic type (which are smaller by $1/MR$). The first type of transitions with the parity selection rule $\pi_i\pi_f=(-1)^{K+1}$ are usually described as $(K-1)$th unique forbidden transitions (the name 'unique' has its basis in the fact that only one form factor coefficient is of importance). The latter types with the parity selection rule $\pi_i\pi_f=(-1)^K$ (including the case $\pi_i\pi_f=-1$ and $K=0,1$) are usually called Kth non-unique forbidden transitions.

Thus the various types of beta-transitions can be summarized as follows:

$K=0,1$	$\pi_i\pi_f=+1$	allowed transitions	(14.29)
$K=0,1$	$\pi_i\pi_f=-1$	first non-unique forbidden transitions	(14.30)
$K>1$	$\pi_i\pi_f=(-1)^K$	Kth non-unique forbidden transitions	(14.31)
$K>1$	$\pi_i\pi_f=(-1)^{K-1}$	$(K-1)$th unique forbidden transitions	(14.32)

with $K=K_{\min}=|J_i-J_f|$.

Let us now make also some explicit estimations for the ft-values and a comparison with corresponding experimental values.

The beta-decay coupling constant G' including the inner radiative correction† has the value‡ (see Raman *et al.* 1975; Hardy and Towner 1975; Wilkinson 1975; Towner *et al.* 1977; Wilkinson 1977; Wilkinson 1978; Szybisz 1979)

$$G'_\beta=(2{\cdot}996\pm0{\cdot}002)10^{-12} \tag{14.33}$$

Therefore we have

$$\frac{2\pi^3\ln 2}{G'^2_\beta}=4{\cdot}790\times10^{24}. \tag{14.34}$$

For historical reasons the ft-values are, however, always given in seconds (see the discussion in Section 7.2.2.2). We have, therefore, to express the time t in seconds instead of natural units. The natural unit of time is $\hbar/m_ec^2=1{\cdot}288630\times10^{-21}$ s.

Thus we have

$$\frac{2\pi^3\ln 2}{G'^2_\beta}=6{\cdot}173\times10^3\,[\mathrm{s}]. \tag{14.35}$$

We then obtain:§

For the superallowed Fermi transitions ($J^\pi=0^+\rightarrow0^+$, $T=1\rightarrow1$)

$$\log ft=3{\cdot}49 \tag{14.36}$$

(see Section 8.1.1.4 and especially eqn (8.140) which gives ${}^{\mathrm{V}}F^{(0)}_{000}=\sqrt{2}$).

For the superallowed mixed transitions ($J^\pi\rightarrow J^\pi$ $(J\neq0)$, $\Delta T=0$)

$$\log ft=3{\cdot}0-3{\cdot}8 \tag{14.37}$$

† $G'^2_\beta=G^2_\beta(1+\Delta_R)$ with $\Delta_R=$ inner radiative correction.

‡ The error has been increased by a factor of about 2. See the discussion in Section 14.2.2.1.

§ Transitions are called superallowed if they occur between members of the same supermultiplet (see Section 8.1.1.4).

(see Section 8.1.1.4 and especially the eqns (8.140) and (8.142). Here we have ${}^{V}F_{000}^{(0)} = 1$ or ${}^{V}F_{000}^{(0)} = \sqrt{2}$ and ${}^{A}F_{101}^{(0)} \simeq 1$).

For allowed transitions

$$\log ft \simeq 3{\cdot}8 \tag{14.38}$$

(here we have assumed ${}^{A}F_{101}^{(0)} \simeq 1$ as it follows from the independent single particle model (see Section 8.1.1.3)).

For first non-unique forbidden transitions

$$\log ft \simeq 6{\cdot}1 \tag{14.39}$$

(here we have assumed B_0 and B_1 to be of the order $1/MR$).

For first unique forbidden transitions

$$\log ft \simeq 7{\cdot}7 \tag{14.40}$$

(here we have also assumed ${}^{A}F_{211}^{(0)} \simeq 1$).

Comparing these estimations with the ft-values realized in nature (see Fig. 14.1) we recognize that we have broad distributions of the ft-values over a wide range on the one side and that our single-particle estimation

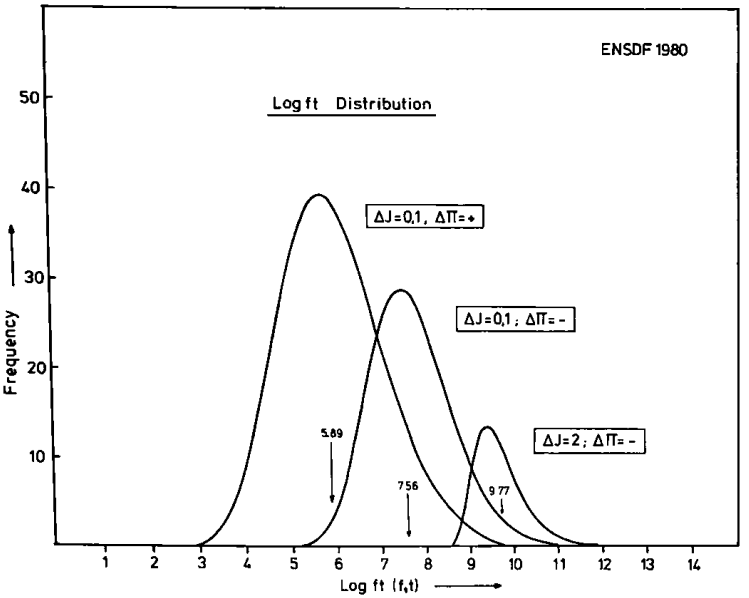


FIG. 14.1. Experimental log ft distribution (taken from Behrens *et al.* 1980). For the first unique forbidden transitions $f_1 t$ has been listed where

$$f_1 = \int_1^{W_0} (p_\nu^2 + \lambda_2 p_e^2) p_e W_e (W_0 - W_e)^2 \, dW_e.$$

is on the other side too low in the average (with exception of the superallowed transitions).

From the experimental ft-values shown in Fig. 14.1 we can immediately deduce that for the usual (not the superallowed) transitions

$$^{A}F_{101}^{(0)} \simeq \tfrac{1}{10} \tag{14.41}$$

$$^{A}F_{211}^{(0)} \simeq \tfrac{1}{6} \tag{14.42}$$

and

$$A_0, C_0 \simeq \tfrac{1}{80}\,. \tag{14.43}$$

That means the form factor coefficients or nuclear matrix elements, respectively, are additionally hindered by other effects, i.e. by nuclear structure effects. The reason is that most of the allowed transitions would be forbidden if the residual interaction between the nucleons is switched off because the initial and final nuclear states belong mostly to different supermultiplets. As discussed in Section 8.1.1.4 the selection rules for Fermi and Gamow–Teller matrix elements require, however, initial and final nuclear states to be members of the same supermultiplet.

First, the switching on of the residual interaction allows that these transitions take place since the irreducible representations P, P', P'' of the supermultiplets are not good quantum numbers any more. Of course, they remain, however, always hindered.

Concluding this section we can summarize the above discussion in the following very rough approximations for the usual (not superallowed) log ft-values:

for allowed and $(K-1)$th unique forbidden transistions

$$\log ft \simeq 3{\cdot}8 \;+\; \simeq 2 \;-\log(W_0R)2(K-1)+\log[\{(2K-1)!!\}^2K]$$

$$\frac{2\pi^3 \ln 2}{G_\beta'^2}: \text{nuclear structure effects}; \qquad \frac{\{(2K-1)!!\}^2K}{(W_0R)^{2K-2}}: \text{retardation factor} \tag{14.44}$$

for Kth non-unique forbidden transitions

$$\log ft \simeq 3{\cdot}8 \;+\; \simeq 2 \;+\; \simeq 2 \;-\log(W_0R)2(K-1)+\log[\{(2K-1)!!\}^2K]$$

$$\frac{2\pi^3 \ln 2}{G_\beta'^2}: \text{nuclear structure effects}; \quad \text{relativistic matrix elements}; \quad \frac{\{(2K-1)!!\}^2K}{(W_0R)^{2K-2}}: \text{retardation factor}\,. \tag{14.45}$$

14.2 Allowed transitions

14.2.1. General formulae

14.2.1.1. The quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$

In Chapter 7 expressions for all kinds of observables have been derived. In that context the final formulae had always been written down in such a way that the observables were functions of the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$. Thus, for a detailed discussion of the allowed transitions, which represent the majority of beta-transitions found in nature, we have first to specialize these quantities for this type of transitions, but, if dominant terms are exclusively considered, we have seen in the foregoing chapter that the quantities $M_0(1, 1)$ and $M_1(1, 1)$ are of importance only. Then we have

$$M_0(1, 1) = {}^{V}F^{(0)}_{000} \tag{14.46}$$

$$M_1(1, 1) = -{}^{A}F^{(0)}_{101} \tag{14.47}$$

i.e. the discussion could be finished at this stage.

For more precise investigations of allowed beta-decays it is, however, particularly the contribution of the higher order terms which is of special interest. For example, if the validity of the CVC theory or the possible existence of second class currents is to be tested these higher order terms, or more exactly, observables which are sensitive to these terms, have especially to be looked for. Therefore, for a comprehensive discussion, we have additionally to include higher order terms which are functions of other form factor coefficients of tensor rank 0, 1 and 2. Those of rank 2 are the same as those which represent the leading terms in the case of second non-unique forbidden transitions. It is, therefore, sometimes customary to denote the whole higher order corrections as second forbidden corrections, or as so-called second forbidden corrections (although the most important ones are not of the second forbidden type).

Since we later on intend to give the electron (positron) energy dependence for the observables explicitly we write the linear combinations $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ in the form (see eqns (6.171) and (6.172))

$$M_0(1, 1) = A_0 + W_e R A_1 + (W_e R)^2 A_2 \tag{14.48}$$

$$m_0(1, 1) = R(B_0 + W_e R B_1) \tag{14.49}$$

$$M_1(1, 1) = C_0 + W_e R C_1 + (W_e R)^2 C_2 \tag{14.50}$$

$$m_1(1, 1) = R(D_0 + W_e R D_1) \tag{14.51}$$

$$M_1(1, 2) = \tfrac{1}{3} p_\nu R(E_0 + W_e R E_1) \tag{14.52}$$

$$m_1(1,2) = \tfrac{1}{9} p_\nu R^2 \bar{E}_0 \tag{14.53}$$

$$M_1(2,1) = \tfrac{1}{3} p_e R (F_0 + W_e R F_1) \tag{14.54}$$

$$m_1(2,1) = \tfrac{1}{15} p_e R^2 \bar{F}_0 \tag{14.55}$$

$$M_2(1,2) = \tfrac{1}{3} p_\nu R (G_0 + W_e R G_1) \tag{14.56}$$

$$m_2(1,2) = \tfrac{1}{9} p_\nu R^2 \bar{G}_0 \tag{14.57}$$

$$M_2(2,1) = \tfrac{1}{3} p_e R (H_0 + W_e R H_1) \tag{14.58}$$

$$m_2(2,1) = \tfrac{1}{15} p_e R^2 \bar{H}_0 \tag{14.59}$$

$$M_0(2,2) = \tfrac{1}{9} (p_e R)(p_\nu R) M_0 \tag{14.60}$$

$$M_1(2,2) = \tfrac{1}{9} (p_e R)(p_\nu R) N_0 \tag{14.61}$$

$$M_2(3,1) = \tfrac{1}{15} (p_e R)^2 X_0 \tag{14.62}$$

$$M_3(3,1) = \tfrac{1}{15} (p_e R)^2 Y_0. \tag{14.63}$$

The quantity R denotes, as always in this book, the nuclear (equivalent charge) radius. Note that $m_e = 1$ in our units.

In the formulae given above terms containing the factor R^n have only been taken into account up to $n = 2$. All terms with $n > 2$ have been omitted because they are responsible for the next higher (and therefore unmeasurably small) order of correction terms beyond the so-called second forbidden terms. The quantities A_0, A_1, A_2, B_0, B_1, C_0, C_1, C_2, D_0, D_1, E_0, E_1, $\bar{E}_0$, F_0, F_1, $\bar{F}_0$, G_0, G_1, $\bar{G}_0$, H_0, H_1, $\bar{H}_0$, M_0, N_0, X_0, and Y_0 are electron and neutrino energy independent linear combinations of form factor coefficients. They can be derived from the eqns (6.171) and (6.172) and come out to be (see also Behrens *et al.* 1978)

$$A_0 = {}^{\vee}F^{(0)}_{000} - \frac{W_0 R}{3} {}^{\vee}F^{(0)}_{011} \mp \frac{\alpha Z}{3} {}^{\vee}F^{(0)}_{011}(1,1,1,1) - \frac{(W_0 R)^2}{6} \times {}^{\vee}F^{(1)}_{000} \mp \frac{\alpha Z}{9} W_0 R \, {}^{\vee}F^{(1)}_{000}(1,1,1,1) - \frac{(\alpha Z)^2}{6} {}^{\vee}F^{(1)}_{000}(1,2,2,2) \tag{14.64}$$

$$A_1 = \tfrac{2}{9} W_0 R \, {}^{\vee}F^{(1)}_{000} \mp \frac{\alpha Z}{3} [{}^{\vee}F^{(1)}_{000}(1,2,2,1) - \tfrac{1}{3} {}^{\vee}F^{(1)}_{000}(1,1,1,1)] \tag{14.65}$$

$$A_2 = -\tfrac{2}{9} {}^{\vee}F^{(1)}_{000} \tag{14.66}$$

$$B_0 = -\tfrac{1}{3} {}^{\vee}F^{(0)}_{011} - \frac{W_0 R}{9} {}^{\vee}F^{(1)}_{000} \pm \frac{\alpha Z}{6} {}^{\vee}F^{(1)}_{000}(1,2,1,1) \tag{14.67}$$

$$B_1 = \tfrac{1}{9} {}^{\vee}F^{(1)}_{000} \tag{14.68}$$

$$C_0 = -{}^{A}F^{(0)}_{101} + \tfrac{1}{3}W_0R[-\sqrt{\tfrac{1}{3}}\,{}^{A}F^{(0)}_{110} + \sqrt{\tfrac{2}{3}}\,{}^{V}F^{(0)}_{111}]$$
$$+ \tfrac{1}{6}(W_0^2-1)R^2\,{}^{A}F^{(1)}_{101} \pm \tfrac{1}{3}\alpha Z\{-\sqrt{\tfrac{1}{3}}\,{}^{A}F^{(0)}_{110}(1,1,1,1)$$
$$-\sqrt{\tfrac{2}{3}}\,{}^{V}F^{(0)}_{111}(1,1,1,1) + \tfrac{1}{9}W_0R[-{}^{A}F^{(1)}_{101}(1,1,1,1)$$
$$+ 2\sqrt{2}\,{}^{A}F^{(0)}_{121}(1,1,1,1)]\} + \tfrac{1}{6}(\alpha Z)^2\,{}^{A}F^{(1)}_{101}(1,2,2,2) \qquad (14.69)$$

$$C_1 = -\tfrac{2}{3}\sqrt{\tfrac{2}{3}}\,{}^{V}F^{(0)}_{111} + \tfrac{2}{27}W_0R[-5\,{}^{A}F^{(1)}_{101} + \sqrt{2}\,{}^{A}F^{(0)}_{121}]$$
$$\pm \tfrac{1}{3}\alpha Z[\tfrac{1}{9}\,{}^{A}F^{(1)}_{101}(1,1,1,1) - \tfrac{2}{9}\sqrt{2}\,{}^{A}F^{(0)}_{121}(1,1,1,1)$$
$$+ {}^{A}F^{(1)}_{101}(1,2,2,1)] \qquad (14.70)$$

$$C_2 = \tfrac{10}{27}\,{}^{A}F^{(1)}_{101} - \tfrac{2}{27}\sqrt{2}\,{}^{A}F^{(0)}_{121} \qquad (14.71)$$

$$D_0 = -\tfrac{1}{3}[\sqrt{\tfrac{1}{3}}\,{}^{A}F^{(0)}_{110} + \sqrt{\tfrac{2}{3}}\,{}^{V}F^{(0)}_{111}] + \frac{W_0R}{27}[-{}^{A}F^{(1)}_{101} + 2\sqrt{2}\,{}^{A}F^{(0)}_{121}]$$
$$\pm \frac{\alpha Z}{6}\,{}^{A}F^{(1)}_{101}(1,2,1,1) \qquad (14.72)$$

$$D_1 = \tfrac{1}{18}[\tfrac{11}{3}\,{}^{A}F^{(1)}_{101} - \tfrac{4}{3}\sqrt{2}\,{}^{A}F^{(0)}_{121}] \qquad (14.73)$$

$$E_0 = \sqrt{\tfrac{2}{3}}\,{}^{A}F^{(0)}_{110} + \sqrt{\tfrac{1}{3}}\,{}^{V}F^{(0)}_{111} - \frac{W_0R}{5}\,{}^{A}F^{(0)}_{121}$$
$$\pm \frac{\alpha Z}{3}[-\tfrac{2}{3}\sqrt{2}\,{}^{A}F^{(1)}_{101}(1,1,1,1) - \tfrac{1}{3}\,{}^{A}F^{(0)}_{121}(1,1,1,1)] \qquad (14.74)$$

$$E_1 = -\tfrac{2}{9}\sqrt{2}\,{}^{A}F^{(1)}_{101} + \tfrac{4}{45}\,{}^{A}F^{(0)}_{121} \qquad (14.75)$$

$$\bar{E}_0 = -\tfrac{1}{3}[{}^{A}F^{(0)}_{121} + 2\sqrt{2}\,{}^{A}F^{(1)}_{101}] \qquad (14.76)$$

$$F_0 = \sqrt{\tfrac{2}{3}}\,{}^{A}F^{(0)}_{110} - \sqrt{\tfrac{1}{3}}\,{}^{V}F^{(0)}_{111} - \frac{W_0R}{9}[2\sqrt{2}\,{}^{A}F^{(1)}_{101} + {}^{A}F^{(0)}_{121}]$$
$$\mp \frac{\alpha Z}{5}\,{}^{A}F^{(0)}_{121}(2,1,1,1) \qquad (14.77)$$

$$F_1 = \tfrac{2}{9}\sqrt{2}\,{}^{A}F^{(1)}_{101} - \tfrac{4}{45}\,{}^{A}F^{(0)}_{121} \qquad (14.78)$$

$$\bar{F}_0 = -{}^{A}F^{(0)}_{121} \qquad (14.79)$$

$$G_0 = -{}^{V}F^{(0)}_{211} + \frac{W_0R}{5}[-\sqrt{\tfrac{2}{5}}\,{}^{V}F^{(0)}_{220} + \sqrt{\tfrac{3}{5}}\,{}^{A}F^{(0)}_{221}] \mp \frac{\alpha Z}{3}$$
$$\times[\sqrt{\tfrac{2}{5}}\,{}^{V}F^{(0)}_{220}(1,1,1,1) + \sqrt{\tfrac{3}{5}}\,{}^{A}F^{(0)}_{221}(1,1,1,1)] \qquad (14.80)$$

$$G_1 = -\tfrac{2}{15}[\sqrt{\tfrac{2}{5}}\,{}^{V}F^{(0)}_{220} + 4\sqrt{\tfrac{3}{5}}\,{}^{A}F^{(0)}_{221}] \qquad (14.81)$$

$$\bar{G}_0 = -\sqrt{\tfrac{2}{5}}\,{}^{V}F^{(0)}_{220} - \sqrt{\tfrac{3}{5}}\,{}^{A}F^{(0)}_{221} \qquad (14.82)$$

$$H_0 = -{}^{V}F^{(0)}_{211} + \frac{W_0R}{3}[-\sqrt{\tfrac{2}{5}}\,{}^{V}F^{(0)}_{220} + \sqrt{\tfrac{3}{5}}\,{}^{A}F^{(0)}_{221}]$$
$$\mp \frac{\alpha Z}{5}[\sqrt{\tfrac{2}{5}}\,{}^{V}F^{(0)}_{220}(2,1,1,1) + \sqrt{\tfrac{3}{5}}\,{}^{A}F^{(0)}_{221}(2,1,1,1)] \qquad (14.83)$$

$$H_1 = \tfrac{2}{15}[\sqrt{\tfrac{2}{5}}\,{}^{V}F^{(0)}_{220} - 4\sqrt{\tfrac{3}{5}}\,{}^{A}F^{(0)}_{221}] \tag{14.84}$$

$$\bar{H}_0 = -\sqrt{\tfrac{2}{5}}\,{}^{V}F^{(0)}_{220} - \sqrt{\tfrac{3}{5}}\,{}^{A}F^{(0)}_{221} \tag{14.85}$$

$$M_0 = \sqrt{2}\,{}^{V}F^{(1)}_{000} \tag{14.86}$$

$$N_0 = -\tfrac{1}{3}\sqrt{(10)}\,{}^{A}F^{(1)}_{101} \tag{14.87}$$

$$X_0 = \sqrt{\tfrac{3}{5}}\,{}^{V}F^{(0)}_{220} - \sqrt{\tfrac{2}{5}}\,{}^{A}F^{(0)}_{221} \tag{14.88}$$

$$Y_0 = -{}^{A}F^{(0)}_{321} \tag{14.89}$$

(upper sign β^--decay, lower sign β^+-decay) W_0 is the maximum electron energy (see eqns (5.24) and (5.25)) and Z the atomic number of the daughter nucleus.

14.2.1.2. The form factor coefficients in terms of nuclear matrix elements

The representation of the observables in terms of form factors or form factor coefficients, respectively, has general validity, i.e. it is independent of details of strong and weak interaction theories and of effects belonging to special nuclear models. The same is true, of course, for the quantities A_0, A_1, A_2 etc. However, if beta-decay observables are intended to be calculated theoretically, it is necessary to relate form factors (or form factor coefficients) to one- (or more-) body nuclear transition matrix elements. As explained in Chapter 8, approximations then become unavoidable. The simplest approximation usually applied is the impulse approximation discussed in detail in Section 8.1. How this approximation can be introduced in general and how the form factors or form factor coefficients, respectively, are then related to the corresponding nuclear matrix elements can be found there. We do not, therefore, repeat the whole discussion here, but, because of the great importance of the allowed transitions, we will specialize the general formulae given in eqns (8.263) to (8.265) to this case and even present explicit results for the convenience of the reader at this place† (see also Behrens *et al.* 1978).

$${}^{V}F^{(0)}_{000} = {}^{V}\mathfrak{M}^{(0)}_{000} \pm \frac{f_S}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{V}\mathfrak{M}^{(0)}_{000} \tag{14.90}$$

$${}^{V}F^{(1)}_{000} = {}^{V}\mathfrak{M}^{(1)}_{000} + 2\frac{f_M}{R}\,{}^{C}\mathfrak{N}^{(0)}_{111} \pm \frac{f_S}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{V}\mathfrak{M}^{(1)}_{000} \tag{14.91}$$

$$\begin{aligned}{}^{V}F^{(1)}_{000}(k_e, m, n, \rho) = {}& {}^{V}\mathfrak{M}^{(1)}_{000}(k_e, m, n, \rho) \\ & + \frac{f_M}{R}\left(\int\left(\frac{r}{R}\right)[2I(r) + rI'(r)]\beta T_{011}\right) \\ & \pm \frac{f_S}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{V}\mathfrak{M}^{(1)}_{000}(k_e, m, n, \rho)\end{aligned} \tag{14.92}$$

† Note that for the decay of the neutron the necessary form factors can simply be calculated by making use of eqns (9.29*a–f*). In that case we have to set $F_V(0) = 1, f_S = F_S(0)$, $f_M = F_M(0), \lambda = F_A(0), f_T = F_T(0)$ and $f_P = F_P(0)$.

$$^{V}F^{(0)}_{011} = -{}^{V}\mathfrak{M}^{(0)}_{001} - \frac{f_M}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{C}\mathfrak{N}^{(0)}_{011} \pm 3\frac{f_S}{R}\,{}^{V}\mathfrak{M}^{(0)}_{000} \quad (14.93)$$

$$^{V}F^{(0)}_{011}(1,1,1,1) = -{}^{V}\mathfrak{M}^{(0)}_{011}(1,1,1,1) - \frac{f_M}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{C}\mathfrak{N}^{(0)}_{011}(1,1,1,1) \pm 3\frac{f_S}{R}\,{}^{V}\mathfrak{M}^{(0)}_{000}(1,1,1,1) \quad (14.94)$$

$$^{A}F^{(0)}_{101} = \mp\lambda\,{}^{A}\mathfrak{M}^{(0)}_{101} + \frac{f_T}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{A}\mathfrak{M}^{(0)}_{101} \quad (14.95)$$

$$^{A}F^{(1)}_{101} = \mp\lambda\,{}^{A}\mathfrak{M}^{(1)}_{101} + \frac{f_T}{R}[2\sqrt{\tfrac{2}{3}}\,{}^{C}\mathfrak{N}^{(0)}_{111} + (W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{A}\mathfrak{M}^{(1)}_{101}] \mp \frac{f_P}{R}2\sqrt{\tfrac{1}{3}}\,{}^{D}\mathfrak{N}^{(0)}_{110} \quad (14.96)$$

$$^{A}F^{(1)}_{101}(k_e, m, n, \rho) = \mp\lambda\,{}^{A}\mathfrak{M}^{(1)}_{101}(k_e, m, n, \rho) + \frac{f_T}{R}\left\{\sqrt{\tfrac{2}{3}}\left(\int\left(\frac{r}{R}\right)[2I(r) + rI'(r)]\,\beta T_{111}\right) + (W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{A}\mathfrak{M}^{(1)}_{101}(k_e, m, n, \rho)\right\} \mp \frac{f_P}{R}\sqrt{\tfrac{1}{3}}\left(\int\left(\frac{r}{R}\right)[2I(r) + rI'(r)]\beta\gamma_5 T_{110}\right) \quad (14.97)$$

$$^{V}F^{(0)}_{111} = {}^{V}\mathfrak{M}^{(0)}_{111} - \frac{f_M}{R}[\sqrt{6}\,{}^{A}\mathfrak{M}^{(0)}_{101} - (W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{C}\mathfrak{N}^{(0)}_{111}] \quad (14.98)$$

$$^{V}F^{(0)}_{111}(1,1,1,1) = {}^{V}\mathfrak{M}^{(0)}_{111}(1,1,1,1) + \frac{f_M}{R}\left\{\sqrt{(\tfrac{2}{3})}\left(\int[3I(r) + rI'(r)]\beta\gamma_5 T_{101}\right) - \sqrt{\tfrac{1}{3}}\left(\int rI'(r)\beta\gamma_5 T_{121}\right) + (W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{C}\mathfrak{N}^{(0)}_{011}(1,1,1,1)\right\} \quad (14.99)$$

$$^{A}F^{(0)}_{110} = \pm\lambda\,{}^{A}\mathfrak{M}^{(0)}_{110} - \frac{f_T}{R}\sqrt{3}\,{}^{A}\mathfrak{M}^{(0)}_{101} \mp \frac{f_P}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{D}\mathfrak{N}^{(0)}_{110} \quad (14.100)$$

$$^{A}F^{(0)}_{110}(1,1,1,1) = \pm\lambda\,{}^{A}\mathfrak{M}^{(0)}_{110}(1,1,1,1) + \frac{f_T}{R}\left\{\sqrt{\tfrac{1}{3}}\left(\int[3I(r) + rI'(r)]\beta\gamma_5 T_{101}\right)\right.$$

$$+\sqrt{\tfrac{2}{3}}\left(\int rI'(r)\beta\gamma_5 T_{121}\right)\Big\}$$

$$\mp\frac{f_P}{R}(W_0R\pm\tfrac{6}{5}\alpha Z)\,{}^{D}\mathfrak{N}^{(0)}_{110}(1,1,1,1) \tag{14.101}$$

$${}^{A}F^{(0)}_{121}=\mp\lambda\,{}^{A}\mathfrak{M}^{(0)}_{121}-\frac{f_T}{R}\left[\frac{5}{\sqrt{3}}\,{}^{C}\mathfrak{N}^{(0)}_{111}-(W_0R\pm\tfrac{6}{5}\alpha Z)\,{}^{A}\mathfrak{M}^{(0)}_{121}\right]\mp\frac{f_P}{R}\,5\sqrt{\tfrac{2}{3}}\,{}^{D}\mathfrak{N}^{(0)}_{110} \tag{14.102}$$

$${}^{A}F^{(0)}_{121}(1,1,1,1)=\mp\lambda\,{}^{A}\mathfrak{M}^{(0)}_{121}(1,1,1,1)$$
$$-\frac{f_T}{R}\left\{\sqrt{\tfrac{1}{3}}\left(\int\left(\frac{r}{R}\right)[5I(r)+rI'(r)]\beta T_{111}\right)\right.$$
$$\left.-(W_0R\pm\tfrac{6}{5}\alpha Z)\,{}^{A}\mathfrak{M}^{(0)}_{121}(1,1,1,1)\right\}$$
$$\mp\frac{f_P}{R}\sqrt{\tfrac{2}{3}}\left(\int\left(\frac{r}{R}\right)[5I(r)+rI'(r)]\beta\gamma_5 T_{110}\right) \tag{14.103}$$

$${}^{V}F^{(0)}_{211}=-{}^{V}\mathfrak{M}^{(0)}_{211}-\frac{f_M}{R}(W_0R\pm\tfrac{6}{5}\alpha Z)\,{}^{C}\mathfrak{N}^{(0)}_{211} \tag{14.104}$$

$${}^{V}F^{(0)}_{220}={}^{V}\mathfrak{M}^{(0)}_{220}+\frac{f_M}{R}\sqrt{(10)}\,{}^{C}\mathfrak{N}^{(0)}_{211}\pm\frac{f_S}{R}(W_0R\pm\tfrac{6}{5}\alpha Z)\,{}^{V}\mathfrak{M}^{(0)}_{220} \tag{14.105}$$

$${}^{V}F^{(0)}_{220}(1,1,1,1)={}^{V}\mathfrak{M}^{(0)}_{220}(1,1,1,1)$$
$$+\frac{f_M}{R}\left\{\sqrt{\tfrac{2}{5}}\left(\int\left(\frac{r}{R}\right)[5I(r)+rI'(r)]\beta T_{211}\right)\right.$$
$$\left.+\sqrt{\tfrac{3}{5}}\left(\int\left(\frac{r}{R}\right)rI'(r)\beta T_{231}\right)\right\}$$
$$\pm\frac{f_S}{R}(W_0R\pm\tfrac{6}{5}\alpha Z)\,{}^{V}\mathfrak{M}^{(0)}_{220}(1,1,1,1) \tag{14.106}$$

$${}^{A}F^{(0)}_{221}=\pm\lambda\,{}^{A}\mathfrak{M}^{(0)}_{221}+\frac{f_T}{R}[\sqrt{(15)}\,{}^{C}\mathfrak{N}^{(0)}_{211}-(W_0R\pm\tfrac{6}{5}\alpha Z)\,{}^{A}\mathfrak{M}^{(0)}_{221}] \tag{14.107}$$

$${}^{A}F^{(0)}_{221}(1,1,1,1)=\pm\lambda\,{}^{A}\mathfrak{M}^{(0)}_{221}(1,1,1,1)$$
$$+\frac{f_T}{R}\left\{\sqrt{\tfrac{3}{5}}\left(\int\left(\frac{r}{R}\right)[5I(r)+rI'(r)]\beta T_{211}\right)\right.$$
$$-\sqrt{\tfrac{2}{5}}\left(\int\left(\frac{r}{R}\right)rI'(r)\beta T_{231}\right)$$
$$\left.-(W_0R\pm\tfrac{6}{5}\alpha Z)\,{}^{A}\mathfrak{M}^{(0)}_{221}(1,1,1,1)\right\} \tag{14.108}$$

$$^{A}F^{(0)}_{321} = \mp\lambda\ {}^{A}\mathfrak{M}^{(0)}_{321} + \frac{f_T}{R}(W_0 R \pm \tfrac{\xi}{5}\alpha Z)\ {}^{A}\mathfrak{M}^{(0)}_{321} \tag{14.109}$$

(upper sign β^--decay, lower sign β^+-decay).

Here, we have $I'(r) = \mathrm{d}I/\mathrm{d}r$, and we have written $I(r)$ instead of $I(k_e, m, n, \rho; r)$. The functions $I(k_e, m, n, \rho; r)$ are, by the way, discussed in detail in Section 4.3. There explicit expressions for some special cases can also be found. The nuclear transition matrix elements $\mathfrak{M}^{(N)}_{KLs}$ and $\mathfrak{M}^{(0)}_{KLs}(k_e, m, n, \rho)$ used above are based on one-body operator matrix elements. Notwithstanding the complexity of nuclear structure, it is then possible to expand the many-particle matrix elements in terms of single-particle matrix elements.

This has been shown in detail in Sections 8.1.1.3 and 8.1.1.4. General formulae for the calculation of these single particle matrix elements have also been derived in Section 8.1.1.2. Because of the great importance of these matrix elements we will, however, additionally present more explicit special expressions for allowed transitions. For the derivation of these expressions use has been made of eqns (8.21*a–d*) and (8.30*a–b*).

Thus in Table 14.1 single particle matrix elements (in j–j coupling) for the different possible initial and final spin values j and orbital angular momentum values l of the nuclear wave functions are summarized (see also Behrens *et al.* 1978).

In Section 8.1.1.4 we saw (see eqn (8.140)) that $^{V}\mathfrak{M}^{(0)}_{000}$ is only different from zero if the selection rules $\Delta J = 0$ and $\Delta T = 0$ are fulfilled. Assuming CVC to be valid (see Section 10.2) this leads to the consequence that also

$$^{V}F^{(0)}_{000} \neq 0 \quad \text{only if} \quad \Delta J = 0 \quad \text{and} \quad \Delta T = 0. \tag{14.110}$$

Even if isospin is not a perfect good quantum number in nature, for transitions with $\Delta J = 0$ and $\Delta T = 1$ we have nevertheless

$$^{V}F^{(0)}_{000} \simeq 0 \tag{14.111}$$

(see Raman *et al.* 1975).

In the case where the single-particle matrix element $\mathfrak{M}^{(0)}_{000}$ differs from zero the above result for the many-particle matrix element $^{V}\mathfrak{M}^{(0)}_{000}$ requires that the many-particle factor $C(K)$ follows the same selection rules for $K = 0$, i.e.

$$C(0) \neq 0 \quad \text{only if} \quad \Delta J = 0 \quad \text{and} \quad \Delta T = 0. \tag{14.112}$$

That means in the limit of isospin being a good quantum number we have also $^{V}F^{(1)}_{000}$, $^{V}F^{(1)}_{000}(k_e, m, n, \rho)$, $^{V}F^{(0)}_{011}$ and $^{V}F^{(0)}_{011}(1, 1, 1, 1)$ being $\simeq 0$ if $\Delta T \neq 0$.

Of course, if $\Delta J \neq 0$ all rank 0 form factor coefficients exactly equal zero.

TABLE 14.1 *Single-particle matrix elements (in j–j coupling) for allowed β-transitions (j = spin and l = orbital angular momentum of the nuclear wave function) (Behrens et al. 1978)*

$j_i = j_f = l + \frac{1}{2}$

$$^{V}\mathfrak{M}^{0}_{000} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}(l+1)^{1/2}\,\Omega$$

$$^{V}\mathfrak{M}^{0}_{011} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}(l+1)^{1/2}\frac{R}{2}\int_0^\infty g_f\{(E_i-E_f)-(V_i-V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{A}\mathfrak{M}^{0}_{101} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{(l+1)(2l+3)}{2l+1}\right\}^{1/2}\Omega$$

$$^{A}\mathfrak{M}^{0}_{121} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\,l\left\{\frac{2(l+1)}{(2l+1)(2l+3)}\right\}^{1/2}\frac{\langle r^2\rangle}{R^2}$$

$$^{V}\mathfrak{M}^{0}_{111} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}(l+1)\left\{\frac{6(l+1)(2l+3)}{2l+1}\right\}^{1/2}\frac{1}{2MR}\,\Omega$$

$$^{A}\mathfrak{M}^{0}_{110} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3(l+1)}{(2l+1)(2l+3)}\right\}^{1/2}\frac{R}{2}\int_0^\infty g_f\{(E_i-E_f)-(V_i-V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{0}_{211} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{2l(l+1)(l+2)}{(2l+1)(2l+3)}\right\}^{1/2}\frac{R}{2}\int_0^\infty g_f\{(E_i-E_f)-(V_i-V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{0}_{220} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{5l(l+1)(l+2)}{(2l+1)(2l+3)}\right\}^{1/2}\frac{\langle r^2\rangle}{R^2}$$

$$^{A}\mathfrak{M}^{0}_{221} = 0$$

$j_i = j_f = l - \frac{1}{2}$

$$^{V}\mathfrak{M}^{0}_{000} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\sqrt{l}\,\Omega$$

$$^{V}\mathfrak{M}^{0}_{011} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\sqrt{l}\frac{R}{2}\int_0^\infty g_f\{(E_i-E_f)-(V_i-V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{A}\mathfrak{M}^{0}_{101} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{l(2l-1)}{2l+1}\right\}^{1/2}\Omega$$

$$^{A}\mathfrak{M}^{0}_{121} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}(l+1)\left\{\frac{2l}{(2l-1)(2l+1)}\right\}^{1/2}\frac{\langle r^2\rangle}{R^2}$$

$$^{V}\mathfrak{M}^{0}_{111} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\,l\left\{\frac{6l(2l-1)}{2l+1}\right\}^{1/2}\frac{1}{2MR}\,\Omega$$

$$^{A}\mathfrak{M}^{0}_{110} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l}{(2l+1)(2l-1)}\right\}^{1/2}\frac{R}{2}\int_0^\infty g_f\{(E_i-E_f)-(V_i-V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{0}_{211} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{2(l-1)l(l+1)}{(2l-1)(2l+1)}\right\}^{1/2}\frac{R}{2}\int_0^\infty g_f\{(E_i-E_f)-(V_i-V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{0}_{220} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{5l(l+1)(l-1)}{(2l-1)(2l+1)}\right\}^{1/2}\frac{\langle r^2\rangle}{R^2}$$

$$^{A}\mathfrak{M}^{0}_{221} = 0$$

TABLE 14.1 (*Continued*)

$j_f = l \pm \frac{1}{2} \quad j_i = l \mp \frac{1}{2}$

$$^{V}\mathfrak{M}^{0}_{000} = 0$$

$$^{V}\mathfrak{M}^{0}_{011} = 0$$

$$^{A}\mathfrak{M}^{0}_{101} = \mp \frac{\sqrt{2}}{(2J_i+1)^{1/2}} 2\left\{\frac{l(l+1)}{2l+1}\right\}^{1/2} \Omega$$

$$^{A}\mathfrak{M}^{0}_{121} = \mp \frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{l(l+1)}{2(2l+1)}\right\}^{1/2} \frac{\langle r^2 \rangle}{R^2}$$

$$^{V}\mathfrak{M}^{0}_{111} = \pm \frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{3l(l+1)}{2(2l+1)}\right\}^{1/2} \frac{1}{MR} \Omega$$

$$^{A}\mathfrak{M}^{0}_{110} = \pm \frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{3l(l+1)}{2l+1}\right\}^{1/2} \times \left\{\pm \frac{2l+1}{2MR} \Omega + \frac{R}{2} \int_0^\infty g_f\{(E_i - E_f) - (V_i - V_f)\} \left(\frac{r}{R}\right)^2 g_i r^2 \, dr\right\}$$

$$^{V}\mathfrak{M}^{0}_{220} = \pm \frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{15l(l+1)}{(2l-1)(2l+1)(2l+3)}\right\}^{1/2} \frac{\langle r^2 \rangle}{R^2}$$

$$^{A}\mathfrak{M}^{0}_{221} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{5l(l+1)(2l+1)}{2(2l-1)(2l+3)}\right\}^{1/2} \frac{\langle r^2 \rangle}{R^2}$$

$$^{V}\mathfrak{M}^{0}_{211} = \pm \frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{3l(l+1)(2l+3)}{2(2l-1)(2l+1)}\right\}^{1/2} \frac{1}{2l+3} R \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\} \left(\frac{r}{R}\right)^2 g_i r^2 \, dr$$

$j_f = l_f - \frac{1}{2} \quad j_i = l_i + \frac{1}{2}$
$l_f = l + 2 \quad l_i = l$

$$^{A}\mathfrak{M}^{0}_{121} = \frac{3\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{(l+1)(l+2)}{2(2l+3)}\right\}^{1/2} \int_0^\infty g_f \left(\frac{r}{R}\right)^2 g_i r^2 \, dr$$

$$^{V}\mathfrak{M}^{0}_{220} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{15(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)}\right\}^{1/2} \int_0^\infty g_f \left(\frac{r}{R}\right)^2 g_i r^2 \, dr$$

$$^{A}\mathfrak{M}^{0}_{221} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{5(l+1)(l+2)(2l+3)}{2(2l+1)(2l+5)}\right\}^{1/2} \int_0^\infty g_f \left(\frac{r}{R}\right)^2 g_i r^2 \, dr$$

$$^{V}\mathfrak{M}^{0}_{211} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{3(l+1)(l+2)(2l+1)}{2(2l+3)(2l+5)}\right\}^{1/2} \frac{1}{2l+1} \times R \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\} \left(\frac{r}{R}\right)^2 g_i r^2 \, dr$$

$j_f = l_f + \frac{1}{2} \quad j_i = l_i + \frac{1}{2}$
$l_f = l + 2 \quad l_i = l$

$$^{V}\mathfrak{M}^{0}_{220} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{15(l+1)(l+2)(l+3)}{2(2l+3)(2l+5)}\right\}^{1/2} \int_0^\infty g_f \left(\frac{r}{R}\right)^2 g_i r^2 \, dr$$

$$^{A}\mathfrak{M}^{0}_{221} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}} \left\{\frac{5(l+1)(l+2)(l+3)}{(2l+3)(2l+5)}\right\}^{1/2} \int_0^\infty g_f \left(\frac{r}{R}\right)^2 g_i r^2 \, dr$$

TABLE 14.1 (*Continued*)

$j_f = l_f + \frac{1}{2} \qquad j_i = l_i + \frac{1}{2}$
$l_f = l + 2 \qquad l_i = l$

$$^{V}\mathfrak{M}^{0}_{211} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{12(l+1)(l+2)(l+3)}{(2l+3)(2l+5)}\right\}^{1/2}\frac{R}{4}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$j_f = l_f + \frac{1}{2} \qquad j_i = l_i - \frac{1}{2}$
$l_f = l + 2 \qquad l_i = l$

$$^{V}\mathfrak{M}_{220} = 0$$
$$^{A}\mathfrak{M}_{221} = 0$$
$$^{V}\mathfrak{M}_{211} = 0$$

$j_f = l_f - \frac{1}{2} \qquad j_i = l_i - \frac{1}{2}$
$l_f = l + 2 \qquad l_i = l$

$$^{V}\mathfrak{M}^{0}_{211} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{12l(l+1)(l+2)}{(2l+1)(2l+3)}\right\}^{1/2}\frac{R}{4}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{0}_{220} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{15l(l+1)(l+2)}{2(2l+1)(2l+3)}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{A}\mathfrak{M}^{0}_{221} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{5l(l+1)(l+2)}{(2l+1)(2l+3)}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$j_i = j_f = l + \frac{1}{2}$

$$^{C}\mathfrak{N}^{0}_{011} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}(l+1)^{1/2}\frac{2l+3}{2MR}\Omega$$

$$^{C}\mathfrak{N}^{0}_{111} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}(l+1)\left\{\frac{6(l+1)}{(2l+1)(2l+3)}\right\}^{1/2}\frac{R}{2}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{D}\mathfrak{N}^{0}_{110} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3(l+1)(2l+3)}{(2l+1)}\right\}^{1/2}\frac{1}{2MR}\Omega$$

$$^{C}\mathfrak{N}^{0}_{211} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{2l(l+1)(l+2)(2l+3)}{2l+1}\right\}^{1/2}\frac{1}{2MR}\Omega$$

$j_i = j_f = l - \frac{1}{2}$

$$^{C}\mathfrak{N}^{0}_{011} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\sqrt{l}\,\frac{2l-1}{2MR}\Omega$$

$$^{C}\mathfrak{N}^{0}_{111} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\,l\left\{\frac{6l}{(2l-1)(2l+1)}\right\}^{1/2}\frac{R}{2}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$$^{D}\mathfrak{N}^{0}_{110} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l(2l-1)}{2l+1}\right\}^{1/2}\frac{1}{2MR}\Omega$$

$$^{C}\mathfrak{N}^{0}_{211} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{2(l-1)l(l+1)(2l-1)}{2l+1}\right\}^{1/2}\frac{1}{2MR}\Omega$$

TABLE 14.1 (*Continued*)

$j_f = l \pm \frac{1}{2} \qquad j_i = l \mp \frac{1}{2}$

$$^{C}\mathfrak{N}^{0}_{011} = 0$$

$$^{C}\mathfrak{N}^{0}_{111} = \pm\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l(l+1)}{2(2l+1)}\right\}^{1/2}\left\{\pm\frac{2l+1}{2MR}\Omega + \frac{R}{2}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr\right\}$$

$$^{D}\mathfrak{N}^{0}_{110} = \pm\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l(l+1)}{2l+1}\right\}^{1/2}\frac{1}{MR}\Omega$$

$j_f = l + \frac{1}{2} \qquad j_i = l - \frac{1}{2}$

$$^{C}\mathfrak{N}^{0}_{211} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l(l+1)(2l+3)}{2(2l-1)(2l+1)}\right\}^{1/2} \times\left[\frac{2l-1}{2MR}\Omega + \frac{2l+1}{2l+3}\frac{R}{2}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr\right]$$

$j_f = l - \frac{1}{2} \qquad j_i = l + \frac{1}{2}$

$$^{C}\mathfrak{N}^{0}_{211} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l(l+1)(2l+3)}{2(2l-1)(2l+1)}\right\}^{1/2} \times\left[-\frac{2l-1}{2MR}\Omega + \frac{2l+1}{2l+3}\frac{R}{2}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr\right]$$

$j_f = l_f + \frac{1}{2} \qquad j_i = l_i + \frac{1}{2}$
$l_f = l + 2 \qquad l_i = l$

$$^{C}\mathfrak{N}^{0}_{211} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{12(l+1)(l+2)(l+3)}{(2l+3)(2l+5)}\right\}^{1/2}\frac{R}{4}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$j_f = l_f + \frac{1}{2} \qquad j_i = l_i - \frac{1}{2}$
$l_f = l + 2 \qquad l_i = l$

$$^{C}\mathfrak{N}^{0}_{211} = 0$$

$j_f = l_f - \frac{1}{2} \qquad j_i = l_i + \frac{1}{2}$
$l_f = l + 2 \qquad l_i = l$

$$^{C}\mathfrak{N}^{0}_{211} = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3(l+1)(l+2)(2l+1)}{2(2l+3)(2l+5)}\right\}^{1/2}\frac{(2l+6)}{(2l+1)} \times\frac{R}{4}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr$$

$j_f = l_f - \frac{1}{2} \qquad j_i = l_i - \frac{1}{2}$
$l_f = l + 2 \qquad l_i = l$

$$^{C}\mathfrak{N}^{0}_{211} = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{12l(l+1)(l+2)}{(2l+1)(2l+3)}\right\}^{1/2}\frac{R}{4}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)^2 g_i r^2\,dr.$$

TABLE 14.1 (*Continued*)

Note that

$$^{D}\mathfrak{M}^{0}_{110} = -\frac{\sqrt{3}}{2MR}\,{}^{A}\mathfrak{M}^{0}_{101}.$$

For the so-called non-relativistic nuclear matrix elements ($\mathfrak{M}_{000}$, $\mathfrak{M}_{101}$, $\mathfrak{M}_{121}$, $\mathfrak{M}_{220}$, $\mathfrak{M}_{221}$) the calculation of

$$\mathfrak{M}^{N}_{KLs}(k, m, n, \rho)$$

can simply be carried out by replacing in $\mathfrak{M}_{000}$ and $\mathfrak{M}_{101}$

$$\Omega \to \int_0^\infty g_f\left(\frac{r}{R}\right)^{2N} I(k, m, n, \rho; r) g_i r^2\,dr$$

and in $\mathfrak{M}_{121}$, $\mathfrak{M}_{220}$, and $\mathfrak{M}_{221}$

$$\int_0^\infty g_f\left(\frac{r}{R}\right)^{2} g_i r^2\,dr \to \int_0^\infty g_f\left(\frac{r}{R}\right)^{2N+2} I(k, m, n, \rho; r) g_i r^2\,dr.$$

For the so-called relativistic nuclear matrix elements ($\mathfrak{M}_{011}, \mathfrak{M}_{111}, \mathfrak{M}_{110}$) the values of

$$\mathfrak{M}^{N}_{KLs}(k, m, n, \rho)$$

are obtained in the following way:

$j_i = j_f = l + \frac{1}{2}$

$$^{V}\mathfrak{M}^{0}_{011}(1,1,1,1) = \frac{\sqrt{2}}{(2J_i+1)^{1/2}}(l+1)^{1/2} \times \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\int_0^r \frac{x}{R} I(1,1,1,1;x)\,dx\, g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{0}_{111}(1,1,1,1) = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}(l+1)\left\{\frac{6(l+1)}{(2l+1)(2l+3)}\right\}^{1/2}\left\{\frac{l+1}{MR}\int_0^\infty g_f I(1,1,1,1;r) g_i r^2\,dr + \frac{1}{2MR}\int_0^\infty g_f\{I(1,1,1,1;r) + rI'(1,1,1,1;r)\} g_i r^2\,dr\right\}$$

$$^{A}\mathfrak{M}^{0}_{110}(1,1,1,1) = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3(l+1)}{(2l+1)(2l+3)}\right\}^{1/2} \times \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\int_0^r \frac{x}{R} I(1,1,1,1;x)\,dx\, g_i r^2\,dr$$

$j_i = j_f = l - \frac{1}{2}$

$$^{V}\mathfrak{M}^{0}_{011}(1,1,1,1) = \frac{\sqrt{2}}{(2J_i+1)^{1/2}} l^{1/2}\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\int_0^r \frac{x}{R} I(1,1,1,1;x)\,dx\, g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{0}_{111}(1,1,1,1) = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{6l}{(2l-1)(2l+1)}\right\}^{1/2} \times \left\{\frac{l}{MR}\int_0^\infty g_f I(1,1,1,1;r) g_i r^2\,dr - \frac{1}{2MR}\int_0^\infty g_f\{I(1,1,1,1;r) + rI'(1,1,1,1;r)\} g_i r^2\,dr\right\}$$

TABLE 14.1 (*Continued*)

$$^{A}\mathfrak{M}^{0}_{110}(1,1,1,1) = -\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l}{(2l+1)(2l-1)}\right\}^{1/2} \times \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\int_0^r \frac{x}{R} I(1,1,1,1;x)\,dx\, g_i r^2\,dr$$

$j_f = l \pm \frac{1}{2} \qquad j_i = l \mp \frac{1}{2}$

$$^{V}\mathfrak{M}^{0}_{111}(1,1,1,1) = \pm\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l(l+1)}{2(2l+1)}\right\}^{1/2}\left\{\frac{1}{MR}\int_0^\infty g_f I(1,1,1,1;r) g_i r^2\,dr + \frac{1}{2MR}\int_0^\infty g_f r I'(1,1,1,1;r) g_i r^2\,dr\right\}$$

$$^{A}\mathfrak{M}^{0}_{110}(1,1,1,1) = \pm\frac{\sqrt{2}}{(2J_i+1)^{1/2}}\left\{\frac{3l(l+1)}{2l+1}\right\}^{1/2}\left\{\pm\frac{2l+1}{2MR}\int_0^\infty g_f I(1,1,1,1;r) g_i r^2\,dr + \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\int_0^r \frac{x}{R} I(1,1,1,1;x)\,dx\, g_i r^2\,dr\right\}.$$

In the formulae listed above g is defined as the solution of the Schrödinger radial equation (see eqn (8.27))

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2M_N(E-V) - \frac{l(l+1)}{r^2}\right\} g(r) = 0.$$

E and V correspond to the energy and potential in the initial and final bound states.

$$\Omega = \int_0^\infty g_f g_i r^2\,dr \simeq 1$$

$$\frac{\langle r^2\rangle}{R^2} = \int_0^\infty g_f \left(\frac{r}{R}\right)^2 g_i r^2\,dr.$$

We come, therefore, to the conclusion that if $\Delta J \neq 0$

$$\begin{aligned} A_0 &= 0 \\ A_1 &= 0 \\ A_2 &= 0 \\ B_0 &= 0 \\ B_1 &= 0 \end{aligned} \qquad (14.113)$$

or if $\Delta J = 0$ but $\Delta T = 1$

$$\begin{aligned} A_0 &\simeq 0 \\ A_1 &\simeq 0 \\ A_2 &\simeq 0 \\ B_0 &\simeq 0 \\ B_1 &\simeq 0 \end{aligned} \qquad (14.114)$$

i.e. all terms connected with A_0, A_1, A_2, B_0, and B_1 can (or have to) be omitted either for pure Gamow–Teller transitions or for mixed transitions where $\Delta T = 1$. In this context it should be remarked that these cases represent the overwhelming majority of allowed beta-transitions.

14.2.1.3. Observables

14.2.1.3.1. Shape factor and ft-value. General expressions for the beta-spectrum have been derived in Section 7.2.2.2. By inserting the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ given in eqns (14.48) to (14.63) into eqns (7.53) and (7.54), special formulae for allowed transition can, therefore, easily be obtained. The beta-spectrum, i.e. the probability for the emission of an electron (positron) in our energy interval between W_e and $W_e + dW_e$ has the form (see eqn (7.55))

$$P(W_e)\,dW_e = \frac{G_\beta'^2}{2\pi^3}[A_0^2 + C_0^2]F_0L_0C(W_e)p_eW_e(W_0 - W_e)^2\,dW_e \quad (14.115)$$

where $F(Z, W_e) = F_0L_0$ is the Fermi function (see Chapter 4) and $C(W_e)$ the so-called shape factor.

It is suitable to write the shape factor of allowed transitions in the form (see eqn (7.56))

$$C(W_e) = 1 + aW_e + \mu_1\gamma_1 b/W_e + cW_e^2 \quad (14.116)$$

which is normalized in a certain sense. The coefficients a, b and c are functions of the quantities A_0, A_1 etc. which have been stated in Section 14.2.1.1 (see eqns (14.64) to (14.89)). They read as† follows (see also Behrens *et al.* 1978):‡§

$$a = R[2A_0A_1 + 2RB_0B_1 + 2C_0C_1 + 2RD_0D_1 - \tfrac{2}{9}W_0R \times (E_0^2 + G_0^2)]/(A_0^2 + C_0^2) \quad (14.117)$$

$$b = -2R(A_0B_0 + C_0D_0)/(A_0^2 + C_0^2) \quad (14.118)$$

† Note that the definition of the coefficient b differs from that in Behrens *et al.* (1978). The function $\mu_1\gamma_1$ has here been split off the quantity b.

‡ If the antineutrino rest mass m_ν is different from zero the statistical factor has to be replaced by

$$[\Delta - W_e][[\Delta - W_e]^2 - m_\nu^2]^{1/2}p_eW_e.$$

In addition, the shape factor given in eqn (14.116) contains terms proportional to $1/W_\nu$ and $1/(W_eW_\nu)$. By an experimental investigation of beta-spectra with very low transition energies Δ, it is, therefore, possible to determine the neutrino rest mass if one exists. Thus, a great number of corresponding measurements on the tritium beta-decay (maximal kinetic energy $E_0 = 18{\cdot}6$ keV) have been carried out (for a compilation see Behrens and Szybisz 1976). An upper limit of $m_\nu < 50$ eV was obtained in this way. A very recent result (Lubimov *et al.* 1980) gives, however, evidence for a non-zero electron antineutrino mass in the range $14 \leqslant m_\nu \leqslant 46$ eV.

§ If in the phase space factor a correction term for the kinetic recoil energy is included we get additional terms of pure kinematical origin which read as $R(W_e) = A_0^2\{1 + 2W_e/M_A\}/(A_0^2 + C_0^2) + C_0^2\{1 - 2W_0/(3M_A) + 10W_e/(3M_A) - 2/(3M_AW_e)\}/(A_0^2 + C_0^2)$.

$$c = R^2[A_1^2 + 2A_0A_2 + C_1^2 + 2C_0C_2 + \tfrac{1}{5}(E_0^2 + \lambda_2 F_0^2 + G_0^2 + \lambda_2 H_0^2)]/(A_0^2 + C_0^2). \tag{14.119}$$

$$\gamma_1 = \sqrt{\{1-(\alpha Z)^2\}}. \tag{14.120}$$

μ_1 and λ_2 are the special Coulomb functions introduced before (see Chapter 4 and also eqn (7.51c)).

The ft-value follows from eqn (14.115) and reads as

$$ft = \frac{2\pi^3 \ln 2}{G_\beta'^2[C_0^2 + A_0^2] \cdot \overline{C(W_e)}} \tag{14.121}$$

or, if higher order terms are neglected, as (see also eqn (14.20))

$$ft = \frac{2\pi^3 \ln 2}{G_\beta'^2[({}^{\mathrm{V}}F_{000}^{(0)})^2 + ({}^{\mathrm{A}}F_{101}^{(0)})^2]} \tag{14.122}$$

because then $A_0 = {}^{\mathrm{V}}F_{000}^{(0)}$ and $C_0 = -{}^{\mathrm{A}}F_{101}^{(0)}$.

The quantities A_0 and C_0 can also be calculated in impulse approximation from the eqns (14.64), (14.69), (14.90), (14.93), (14.95), (14.98) and (14.100). In the case of $\alpha Z \ll 1$, they are related to the nuclear matrix elements as follows:

$$A_0 = {}^{\mathrm{V}}\mathfrak{M}_{000}^{(0)} + \frac{W_0 R}{3} {}^{\mathrm{V}}\mathfrak{M}_{011}^{(0)} \tag{14.123}$$

$$C_0 = \pm\lambda\, {}^{\mathrm{A}}\mathfrak{M}_{101}^{(0)}\left[1 \mp \tfrac{2}{3} W_0 \frac{f_\mathrm{T} + f_\mathrm{M}}{\lambda}\right] + \frac{W_0 R}{3} \times [\mp\lambda\sqrt{\tfrac{1}{3}}\, {}^{\mathrm{A}}\mathfrak{M}_{110}^{(0)} + \sqrt{\tfrac{2}{3}}\, {}^{\mathrm{V}}\mathfrak{M}_{111}^{(0)}] \tag{14.124}$$

or in cartesian notation

$$A_0 = \int 1 - \frac{W_0 R}{3} \int \mathrm{i}\frac{\boldsymbol{\alpha}\mathbf{r}}{R} \tag{14.125}$$

$$C_0 = \pm\lambda \int \boldsymbol{\sigma}\left[1 \mp \tfrac{2}{3} W_0 \frac{f_\mathrm{T} + f_\mathrm{M}}{\lambda}\right] - \frac{W_0 R}{3} \times \left[\pm\lambda \int \gamma_5 \frac{\mathbf{i}\mathbf{r}}{R} + \int \frac{\boldsymbol{\alpha}\times\mathbf{r}}{R}\right]. \tag{14.126}$$

Contradictory to a first look, it should be noted that the ft-value is nearly independent of f_M because the term proportional to f_M cancels in the product $(A_0^2 + C_0^2)\overline{C(W_e)}$ (see Wilkinson 1971a).

For pure Gamow–Teller transitions ($\Delta J = 1$) or for mixed transitions ($\Delta J = 0$) with $\Delta T = 1$, where the quantities A_0, A_1, A_2, B_0, and B_1 have to (or can) be omitted, the linear slope a of the shape factor is of special

interest. In this case a has essentially the form

$$a = -2R\left\{-\tfrac{2}{3}\sqrt{\tfrac{2}{3}}\frac{{}^{V}F^{(0)}_{111}}{{}^{A}F^{(0)}_{101}} + \tfrac{2}{27}W_0 R\left[\frac{-5\,{}^{A}F^{(0)}_{101} + \sqrt{2}\,{}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}}\right]\right.$$
$$\left. \pm \tfrac{1}{27}\alpha Z\left[\frac{{}^{A}F^{(1)}_{101}(1,1,1,1) - 2\sqrt{2}\,{}^{A}F^{(0)}_{121}(1,1,1,1) + 9\,{}^{A}F^{(1)}_{101}(1,2,2,1)}{{}^{A}F^{(0)}_{101}}\right]\right\}. \tag{14.127}$$

The main contribution of the slope a is given by the first term of eqn (14.127) which contains the information on the induced weak magnetic interaction. Taking only into account this first term the slope a is approximately given by

$$a \simeq \tfrac{4}{3}\sqrt{\tfrac{2}{3}}\, R\frac{{}^{V}F^{(0)}_{111}}{{}^{A}F^{(0)}_{101}}. \tag{14.128}$$

Now, we introduce the impulse approximation and express ${}^{V}F^{(0)}_{111}$ and ${}^{A}F^{(0)}_{101}$ in terms of nuclear matrix elements. From the relations (14.95) and (14.98) we then obtain

$$a \simeq \pm\frac{8}{3}\left[-\tfrac{1}{2}\sqrt{\tfrac{2}{3}}\, R\frac{{}^{V}\mathfrak{M}^{(0)}_{111}}{\lambda\,{}^{A}\mathfrak{M}^{(0)}_{101}} + \frac{f_M}{\lambda}\right]. \tag{14.129}$$

Of course, we have further terms (two-body, three-body matrix elements) which are introduced by exchange effects (see Section 8.2). For simplicity these, however, have here (and in the following) been omitted.

In cartesian notation (see Section 8.1.1.6) this equation reads as

$$a \simeq \pm\frac{8}{3}\left[\frac{R\int\frac{\boldsymbol{\alpha}\times\mathbf{r}}{R}}{2\lambda\int\boldsymbol{\sigma}} + \frac{f_M}{\lambda}\right] \tag{14.130}$$

(upper (lower) sign β^- (β^+) decay).

The above expressions show very clearly that the slope a is essentially governed by two terms, the first one strongly dependent on the nuclear structure and the second one proportional to the induced weak magnetic coupling constant f_M, but independent of the special nuclear structure. In the non-relativistic limit (see Section 8.1.1.2) the first term reads as

$$\frac{R\int\frac{\boldsymbol{\alpha}\times\mathbf{r}}{R}}{2\lambda\int\boldsymbol{\sigma}} \to \frac{1}{2M_N\lambda}\left[1 + \frac{\int\mathbf{I}}{\int\boldsymbol{\sigma}}\right] \tag{14.131}$$

where M_N is the nuclear mass and $\mathbf{I}$ denotes the angular momentum

operator. If $\int \mathbf{l}/\int \boldsymbol{\sigma}$ is set equal to zero, one obtains the well known expression

$$a = \pm\frac{8}{3}\left[\frac{1}{2M_N\lambda}+\frac{f_M}{\lambda}\right] \tag{14.132}$$

which is quoted in many textbooks. This latter approximation is, however, not justified, as it can easily be shown by applying the single independent particle model for the description of the nuclear structure.

Then in the ratio ${}^{V}\mathfrak{M}^{(0)}_{111}/{}^{A}\mathfrak{M}^{(0)}_{101}$ the many-particle factor $C(1)$ (see Section 8.1.1.3) cancels and this ratio is simply given by the ratio of the corresponding single particle matrix elements listed in Table 14.1. That means it holds:

for $j_i = j_f = l+\frac{1}{2}$

$$a = \pm\frac{8}{3}\left[\frac{1+l}{2M_N\lambda}+\frac{f_M}{\lambda}\right], \tag{14.133}$$

for $j_i = j_f = l-\frac{1}{2}$

$$a = \pm\frac{8}{3}\left[-\frac{l}{2M_N\lambda}+\frac{f_M}{\lambda}\right], \tag{14.134}$$

and for $j_i = j_f + 1 \quad l_i = l_f = l$

$$a = \pm\frac{8}{3}\left[\frac{1}{4M_N\lambda}+\frac{f_M}{\lambda}\right] \tag{14.135}$$

(j_i, j_f, l_i, l_f are the spin and orbital angular momenta of the corresponding valence nucleons).

Obviously the influence of the nuclear structure is important (with exception of the case $j_i = j_f \pm 1$) and increases with increasing orbital angular momentum l of the participating valence nucleons. The above equations also imply that the coefficients a, b, and c are very small ($a \simeq (10^{-2}\text{–}10^{-3})[m_0c^2]^{-1}$, $b \simeq (10^{-2}\text{–}10^{-3})[m_0c^2]$, and $c \simeq (10^{-4}\text{–}10^{-5}) \times [m_0c^2]^{-2}$). Thus the deviations from the so-called allowed form, i.e. from $C(W_e) = 1$, can very often be neglected.

Up to now we have only presented formulae for β^-- or β^+-decay, respectively. For completeness we will, however, also consider the electron capture at the end of this section (see Bambynek *et al.* 1977). If contributions from higher order (so-called second forbidden) terms are neglected, in allowed transitions electrons can only be captured from orbits with $\kappa_x = \pm 1$, i.e. from the K, L_1, L_2, M_1, M_2 etc. shells (see eqns (7.170) and (7.174)). Capture from orbits with $\kappa_x = \pm 2$, for example, would be governed by the quantities $M_1(2, 1)$, $M_2(2, 1)$, etc. which are smaller than $M_0(1, 1)$ and $M_1(1, 1)$ by at least a factor $p_xR \leqslant 0{\cdot}02$.

Therefore, we have (see eqns (7.174) and (7.175))

$$C_x(\kappa = \pm 2) \leqslant 4 \times 10^{-4}\, C_x(\kappa = \pm 1). \tag{14.136}$$

Thus capture from orbits with $\kappa \neq \pm 1$ can be expected to be difficult to observe. On the other hand, capture from such states in principle offers a possibility of determining the higher forbidden contributions separately from the leading terms. Taking into account the leading terms only, we find (see eqn (7.174) and

$$C_x(\kappa = \pm 1) = ({}^{V}F_{000}^{(0)})^2 + ({}^{A}F_{101}^{(0)})^2 \tag{14.137}$$

and

$$\sum_x n_x f_x t = \frac{2\pi^3 \ln 2}{G_\beta'^2 \{({}^{V}F_{000}^{(0)})^2 + ({}^{A}F_{101}^{(0)})^2\}} \tag{14.138}$$

or explicitly

$$n_K(ft)_K + n_{L_1}(ft)_{L_1} + n_{L_2}(ft)_{L_2} + \ldots = \frac{2\pi^3 \ln 2}{G_\beta'^2 \{({}^{V}F_{000}^{(0)})^2 + ({}^{A}F_{101}^{(0)})^2\}}. \tag{14.139}$$

For capture from atoms with high Z-values we have to replace $C(\kappa = \pm 1)$ by $C(\kappa = \pm 1) = A_0^2 + C_0^2$ with

$$A_0 = {}^{V}F_{000}^{(0)} + \frac{\alpha \tilde{Z}}{3} {}^{V}F_{011}^{(0)}(1, 1, 1, 1) - \frac{(\alpha \tilde{Z})^2}{6} {}^{V}F_{000}^{(1)}(1, 2, 2, 2) \tag{14.140}$$

$$C_0 = -{}^{A}F_{101}^{(0)} + \frac{\alpha \tilde{Z}}{3}[\sqrt{\tfrac{1}{3}}\, {}^{A}F_{110}^{(0)}(1, 1, 1, 1) - \sqrt{\tfrac{2}{3}}\, {}^{V}F_{111}^{(0)}(1, 1, 1, 1)] + \frac{(\alpha \tilde{Z})^2}{6} {}^{A}F_{101}^{(1)}(1, 2, 2, 2) \tag{14.141}$$

($\tilde{Z}$ = atomic number of the parent nucleus).

As mentioned before (see eqn (7.176)) f_x corresponds to the integrated Fermi function and is, here, given by

$$f_K = \frac{\pi}{2} \beta_K^2 p_{\nu_K}^2 B_K \tag{14.142a}$$

$$f_{L_1} = \frac{\pi}{2} \beta_{L_1}^2 p_{\nu_{L_1}}^2 B_{L_1} \tag{14.142b}$$

$$f_{L_2} = \frac{\pi}{2} \beta_{L_2}^2 p_{\nu_{L_2}}^2 B_{L_2} \tag{14.142c}$$

etc. ($p_{\nu_x} = W_0 + W_x'$ with $W_x' = 1 - |E_x'|$, see eqn (7.171)).

The quantity n_x which is equal to the relative occupation number of electrons in a shell (for a filled shell $n_x = 1$) is of importance only for partly filled shells. The factor B_x takes into account the effects of electron exchange and overlap in the atomic cloud (see Section 4.6).

Besides, the ratios of the capture probabilities from different subshells are simply given by

$$\lambda_{L_1}/\lambda_K = \frac{n_{L_1} f_{L_1}}{n_K f_K} \tag{14.143a}$$

$$\lambda_{L_2}/\lambda_K = \frac{n_{L_2} f_{L_2}}{n_K f_K} \tag{14.143b}$$

etc.

The last observable which can be measured in this context is the electron capture to positron decay ratio since electron capture and β^+-decay are always (if energetically possible) competing processes. This ratio for allowed transitions is given by

$$\frac{\lambda_K}{\lambda_{\beta^+}} = \frac{f_K}{f_{\beta^+}}. \tag{14.144}$$

This result is based on the neglect of contributions from higher order terms. If these terms are additionally taken into account we get small corrections which read as (see also Bambynek *et al.* 1977; Firestone *et al.* 1978)

$$\frac{\lambda_K}{\lambda_{\beta^+}} = \frac{f_K}{f_{\beta^+}}(1+\eta) \tag{14.145}$$

with

$$\eta = 1 - a(\bar{W}_e + W_K') - b(1 + \mu_1\gamma_1/\bar{W}_e) + c(W_K^2 - \bar{W}_e^2).$$

a is the linear slope of the shape factor, c the quadratic and b the corresponding coefficient of the $1/W$ term for β^+-decay (see eqns (14.117) to (14.119)). $\bar{W}$ is the average (over the beta-spectrum) energy of the positron.

14.2.1.3.2. Distribution of electrons emitted from oriented nuclei. General formulae for the angular distribution $\omega(\theta_e, W_e)$ of electrons emitted from oriented nuclei have been discussed in Section 7.2.2.1. It has been shown that this distribution can be expressed as a linear combination of Legendre polynomials $P_k(\cos\theta_e)$ where θ_e describes the angle between the orientation axes of the parent nucleus and the electron emission (see eqn (7.38)):

$$\omega(\theta_e, W_e) = 1 + a_1 P_1(\cos\theta_e) + a_2 P_2(\cos\theta_e). \tag{14.146}$$

By making use of the particle parameters $b_{KK'}^{(k)}$ given in Table 7.2 and by inserting the quantities $M_K(k_e, k_\nu)$, $m_K(k_e, k_\nu)$ of eqns (14.48) to (14.63), formulae for the asymmetry coefficient a_1 and the anisotropy coefficient a_2 of allowed transition can easily be derived (see also Behrens *et al.* 1978). Firstly, we obtain for the asymmetry coefficient a_1 for transitions $J \rightarrow J$ and $J \rightarrow J \pm 1$

$$a_1 = A_1^0 f_1(J) \frac{p_e}{W_e} (1 + \alpha_\beta W_e) \tag{14.147}$$

$$A_1^0 = \mp \left(\frac{6J}{J+1}\right)^{1/2} \Lambda_1 \tilde{\Gamma}_{11}(1) \left[C_0^2 - \frac{\tilde{\Gamma}_{01}(1)}{\tilde{\Gamma}_{11}(1)} \sqrt{2}\, C_0 A_0 \right] \Big/ [C_0^2 + A_0^2] \tag{14.148}$$

$$\alpha_\beta = R \frac{A_0 C_1 - C_0 A_1}{\tilde{\Gamma}_{11}(1) C_0^2 - \tilde{\Gamma}_{01}(1) \sqrt{2}\, C_0 A_0} \left[\tilde{\Gamma}_{11}(1) \frac{2 C_0 A_0}{A_0^2 + C_0^2} + \tilde{\Gamma}_{01}(1) \sqrt{2} \frac{C_0^2 - A_0^2}{A_0^2 + C_0^2} \right] + \frac{\sqrt{2}}{3} R \frac{\eta_{12}}{\Lambda_1} \times \frac{\tilde{\Gamma}_{11}(1) C_0 F_0 + \tilde{\Gamma}_{01}(1) \sqrt{2}\, A_0 F_0 - \tilde{\Gamma}_{12}(1) \sqrt{5}\, C_0 H_0}{\tilde{\Gamma}_{11}(1) C_0^2 - \tilde{\Gamma}_{01}(1) \sqrt{2}\, C_0 A_0} \tag{14.149}$$

where Λ_1 and η_{12} are the special Coulomb functions considered in Chapter 4 (see also eqns (7.51*d*) and (7.51g)). The upper (lower) sign stands for β^- (β^+) decay. The orientation parameter $f_1(J)$ is related to $G_1(J)$ by

$$G_1(J) = \left\{ \frac{3J}{(J+1)(2J+1)} \right\}^{1/2} f_1(J) \tag{14.150}$$

From eqn (7.34) $f_1(J)$ can therefore be derived as

$$f_1(J) = \frac{1}{J} \sum_M M a_M \tag{14.151}$$

with $\sum_M a_M = 1$. The coefficients a_M represent the relative population of the initial substates. In the above eqns (14.142) and (14.143) a shorthand notation

$$\tilde{\Gamma}_{KK'}(1) = \tilde{\Gamma}_{KK'}(1, J_f, J_i) \tag{14.152}$$

has been introduced. From eqn (7.40) one obtains for $J \rightarrow J$ transitions

$$\tilde{\Gamma}_{11}(1) = \left\{ \frac{1}{6J(J+1)} \right\}^{1/2} \tag{14.153a}$$

$$\tilde{\Gamma}_{01}(1) = \frac{1}{\sqrt{3}} \tag{14.153b}$$

$$\tilde{\Gamma}_{12}(1) = -\left\{\frac{(2J-1)(2J+3)}{30J(J+1)}\right\}^{1/2}, \tag{14.153c}$$

for $J \to J+1$ transitions

$$\tilde{\Gamma}_{11}(1) = -\left\{\frac{J}{6(J+1)}\right\}^{1/2} \tag{14.154a}$$

$$\tilde{\Gamma}_{12}(1) = -\left\{\frac{J+2}{10(J+1)}\right\}^{1/2}, \tag{14.154b}$$

and for $J \to J-1$ transitions

$$\tilde{\Gamma}_{11}(1) = \left\{\frac{J+1}{6J}\right\}^{1/2} \tag{14.155a}$$

$$\tilde{\Gamma}_{12}(1) = -\left\{\frac{J-1}{10J}\right\}^{1/2}. \tag{14.155b}$$

In the last two cases where we have $\Delta J = 1$ the quantities A_0 and A_1 vanish (see eqn (14.113)).

As shown before (see eqn (14.144)) for $\Delta J = 0$ transitions with isospin selection rule $\Delta T = 1$ the quantities A_0 and A_1 can also be neglected. Then, the above eqns (14.147) and (14.148) simplify to

$$A_1^0 = \mp\left\{\frac{6J}{J+1}\right\}^{1/2} \Lambda_1 \tilde{\Gamma}_{11}(1) \tag{14.156}$$

$$\alpha_\beta = \frac{\sqrt{2}}{3} R \frac{\eta_{12}}{\Lambda_1}\left[\frac{F_0}{C_0} - \frac{\tilde{\Gamma}_{12}(1)}{\tilde{\Gamma}_{11}(1)}\sqrt{5}\frac{H_0}{C_0}\right]. \tag{14.157}$$

In impulse approximation the form factors or form factor coefficients, respectively, can be expressed in terms of nuclear matrix elements (see Section 8.1.2). Applying eqns (14.77) and (14.83) and inserting the eqns (14.98), (14.100), (14.104), (14.105), and (14.107), we get (assuming $\alpha Z \ll 1$)

$$\frac{F_0}{C_0} \simeq \frac{\lambda\sqrt{\frac{2}{3}}\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{110} \mp \frac{1}{\sqrt{3}}\,{}^{\mathrm{V}}\mathfrak{M}^{(0)}_{111}}{\lambda\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{101}} \pm \sqrt{2}\frac{f_{\mathrm{M}} - f_{\mathrm{T}}}{\lambda R} \tag{14.158}$$

$$\frac{H_0}{C_0} \simeq \frac{{}^{\mathrm{V}}\mathfrak{M}^{(0)}_{211}}{\lambda\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{101}} + \tfrac{1}{3}(W_0 R)\frac{-\sqrt{\frac{2}{5}}\,{}^{\mathrm{V}}\mathfrak{M}^{(0)}_{220} + \lambda\sqrt{\frac{3}{5}}\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{221}}{\lambda\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{101}} \tag{14.159}$$

or in cartesian notation

$$\frac{F_0}{C_0} \simeq \frac{\lambda\sqrt{2}\int \gamma_5 \frac{\mathrm{i}\mathbf{r}}{R} \pm \frac{1}{\sqrt{2}}\int \frac{\boldsymbol{\alpha}\times\mathbf{r}}{R}}{\lambda\int \boldsymbol{\sigma}} \pm \sqrt{2}\frac{f_{\mathrm{M}} - f_{\mathrm{T}}}{\lambda R} \tag{14.160}$$

$$\frac{H_0}{C_0} \simeq \pm \frac{\sqrt{3}\int \mathrm{i}\frac{A_{ij}}{R}}{2\lambda \int \boldsymbol{\sigma}} + \tfrac{1}{3}(W_0 R)\left[\pm\sqrt{3}\frac{\int \frac{R_{ij}}{R^2}}{\lambda\int \boldsymbol{\sigma}} - \frac{\sqrt{3}}{2}\frac{\int \frac{\mathrm{i}T_{ij}}{R^2}}{\int \boldsymbol{\sigma}}\right] \tag{14.161}$$

(upper sign β^--decay, lower sign β^+-decay).

Assuming a model† where $E_i - E_f - (V_i - V_f) = 0$ in the single-particle matrix elements (for example the oscillator model, and the s–d shell), some simple estimates for the ratios F_0/C_0 and H_0/C_0 can be made by making use of the independent-particle treatment (see Section 8.1.1.3) and of the single-particle matrix elements listed in Table 14.1. Analogous to eqns (14.133) to (14.135) we then obtain:

for $j_i = j_f = l + \frac{1}{2}$

$$\frac{F_0}{C_0} \simeq \pm \frac{\sqrt{2}}{R}\left[\frac{1+l}{2M_N\lambda} + \frac{f_M - f_T}{\lambda}\right] \tag{14.162}$$

$$\frac{H_0}{C_0} \simeq \mp \frac{C(2)}{C(1)} \frac{\sqrt{2}\,(W_0 R)\{l(l+2)\}^{1/2}}{3(2l+3)\lambda} \frac{\langle r^2\rangle}{R^2}, \tag{14.163}$$

for $j_i = j_f = l - \frac{1}{2}$

$$\frac{F_0}{C_0} \simeq \pm \frac{\sqrt{2}}{R}\left[-\frac{l}{2M_N\lambda} + \frac{f_M - f_T}{\lambda}\right] \tag{14.164}$$

$$\frac{H_0}{C_0} \simeq \pm \frac{C(2)}{C(1)} \frac{\sqrt{2}\,(W_0 R)\{(l+1)(l-1)\}^{1/2}}{3(2l-1)\lambda} \frac{\langle r^2\rangle}{R^2}, \tag{14.165}$$

for $j_f = j_i + 1 \quad l_i = l_f = l$

$$\frac{F_0}{C_0} \simeq \frac{\sqrt{2}}{R}\left[-\frac{2l+1}{4M_N\lambda} \pm \frac{1}{4M_N\lambda} \pm \frac{f_M - f_T}{\lambda}\right] \tag{14.166}$$

$$\frac{H_0}{C_0} \simeq \frac{C(2)}{C(1)} \frac{W_0 R}{3}\left\{\frac{3}{2(2l-1)(2l+3)}\right\}^{1/2}\left[\pm\frac{1}{\lambda} + \frac{2l+1}{2}\right]\frac{\langle r^2\rangle}{R^2}, \tag{14.167}$$

for $j_f = j_i - 1 \quad l_f = l_i = l$

$$\frac{F_0}{C_0} \simeq \frac{\sqrt{2}}{R}\left[\frac{2l+1}{4M_N} \pm \frac{1}{4M_N\lambda} \pm \frac{f_M - f_T}{\lambda}\right] \tag{14.168}$$

$$\frac{H_0}{C_0} \simeq \frac{C(2)}{C(1)} \frac{W_0 R}{3}\left\{\frac{3}{2(2l-1)(2l+3)}\right\}^{1/2}\left[\pm\frac{1}{\lambda} - \frac{2l+1}{2}\right]\frac{\langle r^2\rangle}{R^2}. \tag{14.169}$$

† Then the non relativistic limit of the matrix element $\int \gamma_5 (\mathrm{i}\mathbf{r}/R)$ is given by

$$\int \gamma_5 \frac{\mathrm{i}\mathbf{r}}{R} \to -\frac{1}{2M_N R}\int \mathrm{i}\boldsymbol{\sigma}\times\mathbf{l}.$$

The $C(K)$ are the many particle factors of eqn (8.68) or (8.77), respectively, which do not cancel if the tensor rank K is different in numerator and denominator as it is in the term $C(2)/C(1)$.

Secondly, the anisotropy coefficient a_2 is derived as

$$a_2 = f_2(J)\frac{p_e^2}{W_e}A_2^0(1+\hat{b}W_e) \tag{14.170}$$

$$A_2^0 = -2J^2\left\{\frac{15}{J(J+1)(2J+3)(2J-1)}\right\}^{1/2}\tilde{\Gamma}_{11}(2)\nu_{12}R \times\left[C_0F_0+\sqrt{\left(\frac{2}{3}\right)}\frac{\tilde{\Gamma}_{02}(2)}{\tilde{\Gamma}_{11}(2)}A_0H_0-\frac{\tilde{\Gamma}_{12}(2)}{\tilde{\Gamma}_{11}(2)}C_0H_0\right]\Big/[C_0^2+A_0^2] \tag{14.171}$$

$$\hat{b} = R\left\{C_0F_1+\sqrt{\left(\frac{2}{3}\right)}\frac{\tilde{\Gamma}_{02}(2)}{\tilde{\Gamma}_{11}(2)}A_0H_1-\frac{\tilde{\Gamma}_{12}(2)}{\tilde{\Gamma}_{11}(2)}C_0H_1-\frac{1}{5}\frac{\tilde{\Gamma}_{02}(2)}{\tilde{\Gamma}_{11}(2)}\frac{\eta_{13}}{\nu_{12}}A_0X_0 -\frac{1}{5}\sqrt{\left(\frac{2}{3}\right)}\frac{\tilde{\Gamma}_{12}(2)}{\tilde{\Gamma}_{11}(2)}\frac{\eta_{13}}{\nu_{12}}C_0X_0+\frac{1}{5}\sqrt{\left(\frac{7}{3}\right)}\frac{\tilde{\Gamma}_{13}(2)}{\tilde{\Gamma}_{11}(2)}\frac{\eta_{13}}{\nu_{12}}C_0Y_0\right\} \Big/\left\{C_0F_0+\sqrt{\left(\frac{2}{3}\right)}\frac{\tilde{\Gamma}_{02}(2)}{\tilde{\Gamma}_{11}(2)}A_0H_0-\frac{\tilde{\Gamma}_{12}(2)}{\tilde{\Gamma}_{11}(2)}C_0H_0\right\} \tag{14.172}$$

The statistical tensor $G_2(J)$ (see eqn (7.34)) is related to the orientation parameter $f_2(J)$ by

$$G_2(J) = J^2\left\{\frac{45}{(2J+3)(2J+1)J(J+1)(2J-1)}\right\}^{1/2}f_2(J) \tag{14.173}$$

with

$$f_2(J) = \frac{1}{J^2}\left[\sum_M M^2a_M - \tfrac{1}{3}J(J+1)\right]. \tag{14.174}$$

ν_{12} and η_{13} are Coulomb functions (see Chapter 4 and also eqns (7.51e) and (7.51g)).

The $\tilde{\Gamma}_{KK'}(2) = \Gamma_{KK'}(2, J_f, J_i)$ coefficients have the following values (see eqn (7.40)):

for $J \rightarrow J$ transitions

$$\tilde{\Gamma}_{11}(2) = \left\{\frac{(2J-1)(2J+3)}{30J(J+1)}\right\}^{1/2} \tag{14.175a}$$

$$\tilde{\Gamma}_{02}(2) = \frac{1}{\sqrt{5}} \tag{14.175b}$$

$$\tilde{\Gamma}_{12}(2) = \left\{\frac{3}{10J(J+1)}\right\}^{1/2}, \tag{14.175c}$$

for $J \rightarrow J+1$ transitions

$$\tilde{\Gamma}_{11}(2) = -\left\{\frac{J(2J-1)}{30(J+1)(2J+3)}\right\}^{1/2} \tag{14.176a}$$

$$\tilde{\Gamma}_{12}(2) = -\left\{\frac{(J+2)(2J-1)}{10(J+1)(2J+3)}\right\}^{1/2}, \tag{14.176b}$$

for $J \rightarrow J-1$ transitions

$$\tilde{\Gamma}_{11}(2) = -\left\{\frac{(J+1)(2J+3)}{30J(2J-1)}\right\}^{1/2} \tag{14.177a}$$

$$\tilde{\Gamma}_{12}(2) = \left\{\frac{(J-1)(2J+3)}{10J(2J-1)}\right\}^{1/2}. \tag{14.177b}$$

Analogous to eqns (14.156) and (14.157), for $\Delta J = 1$ or $\Delta J = 0$, $\Delta T = 1$ transitions the above formulae for A_2^0 and $\hat{b}$ become simpler:

$$A_2^0 = -2J^2\left\{\frac{15}{J(J+1)(2J+3)(2J-1)}\right\}^{1/2}\tilde{\Gamma}_{11}(2)\nu_{12}R \times\left[\frac{F_0}{C} - \frac{\tilde{\Gamma}_{12}(2)}{\tilde{\Gamma}_{11}(2)}\frac{H_0}{C_0}\right] \tag{14.178a}$$

$$\hat{b} = R\left\{F_1 - \frac{\tilde{\Gamma}_{12}(2)}{\tilde{\Gamma}_{11}(2)}H_1 - \frac{1}{5}\sqrt{\left(\frac{2}{3}\right)}\frac{\tilde{\Gamma}_{12}(2)}{\tilde{\Gamma}_{11}(2)}\frac{\eta_{13}}{\nu_{12}}X_0 + \frac{1}{5}\sqrt{\left(\frac{7}{3}\right)}\frac{\tilde{\Gamma}_{13}(2)}{\tilde{\Gamma}_{11}(2)}\frac{\eta_{13}}{\nu_{12}}Y_0\right\} \Big/ \left\{F_0 - \frac{\tilde{\Gamma}_{12}(2)}{\tilde{\Gamma}_{11}(2)}H_0\right\} \tag{14.178b}$$

It should also be noted that the coefficients α_β, A_2^0 and $\hat{b}$ are all together multiplied by the factor R. Thus they are small, i.e. $\alpha_\beta \simeq 10^{-3}$, $A_0^2 \simeq 10^{-3}$, and $\hat{b} \simeq 10^{-1}$–10^{-2}.

14.2.1.3.3. β–γ and β–α correlations. Correlations between beta-particles and other subsequent radiations like, for example, gamma- or alpha-radiation have been treated in Section 7.2.2.3 in general. We now have to specialize these results for the case of allowed transitions.

For β–γ correlations we then obtain (see eqn (7.79))

$$\omega(W_e, \theta_{\beta\gamma}) = 1 + \tau\frac{v}{c}\tilde{A}P_1(\cos\theta_{\beta\gamma}) + \varepsilon P_2(\cos\theta_{\beta\gamma}). \tag{14.179}$$

Here, $\theta_{\beta\gamma}$ is the angle between β-particle and γ-quantum. The $P_k(\cos\theta_{\beta\gamma})$ are the Legendre polynomials. The quantity τ, which determines the sign of the circular polarization of the γ-quantum, is equal to

± 1. The quantities v and c denote the velocities of the electrons (positrons) and of light (Note that $v/c = p_e/W_e$). The first term in eqn (14.179) represents, therefore, the β–γ circular polarization correlation and the second the β–γ angular correlation.

The asymmetry coefficient $\tilde{A}$ of the β–γ circular polarization correlation can be derived from eqns (7.75), (7.79), Table 7.2, and eqns (14.48) to (14.63) to

$$\tilde{A} = A_0(1 + \tilde{\alpha} W_e) \tag{14.180}$$

where

$$\tilde{A}_0 = \mp \Gamma_{11}(1)\sqrt{2}\,\Lambda_1 A_1(\gamma)\left[C_0^2 + \frac{\Gamma_{01}(1)}{\Gamma_{11}(1)}\sqrt{2}\,C_0 A_0\right] \Big/ [C_0^2 + A_0^2] \tag{14.181}$$

$$\tilde{\alpha} = R\frac{A_0 C_1 - C_0 A_1}{\Gamma_{11}(1)C_0^2 + \Gamma_{01}(1)\sqrt{2}\,C_0 A_0} \times \left[\Gamma_{11}(1)\frac{2C_0 A_0}{A_0^2 + C_0^2} - \Gamma_{01}(1)\sqrt{2}\,\frac{C_0^2 - A_0^2}{A_0^2 + C_0^2}\right] + \frac{\sqrt{2}}{3} R\frac{\eta_{12}}{\Lambda_1} \times \frac{\Gamma_{11}(1)C_0 F_0 - \Gamma_{01}(1)\sqrt{2}\,A_0 F_0 + \Gamma_{12}(1)\sqrt{5}\,C_0 H_0}{\Gamma_{11}(1)C_0^2 + \Gamma_{01}(1)\sqrt{2}\,C_0 A_0}. \tag{14.182}$$

As before, terms quadratic in W_e have been omitted.

We have used the shorthand notation $\Gamma_{KK'}(1) = \Gamma_{KK'}(1, J_i, J_f)$.

For $J \to J$ β-transitions we have $\Gamma_{11}(1) = \bar{\Gamma}_{11}(1)$, $\Gamma_{01}(1) = \bar{\Gamma}_{01}(1)$ and $\Gamma_{12}(1) = \bar{\Gamma}_{12}(1)$ (see eqns (14.153*a*–*c*)).

For $J \to J+1$ transitions one obtains from eqn (7.40)

$$\Gamma_{11}(1) = \left\{\frac{J+2}{6(J+1)}\right\}^{1/2} \tag{14.183a}$$

$$\Gamma_{12}(1) = -\left\{\frac{J}{10(J+1)}\right\}^{1/2} \tag{14.183b}$$

and for $J \to J-1$ transitions

$$\Gamma_{11}(1) = -\left\{\frac{J-1}{6J}\right\}^{1/2} \tag{14.184a}$$

$$\Gamma_{12}(1) = -\left\{\frac{J+1}{10J}\right\}^{1/2}. \tag{14.184b}$$

The quantities $A_1(\gamma)$ contain all the information about the γ transition (see eqn (7.80)) and have the form†

$$A_1(\gamma) = \sum_{LL'} F_1(LL' J_f J_f)\delta_L \delta_{L'} \tag{14.185}$$

† If we compare our formulae which describe β–γ correlations with those of Rose and Brink (1967) we find that

$$\delta_L \delta_{L'}(\text{Ro}) = (-1)^{L-L'}\delta_L \delta_{L'}\ (\text{ours}).$$

(the δ_L represent the reduced γ matrix elements for the 2^L-pole γ-transition).

For the $F_k(LL'J_{ff}J_f)$ coefficients see eqn (7.78b). Extensive tables for these coefficients can, for example, be found in the book by Appel (1968). J_f is the final state of the beta-transition (in our case $J_f = J, J-1$ or $J+1$) i.e. the initial state of the gamma-transition and J_{ff} the final state of the gamma-transition.

For $\Delta J = 1$ or (approximately) for $\Delta J = 0, \Delta T = 1$ transitions the above formulae simplify to

$$\tilde{A}_0 = \mp \Gamma_{11}(1)\sqrt{2}\,\Lambda_1 A_1(\gamma) \tag{14.186}$$

$$\tilde{\alpha} = \frac{\sqrt{2}}{3} R \frac{\eta_{12}}{\Lambda_1}\left[\frac{F_0}{C_0} + \frac{\Gamma_{12}(1)}{\Gamma_{11}(1)}\sqrt{5}\,\frac{H_0}{C_0}\right] \tag{14.187}$$

(for analogy see eqns (14.156) and (14.157)). Corresponding values for the ratio F_0/C_0 and H_0/C_0 in impulse approximation have been listed elsewhere (see eqns (14.158) to (14.159)). Approximate expressions for the independent particle model have also been mentioned before (see eqns (14.164) to (14.169)).

Next we will consider the anisotropy coefficient ε of the β–γ angular correlation. From eqn (7.75), Table 7.2, eqns (14.48) to (14.63) and eqn (7.79) we obtain

$$\varepsilon = a_{\beta\gamma}\frac{p_e^2}{W_e}(1 + b_{\beta\gamma} W_e) \tag{14.188}$$

where

$$a_{\beta\gamma} = -\tfrac{2}{3}\sqrt{2}\Gamma_{11}(2)\nu_{12} R A_2(\gamma)\left[C_0F_0 + \sqrt{\left(\frac{2}{3}\right)}\frac{\Gamma_{02}(2)}{\Gamma_{11}(2)} A_0H_0 + \frac{\Gamma_{12}(2)}{\Gamma_{11}(2)} C_0H_0\right] \Big/ [C_0^2 + A_0^2] \tag{14.189}$$

$$b_{\beta\gamma} = R\left\{C_0F_1 + \sqrt{\left(\frac{2}{3}\right)}\frac{\Gamma_{02}(2)}{\Gamma_{11}(2)} A_0H_1 + \frac{\Gamma_{12}(2)}{\Gamma_{11}(2)} C_0H_1 - \frac{1}{5}\frac{\Gamma_{02}(2)}{\Gamma_{11}(2)}\frac{\eta_{13}}{\nu_{12}} A_0X_0 + \frac{1}{5}\sqrt{\left(\frac{2}{3}\right)}\frac{\Gamma_{12}(2)}{\Gamma_{11}(2)}\frac{\eta_{13}}{\nu_{12}} C_0X_0 + \frac{1}{5}\sqrt{\left(\frac{7}{3}\right)}\frac{\Gamma_{13}(2)}{\Gamma_{11}(2)}\frac{\eta_{13}}{\nu_{12}} C_0Y_0\right\} \Big/ \left\{C_0F_0 + \sqrt{\left(\frac{2}{3}\right)}\frac{\Gamma_{02}(2)}{\Gamma_{11}(2)} A_0H_0 + \frac{\Gamma_{12}(2)}{\Gamma_{11}(2)} C_0H_0\right\} \tag{14.190}$$

For $J \to J$ β-transitions we have $\Gamma_{11}(2) = \tilde{\Gamma}_{11}(2)$, $\Gamma_{02}(2) = \tilde{\Gamma}_{02}(2)$ and

$\Gamma_{12}(2)=\tilde{\Gamma}_{12}(2)$ so that these coefficients can be taken from eqns (14.175a–c).

For $J\to J+1$ transitions we get from eqn (7.40)

$$\Gamma_{11}(2)=-\left\{\frac{(J+2)(2J+5)}{30(2J+1)(J+1)}\right\}^{1/2} \tag{14.191a}$$

$$\Gamma_{12}(2)=\left\{\frac{J(2J+5)}{10(2J+1)(J+1)}\right\}^{1/2} \tag{14.191b}$$

and for $J\to J-1$ β-transitions

$$\Gamma_{11}(2)=-\left\{\frac{(J-1)(2J-3)}{30J(2J+1)}\right\}^{1/2} \tag{14.192a}$$

$$\Gamma_{12}(2)=-\left\{\frac{(J+1)(2J-3)}{10J(2J+1)}\right\}^{1/2}. \tag{14.192b}$$

Similarly as in eqn (14.185) we derive from eqn (7.80)

$$A_2(\gamma)=\sum_{LL'}F_2(LL'J_{\mathrm{ff}}J_{\mathrm{f}})\delta_{\mathrm{L}}\delta_{L'}. \tag{14.193}$$

For β–α correlations we have to replace $A_2(\gamma)$ by $A_2(\alpha)$ which is given by eqn (7.76). Of course, a β–α circular polarization correlation is impossible. Thus we have no $P_1(\cos\theta_{\beta\alpha})$ term in the correlation.†

Therefore we obtain

$$\omega(W_{\mathrm{e}},\theta_{\beta\alpha})=1+\varepsilon_\alpha P_2(\cos\theta_{\beta\alpha}). \tag{14.194}$$

with

$$\varepsilon_\alpha=a_{\beta\alpha}\frac{p_{\mathrm{e}}^2}{W_{\mathrm{e}}}(1+b_{\beta\gamma}W_{\mathrm{e}}) \tag{14.195}$$

and

$$a_{\beta\alpha}=-\tfrac{2}{3}\sqrt{2}\,\Gamma_{11}(2)\nu_{12}RA_2(\alpha)$$
$$\times\left[F_0C_0+\sqrt{\left(\frac{2}{3}\right)}\frac{\Gamma_{02}(2)}{\Gamma_{11}(2)}A_0H_0+\frac{\Gamma_{12}(2)}{\Gamma_{11}(2)}C_0H_0\right]\Big/[C_0^2+A_0^2]. \tag{14.196}$$

† This is not exactly true. We have a β–α correlation term which is of pure kinematical origin. It has the form

$$\omega(W_{\mathrm{e}},\theta_{\beta\alpha})=1+a_1P_1(\cos\theta_{\beta\alpha})$$

with

$$a_1=O\left(\frac{2p_{\mathrm{e}}}{M_Av}\right).$$

v is the velocity of the α-particle in the centre of mass frame of the β-decay daughter (see, for example, Holstein 1974a).

It should also be noticed that the dynamical beta-decay information obtained from the $\beta-\alpha$ correlation is completely equivalent to that obtained from the $\beta-\gamma$ correlation.

14.2.1.3.4. Electron–neutrino correlations. From our general expressions given in Section 7.2.4.1 (see eqns (7.96–7.98) and Table 7.4) the form of the $\beta-\nu$ correlation for allowed transitions is easily derived as

$$\omega(W_e, \theta_{e\nu}) = 1 + d_1 P_1(\cos\theta_{e\nu}) + d_2 P_2(\cos\theta_{e\nu}). \tag{14.197}$$

The most important correlation coefficient is here d_1 given by†

$$d_1 = D_1^0 \frac{p_e}{W_e} \Lambda_1 (1 + \alpha_{\beta\nu} W_e - \mu_1 \gamma_1 b / W_e) \tag{14.198}$$

and

$$D_1^0 = [A_0^2 - \tfrac{1}{3}C_0^2 - \tfrac{4}{9}\sqrt{2}\,(W_0 R) C_0 E_0]/[C_0^2 + A_0^2] \tag{14.199}$$

$$\alpha_{\beta\nu} = \tfrac{8}{3} R \frac{A_0 C_0 (A_1 C_0 - C_1 A_0)}{(C_0^2 + A_0^2)(A_0^2 - \tfrac{1}{3}C_0^2)} + \tfrac{4}{9}\sqrt{2} R \frac{C_0 E_0 - \dfrac{\eta_{12}}{\Lambda_1} C_0 F_0}{A_0^2 - \tfrac{1}{3}C_0^2}. \tag{14.200}$$

The coefficient b represents the factor of the $1/W$-term in the shape factor (see eqn (14.118)). The correlation coefficient d_2 is very small and reads as

$$d_2 = \frac{2\sqrt{2}}{9} \frac{p_e^2}{W_e} (W_0 - W_e) \nu_{12} R^2 [-A_0 M_0 + \sqrt{\tfrac{1}{5}}\, C_0 N_0]/[C_0^2 + A_0^2]. \tag{14.201}$$

For transitions of the type $\Delta J = 1$ or (approximately) for transitions with $\Delta T = 1$ the above formulae can be simplified in the same way as other formulae have before, i.e. (see also Calaprice 1975)

$$D_1^0 = -\tfrac{1}{3} - \tfrac{4}{9}\sqrt{2}\,(W_0 R) \frac{E_0}{C_0} \tag{14.202}$$

$$\alpha_{\beta\nu} = -\tfrac{4}{3}\sqrt{2}\, R \frac{E_0 - \dfrac{\eta_{12}}{\Lambda_1} F_0}{C_0} \tag{14.203}$$

† If in the phase space factor a correction term for the kinetic recoil energy is additionally taken into account (see Section 5.2) we get a second term of pure kinematical origin. This term has the form

$$\omega(W_e, \theta_{e\nu}) = 1 + d_{1_{kin}} P_1(\cos\theta_{e\nu}) + d_{2_{kin}} P_2(\cos\theta_{e\nu})$$

with

$$d_{1_{kin}} = \frac{p_e}{W_e} \left\{ \frac{2}{3} \frac{W_0}{M_A} - \frac{4 W_e}{M_A} \right\} \frac{C_0^2}{A_0^2 + C_0^2} \qquad d_{2_{kin}} = -\frac{p_e^2}{W_e} \frac{2}{M_A} \frac{A_0^2 - \tfrac{1}{3}C_0^2}{A_0^2 + C_0^2}$$

and

$$d_2 = \frac{2\sqrt{2}}{9\sqrt{5}} \frac{p_e^2}{W_e} (W_0 - W_e) \nu_{12} R^2 \frac{N_0}{C_0}. \tag{14.204}$$

For the special case of a 1–0 transition a similar formula has also been derived by Sato *et al.* (1977). In order to get an impression of how the quantities F_0/C_0 and E_0/C_0 are related to nuclear matrix elements we mentioned in the other preceding sections some relations in impulse approximation. For F_0/C_0 such a relation can be found in eqns (14.158) and (14.160). For E_0/C_0 we obtain similarly (if $\alpha Z \ll 1$)

$$\frac{E_0}{C_0} \simeq \frac{\lambda\sqrt{\frac{2}{3}}\, {}^{A}\mathfrak{M}^{(0)}_{110} \pm \frac{1}{\sqrt{3}}\, {}^{V}\mathfrak{M}^{(0)}_{111}}{\lambda\, {}^{A}\mathfrak{M}^{(0)}_{101}} \mp \sqrt{2} \frac{f_M + f_T}{\lambda R} \tag{14.205}$$

or in a cartesian notation

$$\frac{E_0}{C_0} \simeq \frac{\lambda\sqrt{2}\int \gamma_5 \frac{i\mathbf{r}}{R} \mp \frac{1}{\sqrt{2}} \int \frac{\boldsymbol{\alpha} \times \mathbf{r}}{R}}{\lambda \int \boldsymbol{\sigma}} \mp \sqrt{2} \frac{f_M + f_T}{\lambda R}. \tag{14.206}$$

By assuming a model where $(E_i - E_f) - (V_i - V_f) = 0$ in the single-particle matrix elements, we can make the same simple estimates for E_0/C_0 as before for F_0/C_0 (see eqns (14.162), (14.164), (14.166) and (14.168)). Then we have:

for $j_i = j_f = l + \frac{1}{2}$

$$\frac{E_0}{C_0} \simeq \mp \frac{\sqrt{2}}{R} \left[\frac{1+l}{2M_N \lambda} + \frac{f_M + f_T}{\lambda} \right], \tag{14.207}$$

for $j_i = j_f = l - \frac{1}{2}$

$$\frac{E_0}{C_0} \simeq \mp \frac{\sqrt{2}}{R} \left[-\frac{l}{2M_N \lambda} + \frac{f_M + f_T}{\lambda} \right], \tag{14.208}$$

for $j_f = j_i + 1 \quad l_i = l_f = l$

$$\frac{E_0}{C_0} \simeq \frac{\sqrt{2}}{R} \left[-\frac{2l+1}{4M_N} \mp \frac{1}{4M_N \lambda} \mp \frac{f_M + f_T}{\lambda} \right], \tag{14.209}$$

for $j_f = j_i - 1 \quad l_i = l_f = l$

$$\frac{E_0}{C_0} \simeq \frac{\sqrt{2}}{R} \left[\frac{2l+1}{4M_N \lambda} \mp \frac{1}{4M_N \lambda} \mp \frac{f_M + f_T}{\lambda} \right]. \tag{14.210}$$

Besides, the recoil momentum distribution $P(q)\,dq$ is also strongly correlated to the electron–neutrino correlation as shown in detail in Section 7.2.4.1.

For allowed transitions the corresponding formula for the recoil momentum spectrum can be derived from eqn (7.105). It comes out to

$$P(q)\,\mathrm{d}q = \frac{G_\beta'^2}{4\pi^3}(C_0^2 + A_0^2)\left\{\int_{W_{e_{min}}}^{W_{e_{max}}} F_0 L_0 \left[C(W_e)(W_0 - W_e)W_e + d_1 \frac{W_e}{p_e} \times \frac{q^2 + 1 - W_0^2 + 2W_e(W_0 - W_e)}{2}\right] \mathrm{d}W_e\right\} q\,\mathrm{d}q \tag{14.211}$$

where

$$W_{e_{min}} = \frac{(W_0 - q)^2 + 1}{2(W_0 - q)} \tag{14.212}$$

$$W_{e_{max}} = \frac{(W_0 + q)^2 + 1}{2(W_0 + q)}\,. \tag{14.213}$$

d_1 has to be taken from eqn (14.198) and $C(W_e)$ from eqn (14.116) (see also Nachtmann 1968; Kleppinger *et al.* 1977).

14.2.1.3.5. Electron–neutrino triple correlations. Here, we consider the correlation between electrons and neutrinos emitted from oriented nuclei. The general situation had been eluciated in Section 7.2.4.2. The correlation itself, which we will discuss in the following, has, for allowed transitions, the form (see eqn (7.114))

$$\omega_J(\theta_e, \theta_\nu, W_e) = 1 + \tilde{d}\,\frac{\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)}{J p_e p_\nu}\,. \tag{14.214}$$

The coefficient $\tilde{d}$ can be calculated from eqns (7.116), (7.112), and Table 7.5. It can be written as

$$\tilde{d} = \left\{\frac{J}{2(J+1)}\right\}^{1/2} f_1(J)[\tilde{d}_0 sl_1 + p_e se_{12}\tilde{d}_1] \tag{14.215}$$

The sl_1 and se_{12} are the special Coulomb functions described in Section 4.2.3 (see especially eqns (4.48) and (4.58)). The quantities $\tilde{d}_0$ and $\tilde{d}_1$ read as

$$\tilde{d}_0 = -2\sqrt{6}\,\tilde{\Gamma}_{01}(1)R[C_0 B_0 - A_0 D_0]/[C_0^2 + A_0^2] \tag{14.216}$$

$$\tilde{d}_1 = 2R[\sqrt{\tfrac{1}{3}}\,\tilde{\Gamma}_{01}(1)A_0 F_0 + \sqrt{\tfrac{3}{2}}\,\tilde{\Gamma}_{11}(1)C_0 F_0 + \sqrt{\tfrac{5}{6}}\,\tilde{\Gamma}_{12}(1)C_0 H_0]/[C_0^2 + A_0^2]. \tag{14.217}$$

For $\Delta J = 1$ (or approximately for $\Delta T = 1$ transitions) beta-decays we get simply

$$\tilde{d}_0 = 0 \tag{14.218}$$

$$\tilde{d}_1 = 2R\left[\sqrt{\tfrac{3}{2}}\,\Gamma_{11}(1)\frac{F_0}{C_0} + \sqrt{\tfrac{5}{6}}\,\Gamma_{12}(1)\frac{H_0}{C_0}\right]. \tag{14.219}$$

By making use of eqns (14.164) to (14.169), simple order of magnitude estimations can easily be made for the coefficient $\tilde{d}_1$ given above.

In the limit $\alpha Z \ll 1$ the special Coulomb functions sl_1 and se_{12} are given by (see eqns (4.55) and (4.58))

$$sl_1 \simeq \frac{\alpha Z}{W_e} \tag{14.220}$$

$$se_{12} \simeq \alpha Z \left\{ \frac{W_e}{p_e} - \frac{1}{4} \frac{p_e}{W_e} \right\} = \frac{\alpha Z}{p_e} \left\{ \frac{3}{4} W_e + \frac{1}{4} \frac{1}{W_e} \right\}. \tag{14.221}$$

That means for low Z-values the above correlation coefficients are proportional to αZ. For $Z \to 0$ they vanish. An absolute test of time reversal invariance would, therefore, only be possible in this latter case (see also the discussion in Section 7.2.4.2). If time reversal invariance is violated, in $J \to J$ transitions we would have an additional term which reads as

$$\tilde{d} = -2 \left\{ \frac{J}{J+1} \right\}^{1/2} \frac{p_e}{W_e} \Lambda_1 f_1(J) \frac{|{}^{V}F_{000}^{(0)}| \, |{}^{A}F_{101}^{(0)}|}{|{}^{V}F_{000}^{(0)}|^2 + |{}^{A}F_{101}^{(0)}|^2} \sin \phi_{VA} \tag{14.222}$$

because the form factor coefficients are not purely real any more and are then given by (see Section 10.1)

$${}^{V}F_{000}^{(0)} = |{}^{V}F_{000}^{(0)}| e^{i\phi_V} \tag{14.223a}$$

$${}^{A}F_{101}^{(0)} = |{}^{A}F_{101}^{(0)}| e^{i\phi_A} \tag{14.223b}$$

ϕ_{VA} is the angle difference between the V and A form factor coefficients ${}^{V}F_{000}^{(0)}$ and ${}^{A}F_{101}^{(0)}$, i.e.

$$\phi_{VA} = \phi_V - \phi_A. \tag{14.224}$$

For the decay of the neutron we have $|{}^{V}F_{000}^{(0)}| = 1$ and $|{}^{A}F_{101}^{(0)}| = \lambda\sqrt{3}$ (see eqns (6.47) and (6.48)). Assuming time reversal invariance to be valid we then get $\phi_{VA} = -\pi$. By applying eqns (14.64), (14.67), (14.69), (14.95), (14.98), (14.100), (14.158), and Table 14.1 we obtain for the neutron

$$\tilde{d}_0 = 2\sqrt{\left(\frac{2}{3}\right)} \frac{2\left(\frac{1}{2M} + f_M\right) + f_T + 3\lambda f_S}{1 + 3\lambda^2} \tag{14.225}$$

$$\tilde{d}_1 = 2\sqrt{\left(\frac{2}{3}\right)} \frac{\left(\frac{1}{2M} + f_M - f_T\right)(1 + 3\lambda)}{1 + 3\lambda^2}. \tag{14.226}$$

By inserting the corresponding values for f_M and λ ($f_M = 1{\cdot}008 \times 10^{-3}$ and

$\lambda = 1{\cdot}25$) and making use of the approximate formulae for sl_1 and se_{12} given in eqns (14.220) and (14.221) we then get explicitly for the neutron decay

$$\tilde{d} = 3{\cdot}49 \times 10^{-6} \frac{1}{W_e} + 3{\cdot}90 \times 10^{-6}\, W_e. \tag{14.227}$$

In that context it is also worth mentioning the articles by Holstein (1972) and Igarashi (1978) where similar, but not so general, formulae for $\alpha Z \ll 1$ can be found.

14.2.1.3.6. Distribution of a gamma radiation emitted from an oriented nucleus without detecting the intermediate beta-decay. This observable has been generally treated in Section 7.2.5. We, therefore, now refer to that previous discussion. At this point we will, however, restrict ourselves to allowed transitions and present some explicit results for these kinds of transitions. If the beta-decay itself is not observed the possibility nevertheless exists to determine some beta-decay form factor coefficients by observing the nuclear orientation before and after the beta-decay. This can be done by investigating the distribution of a gamma radiation, which follows a beta-transition, with respect to the orientation axis of a polarized or aligned initial nucleus. This distribution has the form (see eqn (7.131))

$$\omega_\gamma(\theta_\gamma) = 1 - \tau h_1 P_1(\cos\theta_\gamma) + h_2 P_2(\cos\theta_\gamma) \tag{14.228}$$

if θ_γ is the angle between the gamma-quantum and the nuclear orientation axis. The quantity τ determines the sign of the circular polarization of the gamma-quantum ($\tau = \pm 1$). In this context the $J \to J$ transitions are the most important ones. Then the symmetry and anisotropy coefficients read as†

$$h_1 = \left\{\frac{3J}{J+1}\right\}^{1/2} f_1(J) A_1(\gamma) \frac{c_{00}^{(1)} A_0^2 + c_{11}^{(1)} C_0^2}{A_0^2 + C_0^2} \tag{14.229}$$

with

$$c_{00}^{(1)} = 1 \tag{14.230a}$$

$$c_{11}^{(1)} = 1 - \frac{1}{J(J+1)} \tag{14.230b}$$

and

$$h_2 = J^2 \left\{\frac{45J(J+1)}{(2J-1)^3(2J+3)^3}\right\}^{1/2} f_2(J) A_2(\gamma) \frac{c_{00}^{(2)} A_0^2 + c_{11}^{(2)} C_0^2}{A_0^2 + C_0^2} \tag{14.231}$$

with

$$c_{00}^{(2)} = 4 - \frac{3}{J(J+1)} \tag{14.232a}$$

$$c_{11}^{(2)} = 4 - \frac{15}{J(J+1)} + \frac{9}{\{J(J+1)\}^2}. \tag{14.232b}$$

For $A_2(\gamma)$ see eqn (7.80) or eqn (14.193).

† The coefficients $c_{nm}^{(k)}$ used here should not be confused with the neutrino particle parameters of Table 7.3.

It should be noted that terms connected with Legendre polynomials of higher order, i.e. those with h_3, h_4 etc., are of the same order of magnitude as the terms with h_1 and h_2. From eqn (7.132) we have, however, the restriction that the maximal order k_{max} of the Legendre polynomials is limited to $k_{max} = \min\{2J_i, 2J_f\}$. The influence of the higher order terms differs from that seen before in the other observables. The reason is that we have here no interference terms A_0C_0, C_0F_0, C_0H_0 etc. between form factor coefficients of different tensor rank. Terms proportional to F_0^2, E_0^2, H_0^2 etc. are multiplied by factors $(W_0R)^2$ or $(W_eR)^2$ and can, therefore, be neglected against A_0^2 and C_0^2. The only higher order correction η which remains can be written as

$$h_k = h_k^0\{1+\eta_k\} \tag{14.233}$$

with

$$\eta_k = 2\overline{W_eR}\left\{\frac{c_{00}^{(k)}A_0A_1 + c_{11}^{(k)}C_0C_1}{c_{00}^{(k)}A_0^2 + c_{11}^{(k)}C_0^2} - \frac{A_0A_1 + C_0C_1}{A_0^2 + C_0^2}\right\}. \tag{14.234}$$

For $J \to J-1$ and $J \to J+1$ beta-transitions h_k is independent of the form factor coefficients and can be derived from eqn (7.131) too.

For $J \to J-1$

$$h_1 = \left\{\frac{3(J-1)}{J}\right\}^{1/2} A_1(\gamma)f_1(J) \tag{14.235}$$

$$h_2 = \left\{\frac{45J(J-1)(2J-3)}{(2J+1)(2J-1)^2}\right\}^{1/2} A_2(\gamma)f_2(J). \tag{14.236}$$

For $J \to J+1$

$$h_1 = \frac{J}{J+1}\left\{\frac{3(J+2)}{J+1}\right\}^{1/2} A_1(\gamma)f_1(J) \tag{14.237}$$

$$h_2 = \frac{J^2}{(J+1)(2J+3)}\left\{\frac{45(J+2)(2J+5)}{(J+1)(2J+1)}\right\}^{1/2} A_2(\gamma)f_2(J). \tag{14.238}$$

Higher order corrections are unmeasurably small in these latter cases, the reasons being the same as explained above.

It is also worth mentioning that this change of nuclear orientation can be observed for electron capture. With the exception of the *ft*-value all the other observables discussed before cannot be observed in electron capture. In that respect the change of nuclear orientation is of special interest.

14.2.1.3.7. Electron polarization. Electron polarization has been discussed in general before (see Section 7.3). For allowed transitions eqn (7.151) can now be simplified to

$$P_e = \mp\frac{p_e}{W_e}\Lambda_1\left\{1 + \mu_1\gamma_1 g - \mu_1\gamma_1\frac{b}{W_e}\right\} \tag{14.239}$$

(upper sign β^--decay, lower sign β^+-decay).

The coefficients g and b are caused by terms of higher order. The coefficient g reads as

$$g = 2R^2 \frac{A_0 B_1 + C_0 D_1}{A_0^2 + C_0^2}. \tag{14.240}$$

The coefficient b is identical with the b coefficient of the $1/W_e$-term in the shape factor, and, therefore, is given by eqn (14.118). Λ_1 and μ_1 are special Coulomb functions (see Chapter 4 and eqns (7.51c) and (7.51d)).

For $\Delta J = 1$ and (approximately) for $\Delta J = 0$, $\Delta T = 1$ we get simpler expressions which can be written as follows

$$g = 2R^2 \frac{D_1}{C_0} \tag{14.241}$$

$$b = -2R \frac{D_0}{C_0}. \tag{14.242}$$

In order to get a quick order of magnitude impression we apply, as before, impulse approximation to D_0/C_0 and D_1/C_0. Then we obtain

$$\frac{D_0}{C_0} = -\frac{1}{3}\left\{\frac{\lambda\sqrt{\frac{1}{3}}\,{}^{A}\mathfrak{M}^{(0)}_{110} \pm \sqrt{\frac{2}{3}}\,{}^{V}\mathfrak{M}^{(0)}_{111}}{\lambda\,{}^{A}\mathfrak{M}^{(0)}_{101}} \mp \frac{2f_M + f_T}{\lambda R} - \frac{f_P}{\lambda R}\frac{W_0}{2M_N}\right\} \tag{14.243}$$

$$\frac{D_1}{C_0} = -\frac{1}{54}\frac{11\,{}^{A}\mathfrak{M}^{(1)}_{101} - 4\sqrt{2}\,{}^{A}\mathfrak{M}^{(0)}_{121}}{{}^{A}\mathfrak{M}^{(0)}_{101}} - \frac{1}{3}\frac{f_P}{\lambda R^2}\frac{1}{2M_N} \tag{14.244}$$

or in cartesian notation

$$\frac{D_0}{C_0} = -\frac{1}{3}\left\{\frac{\lambda\int \gamma_5 \frac{\mathbf{i}\mathbf{r}}{R} \mp \int \frac{\boldsymbol{\alpha}\times\mathbf{r}}{R}}{\lambda\int \boldsymbol{\sigma}} \mp \frac{2f_M + f_T}{\lambda R} - \frac{f_P}{\lambda R}\frac{W_0}{2M_N}\right\} \tag{14.245}$$

$$\frac{D_1}{C_0} = -\frac{1}{54}\left\{\frac{11\int \boldsymbol{\sigma}\left(\frac{r}{R}\right)^2 - 12\int \frac{(\boldsymbol{\sigma}\mathbf{r})\mathbf{r} - \frac{1}{3}\boldsymbol{\sigma}r^2}{R^2}}{\int \boldsymbol{\sigma}}\right\} - \frac{1}{3}\frac{f_P}{\lambda R^2}\frac{1}{2M_N}. \tag{14.246}$$

In the same way as before we can now also make some simple independent-particle model estimations (assuming $E_i - E_f - (V_i - V_f) = 0$ in the single-particle matrix elements) which result in:
for $j_i = j_f = l + \frac{1}{2}$

$$\frac{D_0}{C_0} \simeq \frac{1}{3R}\left[\pm\frac{1+l}{M_N\lambda} \pm \frac{2f_M + f_T}{\lambda} + \frac{f_P}{\lambda}\frac{W_0}{2M_N}\right] \tag{14.247}$$

$$\frac{D_1}{C_0} \simeq -\frac{1}{54}\left[11 + \frac{8l}{(2l+3)}\right]\frac{\langle r^2\rangle}{R^2} - \frac{1}{3}\frac{f_P}{\lambda R^2}\frac{1}{2M_N}, \tag{14.248}$$

for $j_i = j_f = l - \frac{1}{2}$

$$\frac{D_0}{C_0} \simeq \frac{1}{3R}\left[\mp\frac{l}{M_N\lambda} \pm \frac{2f_M + f_T}{\lambda} + \frac{f_P}{\lambda}\frac{W_0}{2M_N}\right] \tag{14.249}$$

$$\frac{D_1}{C_0} \simeq -\frac{1}{54}\left[11 + \frac{8(l+1)}{(2l-1)}\right]\frac{\langle r^2\rangle}{R^2} - \frac{1}{3}\frac{f_P}{\lambda R^2}\frac{1}{2M_N}, \tag{14.250}$$

for $j_f = j_i + 1 \quad l_i = l_f = l$

$$\frac{D_0}{C_0} \simeq \frac{1}{3R}\left[\frac{2l+1}{4M_N} \pm \frac{1}{2M_N\lambda} \pm \frac{2f_M + f_T}{\lambda} + \frac{f_P}{\lambda}\frac{W_0}{2M_N}\right] \tag{14.251}$$

$$\frac{D_1}{C_0} \simeq -\frac{1}{6}\frac{\langle r^2\rangle}{R^2} - \frac{1}{3}\frac{f_P}{\lambda R^2}\frac{1}{2M_N}, \tag{14.252}$$

for $j_f = j_i - 1 \quad l_i = l_f = l$

$$\frac{D_0}{C_0} \simeq \frac{1}{3R}\left[-\frac{2l+1}{4M_N} \pm \frac{1}{2M_N\lambda} \pm \frac{2f_M + f_T}{\lambda} + \frac{f_P}{\lambda}\frac{W_0}{2M_N}\right] \tag{14.253}$$

$$\frac{D_1}{C_0} \simeq -\frac{1}{6}\frac{\langle r^2\rangle}{R^2} - \frac{1}{3}\frac{f_P}{\lambda R^2}\frac{1}{M_N}. \tag{14.254}$$

In the above equations a term with the induced pseudoscalar interaction f_P has been included because for allowed transitions the quantities D_0 and D_1 are the only ones where f_P has noticeable influence in the limit $\alpha Z \to 0$. If we insert, however, the value $f_P = -6{\cdot}5 \times 10^{-2}$ following from PCAC (see Section 10.3) we recognize that even for D_0 and D_1 the effect of f_P can usually be neglected. For heavier nuclei with non-negligible αZ, on the other hand, f_P influences the observables through terms connected with αZ. Moreover, it should be noticed that the influence of f_P has usually not to be considered in allowed transitions in contrast to the situation in forbidden transitions.

In conclusion, we can state that the deviation of the electron polarization from the value $\mp\Lambda_1 v/c$ is of the order 10^{-3}, and, therefore, at the moment difficult, but not impossible, to measure (see, for instance, van Klinken *et al.* 1978).

14.2.1.3.8. Neutrino observables. In general, the neutrino could also be observed instead of the electron. Then, all observables analogous to those just discussed could be considered in addition, i.e. observables where the electron would always be replaced by the neutrino. The present status of the experimental technique does not, however, allow us to detect the neutrino with a similar precision as the electron.

The same is true if the recoil nucleus is observed instead. All neutrino observables which require high precision cannot, therefore, be experimentally investigated at the moment. Nevertheless, general formulae for

these have been derived in Section 7.2.3. Now we intend therefore to discuss only two neutrino observables which might be of some interest at present.

First, there is the neutrino spectrum for allowed transitions. This spectrum can be derived from eqn (7.85) as

$$P(p_\nu)\,\mathrm{d}p_\nu = \frac{G_\beta'^2}{(2\pi)^3}[A_0^2+C_0^2]F_0L_0C(W_\nu)p_\nu^2(W_0-W_\nu)\sqrt{\{(W_0-W_\nu)^2-1\}}\,\mathrm{d}p_\nu \tag{14.255}$$

where $P(p_\nu)\,\mathrm{d}p_\nu$ is the number of neutrinos emitted in the momentum interval between p_ν and $p_\nu+\mathrm{d}p_\nu$ (note that $p_\nu = W_\nu$). The Fermi function F_0L_0 has now to be taken at $W_e = W_0 - W_\nu$. The neutrino shape factor is, of course, strongly correlated to the electron shape factor $C(W_e)$ and given by

$$C(W_\nu) = 1 + aW_0 + cW_0^2 - (a+2W_0c)W_\nu + cW_\nu^2 + \mu_1\gamma_1 b/(W_0-W_\nu) \tag{14.256}$$

(note that $W_{\nu_{max}} = W_0 - 1$), where the Coulomb function μ_1 has to be taken at $W_0 - W_\nu$. The coefficients a, b, and c are the same as given in eqns (14.116) to (14.119). For reactor antineutrino spectra see also Davis *et al.* (1979).

Secondly, we will also mention the distribution of neutrinos emitted from oriented nuclei, but only the main term of the asymmetry coefficient. It reads as (see Section 7.2.3)

$$\omega(\theta_\nu, W_\nu) = 1 + b_1 P_1(\cos\theta_\nu) \tag{14.257}$$

with

$$b_1 = B_1^0 f_1(J_i)$$

$$B_1^0 = \pm\left(\frac{6J}{J+1}\right)^{1/2}\tilde{\Gamma}_{11}(1)\left[C_0^2+\frac{\tilde{\Gamma}_{01}(1)}{\tilde{\Gamma}_{11}(1)}\sqrt{2}\,C_0A_0\right]/[C_0^2+A_0^2]. \tag{14.258}$$

The $\tilde{\Gamma}_{KK'}(k)$ coefficients have to be taken from eqns (14.153*a*) and (14.155*b*).

Further details concerning higher order terms will not be discussed at this point. Corresponding expressions can, however, easily be derived replacing the particle parameter $b_{KK'}^{(k)}$ by the neutrino parameters $c_{KK'}^{(k)}$. They come out to be very similar to those given for the beta-particles in the sections before.

14.2.1.4. Summary of the observables

The content of the foregoing section can be summarized as follows:

(i) The main terms are represented by

$$A_0 = {}^{\mathrm{V}}F_{000}^{(0)} \tag{14.259}$$

$$C_0 = -{}^{\mathrm{A}}F_{101}^{(0)}. \tag{14.260}$$

They are greater than the quantities $|W_0RE_0|$, $|W_0RF_0|$, $|W_0RG_0|$ etc. by at least a factor of about 10. If only these dominant terms are taken into account the following observables exclusively differ from zero:

ft-value
beta-spectrum
asymmetry coefficient A_1^0 of electrons emitted from oriented nuclei;
asymmetry coefficient $\tilde{A}$ of the $\beta-\gamma$ (CP) correlation;
electron-neutrino correlation asymmetry coefficient D_1^0;
distribution of gamma radiation emitted from an oriented nucleus;
electron polarization;
asymmetry coefficient B_1^0 of neutrinos emitted from oriented nuclei.

From these observables only the ratio ${}^{V}F_{000}^{(0)}/{}^{A}F_{101}^{(0)}$ can then be determined with exception of the *ft*-value which allows a determination of the absolute values of ${}^{V}F_{000}^{(0)}$ and ${}^{A}F_{101}^{(0)}$ (exactly of the square sum of both). The electron polarization and the beta-spectrum is independent of ${}^{V}F_{000}^{(0)}$ and ${}^{A}F_{101}^{(0)}$. For pure Gamow–Teller transistions, i.e. $\Delta J=1$ transistions, with exception of the *ft*-value no dynamic beta-decay information can be obtained since in this case all observables do not depend on ${}^{V}F_{000}^{(0)}$ and ${}^{A}F_{101}^{(0)}$.

(ii) If in addition higher order terms are considered two effects occur. Firstly, we have small corrections to those observables which differ from zero even if only main terms are taken into account. Secondly, some other observables which are only a function of the higher order terms now have finite values. These are the following:
anisotropy coefficient A_2^0 of electrons emitted from oriented nuclei;
anisotropy coefficient $a_{\beta\gamma}$ and $a_{\beta\alpha}$ of the $\beta-\gamma$ or $\beta-\alpha$ correlation, respectively;
electron–neutrino triple correlation.
It is, therefore, very often of advantage to investigate these last observables if higher order contributions are intended to be determined separately from the leading terms.

(iii) Induced interactions influence the observables only over the higher order terms. The linear combinations of form factor coefficients C_1, D_0, E_0, and F_0 are of special interest in that context.

(iv) The mathematical form which describes the dependence of observables from higher order terms can always be represented as a linear combination of different products A_0D_0, A_0F_0, C_0F_0, C_0E_0, A_0H_0, C_0H_0 etc. whereby the factors and signs connected with these products differ in the different possible observables.

It is mostly because of this latter fact that different observables give different information about form factor coefficients, but special attention should be paid to the $J_i = 1 \rightarrow J_f = 0$ (or $J_i = 0 \rightarrow J_f = 1$) transitions in that context since here form factor coefficients of tensor rank 1 only can occur. Then our linear combinations are restricted to the ones with products C_0C_1, C_0D_0, C_0E_0, and C_0F_0. Thus in this case the situation is simpler and different observables like, for example, the energy dependence α_β of the asymmetry of electrons emitted from oriented nuclei and the anisotropy A_2^0 of electrons emitted from oriented nuclei give the same dynamic beta-decay information (in our example they both depend on F_0/C_0 only).

14.2.2. Superallowed transitions

14.2.2.1 $0^+ - 0^+$ transitions

In general, vector and axial vector interaction (i.e. Fermi and Gamow–Teller interaction) both contribute to allowed transitions. The *ft*-values, therefore, usually depend on the two form factor coefficients ${}^{V}F_{000}^{(0)}$ and ${}^{A}F_{101}^{(0)}$ (see eqns (14.121) and (14.122)), but the great exception is represented by beta-transitions where initial and final nuclear states take spin values $J_i^\pi = J_f^\pi = 0^+$.

Then, because of the corresponding spin selection rules the axial form factor coefficient ${}^{A}F_{101}^{(0)}$ is always equal to zero (see, for example, eqn (8.142)). Consequently, the beta-transition is of a pure vector type in this case. As we have seen in Section 10.2, the form factor coefficient ${}^{V}F_{000}^{(0)}$ is not only in impulse approximation but exactly related to the matrix element (Fermi matrix element) ${}^{V}\mathfrak{M}_{000}^{(0)}$ by (see eqns (10.40) to (10.43))

$$ {}^{V}F_{000}^{(0)} = {}^{V}\mathfrak{M}_{000}^{(0)} = M_F \tag{14.261} $$

if CVC is valid.

For $\Delta T = 0$ transitions within one and the same supermultiplet, i.e. for transitions within an isospin multiplet we then have (only β^+-decay is possible)

$$ {}^{V}F_{000}^{(0)} = M_F = \sqrt{\{(T - T_{3i})(T + T_{3i} + 1)\}} \tag{14.262} $$

(see eqn (8.140)).

These $0^+ - 0^+$ beta-transitions between members of an isospin multiplet are usually called superallowed $0^+ - 0^+$ transitions (see Fig. 14.2).

Their great importance lies in the fact that on the one side only the vector interaction is involved and that on the other side the contributing form factor coefficient ${}^{V}F_{000}^{(0)}$ can be evaluated in a nuclear structure independent way. Therefore these transitions offer the exceptional possibility to determine the coupling constant G_β. Until now, only transitions

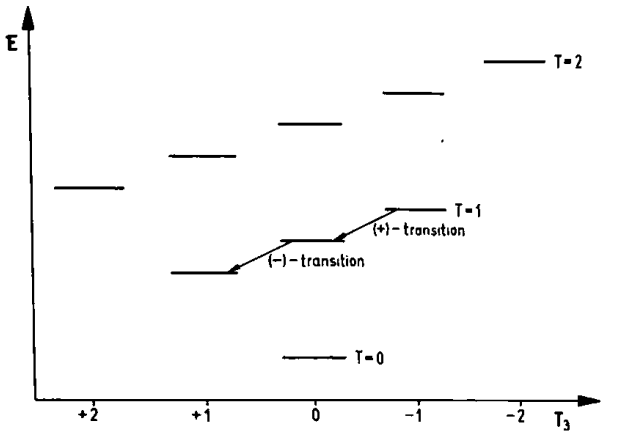


FIG. 14.2. Members of an isospin multiplet. The two possible β^+-transitions between members of an isospin triplet are indicated by (+)- and (−)-transition.

between isospin triplet states have experimentally been investigated with the accuracy required. For $T=1$, eqn (14.262) reduces to

$$^{V}F_{000}^{(0)} = M_F = \sqrt{2}. \tag{14.263}$$

Then the ft-value takes the simple form (see eqn (14.122))

$$ft = \frac{K}{2G_\beta^2} \tag{14.264}$$

where (see Section 7.2.2.2)

$$K = \frac{2\pi^3(\ln 2)\hbar^7}{m_e^5 c^4}.$$

Explicitly, we have

$$K = 1{\cdot}230618 \times 10^{-94}\ \text{erg}^2\ \text{cm}^6\ \text{s (c.g.s. units)}, \tag{14.265}$$

$$K = 1{\cdot}230618 \times 10^{-120}\ \text{J}^2\ \text{m}^6\ \text{s (SI units)}, \tag{14.266}$$

$$K = 2\pi^3 \ln 2 \text{ (natural units)}. \tag{14.267}$$

That means the ft-values should be identical, independent of the particular decaying nucleus. On the other hand, an experimental proof of this fact is a test of the CVC theory at the same time.

As is very often the case in physics, the above simple and beautiful equation, eqn (14.264), is not entirely exact. The reason is that the Coulomb interaction has not been taken into account in all stages until now and that $A_0^2\overline{C(W_e)}$ was set equal to 1. Of the Coulomb interaction we

have up to now only taken into consideration the distortion of the electron radial wave function by the electrostatic part. This latter effect is included in the Fermi function F_0L_0 (see Chapter 4) or in the integrated Fermi function f (see eqn (7.61)), respectively, but, a complete inclusion of the Coulomb interaction and an exact consideration of the shape factor $C(W_e)$ introduces small additional corrections. Equation (14.264) then has to be modified to

$$f^R t = \frac{K}{G_\beta'^2 M_F'^2} \tag{14.268}$$

where

$$G_\beta'^2 = G_\beta^2(1+\Delta_R^{(V)}) \tag{14.269}$$

$$M_F'^2 = 2(1-\delta_c) \tag{14.270}$$

$$\begin{aligned} f^R &= f\overline{C(W_e)}\,\overline{R(W_e)}\,(1+\overline{\delta_R(W_e)}) \\ &= \int_1^{W_0} F_0(Z, W_e)L_0(Z, W_e)p_e W_e(W_0-W_e)^2 C(W_e) \\ &\quad \times(1+\delta_R(Z, W_e))R(W_e)\,\mathrm{d}W_e. \end{aligned} \tag{14.271}$$

The two types of radiative corrections, i.e. the outer corrections $\delta_R(Z, W_e)$ and the inner corrections $\Delta_R^{(V)}$ have been discussed in Chapter 13 in detail. Thus corresponding formulae can be found in that chapter. δ_c is the isospin impurity correction which takes into account the modification of the Fermi matrix element due to isospin impurities of the initial and final nuclear states. $C(W_e)$ is, as before, the shape factor and $R(W_e)$ a recoil correction.

$R(W_e)$ is a very small correction, which in first order takes into account the kinetic recoil energy of the final nucleus in the phase space, and is given by† (see Section 5.2)

$$R(W_e) = 1+\frac{2W_e}{M_A}. \tag{14.272}$$

The shape factor $C(W_e)$ is here written as (see eqns (14.116), (14.117), and (14.118))

$$C(W_e) = \frac{A_0^2}{\{{}^V F_{000}^{(0)}\}^2}[1+aW_e+\mu_1\gamma_1 b/W_e+cW_e^2]. \tag{14.273}$$

† Usually this correction is neglected, but in the case of the superallowed 0^+-0^+ transitions the experimental accuracy is nowadays so high that effects of the 0·1–0·01% level also have to be included. It should also be noted that

$$X = \int_1^{W_0} R(W_e)p_e W_e(W_0-W_e)^2\,\mathrm{d}W_e \simeq 1-\frac{3}{2}\frac{W_0}{M_A}.$$

For a further calculation we have to consider the quantities A_0, A_1, A_2, B_0, and B_1 (given in eqns (14.64) to (14.68)) in more detail. Two kinds of form factor coefficients are involved, i.e.

$$^{V}F^{(N)}_{000}(1, m, n, \rho)$$
$$^{V}F^{(N)}_{011}(1, m, n, \rho).$$

Fortunately, in the case of the superallowed transitions under consideration these two form factor coefficients can be calculated without reference to the nuclear structure by making use of CVC and by assuming isospin to be a good quantum number.†

From our discussion in Section 10.2 it follows (see eqn (10.86)) that‡

$$^{V}F^{(N)}_{000}(k_e, m, n, \rho) = \sqrt{2}\int_0^\infty \left(\frac{r}{R}\right)^{2N} I(k_e, m, n, \rho; r)\rho_c(r)r^2\,dr \tag{14.274}$$

(this method was first introduced by Blin-Stoyle 1969) where $\rho_c(r)$ is the nuclear charge distribution normalized to unity.§

Because $I(k_e, m, n, 0) = 1$ this equation also contains as a special case the above result for the Fermi matrix element $M_F = \sqrt{2}$. For the uniform distribution and the modified Gaussian distribution (see eqn (3.197)) the values of the nuclear matrix elements relative to the Fermi matrix element are shown in Table 14.2.

In Section 10.2 we also derived the following relation‖ (see eqns (10.62 to (10.64))

$$-2N\,^{V}F^{(N-1)}_{011} = (\Delta + 2{\cdot}5 - \Delta E_C)R\,^{V}F^{(N)}_{000} \tag{14.275}$$

which is based on the CVC theory. In impulse approximation we obtain

† Because we have $|\overline{C(W_e)} - 1| < 10^{-2}$ for the known superallowed transitions, isospin impurity effects can be here neglected since their influence is two to three orders of magnitude smaller than $\overline{C(W_e)} - 1$ itself, i.e. of the order 10^{-4} to 10^{-5}.

‡ For $J_i = J_f = 0$, $T_i = T_f = T$

$$4\pi B(C0, q) = \left[\int_0^\infty \rho_c(r) j_0(qr) r^2\,dr\right]^2.$$

§ We have $\int_0^\infty \rho_c(r)r^2\,dr = 1$. Identical distributions for neutrons and protons have been assumed in eqn (14.274). More generally $\rho_c(r)$ may be replaced by $\frac{1}{2}[N\rho_N(r) - Z\rho_c(r)]$, where $\rho_N(r)$ and $\rho_c(r)$ are the distributions of neutrons and protons, respectively, and N and Z are the neutron and proton numbers of the neutron rich nucleus in the $T = 1$ multiplet.

‖
$$W_0 = \Delta\left\{1 - \frac{\Delta^2 - 1}{2M_A\Delta}\right\} = \Delta$$

(see eqn (5.24)).

TABLE 14.2 *Values of the nuclear matrix elements relative to the Fermi matrix element for several different nuclear charge distributions (from Behrens and Bühring 1972)*

	Uniform distribution	Modified Gaussian distribution		
		$A=0$	$A=1$	$A=2$
$\frac{{}^{V}F^{1}_{000}}{{}^{V}F^{0}_{000}}$	$\frac{3}{5}=0{\cdot}6$	0·6	0·6	0·6
$\frac{{}^{V}F^{1}_{000}(1,1,1,1)}{{}^{V}F^{0}_{000}}$	$\frac{27}{35}=0{\cdot}77143$	0·75693	0·76127	0·76288
$\frac{{}^{V}F^{1}_{000}(1,2,2,1)}{{}^{V}F^{0}_{000}}$	$\frac{57}{70}=0{\cdot}81429$	0·84351	0·83629	0·83194
$\frac{{}^{V}F^{1}_{000}(1,2,2,2)}{{}^{V}F^{0}_{000}}$	$\frac{233}{210}=1{\cdot}10952$	1·21868	1·18825	1·17229

analogously (see Behrens and Bühring 1971)

$$-2N\,{}^{V}F^{(N-1)}_{011}(k_e, m, n, \rho)=(\Delta+2{\cdot}5-\Delta E_C)R \times\left\{\int\int_0^r\left(\frac{x}{R}\right)^{2N-1} I(k_e, m, n, \rho; x)\,\mathrm{d}x T_{000} j_0^{V}\right\}. \quad (14.276)$$

ΔE_C was the Coulomb displacement energy which takes the semi classical value $\frac{6}{5}(\alpha Z/R)$ for the uniform charge distribution. Since we consider transitions within an isospin triplet the energy difference Δ, which is at the disposal of the decay, is exactly equal to the Coulomb displacement energy minus the neutron–proton mass difference, i.e. we have (see also Damgaard 1969)

$$\Delta+2{\cdot}5-\Delta E_C=0 \quad (14.277)$$

and therefore

$${}^{V}F^{(N)}_{011}=0 \quad (14.278a)$$

$${}^{V}F^{(N)}_{011}(k_e, m, n, \rho)=0. \quad (14.278b)$$

Thus the shape factor, or more exactly, the corresponding coefficients $A_0^2/\{{}^{V}F^{(0)}_{000}\}^2$, a, b, and c can be written explicitly as (see also Behrens and Bühring 1968, 1972)

$$\frac{A_0^2}{\{{}^{V}F^{(0)}_{000}\}^2}=1-\tfrac{1}{3}(\alpha Z)^2\frac{{}^{V}F^{1}_{000}(1,2,2,2)}{{}^{V}F^{(0)}_{000}}-\tfrac{1}{3}(W_0R)^2 \times\frac{{}^{V}F^{(1)}_{000}}{{}^{V}F^{(0)}_{000}}+\frac{2\alpha\,|Z|}{9}W_0R\frac{{}^{V}F^{(1)}_{000}(1,1,1,1)}{{}^{V}F^{(0)}_{000}} \quad (14.279)$$

$$a = \tfrac{2}{3}R\left[\alpha\,|Z|\left(\frac{{}^{V}F_{000}^{(1)}(1,2,2,1)}{{}^{V}F_{000}^{(0)}} - \frac{1}{3}\frac{{}^{V}F_{000}^{(1)}(1,1,1,1)}{{}^{V}F_{000}^{(0)}}\right) + \tfrac{2}{3}(W_0R)\frac{{}^{V}F_{000}^{(1)}}{{}^{V}F_{000}^{(0)}}\right] \tag{14.280}$$

$$c = -\tfrac{4}{9}R^2\frac{{}^{V}F_{000}^{(1)}}{{}^{V}F_{000}^{(0)}} \tag{14.281}$$

$$b = \tfrac{2}{3}R\left[\tfrac{1}{3}(W_0R)\frac{{}^{V}F_{000}^{(1)}}{{}^{V}F_{000}^{(0)}} + \frac{\alpha\,|Z|}{2}\frac{{}^{V}F_{000}^{(1)}(1,2,1,1)}{{}^{V}F_{000}^{(0)}}\right]. \tag{14.282}$$

The above equations are given for β^+-decay as we need them in this case. For an experimental determination of a superallowed shape factor see Thies (1977) and Thies *et al.* (1978). For the convenience of the reader we have additionally listed the values $\overline{C(W_e)}-1$, which have been obtained by using the above (b has been set equal to zero) eqns (14.273) to (14.282), for the known superallowed transitions in Table 14.3.

We now turn to the calculation of the isospin impurity correction δ_c, which has, by the way, a long history (summaries can be found in the articles by Blin-Stoyle 1969; Behrens 1973; Blin-Stoyle 1973). Following a suggestion made by Wilkinson (1975, 1976, 1978) we can use the Behrends–Sirlin–Ademollo–Gatto (BSAG) theorem (see Behrends and Sirlin 1960; Ademollo and Gatto 1964) to derive an estimation of the

TABLE 14.3 *Values of* $\overline{C(W_e)}-1$ *for superallowed Fermi transitions (Raman et al. 1975)*

Decay	$1-\overline{C(W_e)}$ (%)
$^{10}C \rightarrow {}^{10}B$	0·024
$^{14}O \rightarrow {}^{14}N$	0·043
$^{18}Ne \rightarrow {}^{18}F$	0·071
$^{22}Mg \rightarrow {}^{22}Na$	0·101
$^{26m}Al \rightarrow {}^{26}Mg$	0·123
$^{26}Si \rightarrow {}^{26m}Al$	0·134
$^{30}S \rightarrow {}^{30}P$	0·184
$^{34}Cl \rightarrow {}^{34}S$	0·211
$^{34}Ar \rightarrow {}^{34}Cl$	0·234
$^{38m}K \rightarrow {}^{38}Ar$	0·268
$^{38}Ca \rightarrow {}^{38m}K$	0·294
$^{42}Sc \rightarrow {}^{42}Ca$	0·333
$^{42}Ti \rightarrow {}^{42}Sc$	0·360
$^{46}V \rightarrow {}^{46}Ti$	0·401
$^{46}Cr \rightarrow {}^{46}V$	0·431
$^{50}Mn \rightarrow {}^{50}Cr$	0·474
$^{54}Co \rightarrow {}^{54}Fe$	0·553
$^{62}Ga \rightarrow {}^{62}Zn$	0·747

general trend for δ_c. The BSAG theorem tells us that the renormalization of coupling constants goes as the square of the mass splitting within the isospin multiplet (see also the discussion in Section 10.2). As we have mentioned before (see Section 10.2) the mass splitting within an isospin multiplet, i.e. the displacement energy between any two members of an isobaric multiplet, goes roughly (for the model of a homogeneously charged sphere) as

$$\Delta E_C = \frac{6}{5}\frac{\alpha Z}{R}. \tag{14.283}$$

We expect therefore that the isospin impurity correction goes roughly also as $(\alpha Z)^2$. For our superallowed transitions the squares of the mass splitting go, however, exactly as $Z^{1.86}$ (see Wilkinson and Alburger 1976).

Thus the BSAG theorem gives us the result

$$\delta_c = aZ^{1.86}, \tag{14.284}$$

a result which has the advantage that it is nuclear structure independent. However, it should be pointed out that the above formula only makes allowance for the average behaviour of δ_c as a function of Z. Small deviations because of shell effects, binding energy fluctuation effects, etc. should be expected (see Towner and Hardy 1978).

In order to account for nuclear structure effects microscopic calculation methods have, of course, to be applied. Let us now begin with the simplest microscopic approach, i.e. the use of the independent particle shell model. Assuming the initial and final nuclear states to be of a j^N configuration with seniority zero,† i.e.

$$i\rangle = f\rangle = |j^N, v=0, J=0, T=0\rangle \tag{14.285}$$

we can make use of eqn (8.68) and of the matrix elements in Table 14.1. We then obtain (see also Towner *et al.* 1977)

$${}^{V}F_{000}^{(0)} = M_F' = M_F \frac{(2T-N)\Omega_j^{T_1=T+\frac{1}{2}} + (2T+N+2)\Omega_j^{T_1=T-\frac{1}{2}}}{2(2T+1)} \tag{14.286}$$

where M_F is the Fermi matrix element without Coulomb distortion given by eqn (14.262).

† The corresponding fractional parentage coefficients are (see de-Shalit and Talmi 1963)

$$[j^{N-1}v_1=1, T_1=T+\tfrac{1}{2}, J_1=j|\}j^N v=0, T, J=0]^2 = \frac{(N-2T)(T+1)}{N(2T+1)}$$

$$[j^{N-1}v_1=1, T_1=T-\tfrac{1}{2}, J_1=j|\}j^N v=0, T, j=0]^2 = \frac{(N+2T+2)T}{N(2T+1)}.$$

The radial matrix element, or in this case the overlap integral

$$\Omega_j^{T_1=T\pm\frac{1}{2}} = \int_0^\infty g_f(r, \kappa_f) g_i(r, \kappa_i) r^2 \, dr \tag{14.287}$$

can be different for the two cases $T_1 = T + \frac{1}{2}$ and $T_1 = T - \frac{1}{2}$. In the extreme single particle model, however, we have

$$\Omega_j^{T_1=T-\frac{1}{2}} = \Omega_j^{T_1=T+\frac{1}{2}} = \Omega_j \tag{14.288}$$

and

$$\Omega_j = 1 - \frac{\varepsilon_1^0}{2}. \tag{14.289}$$

Then the isospin impurity correction is simply the deviation of the square of the overlap integral from unity, i.e.

$$\delta_c = \varepsilon_1^0. \tag{14.290}$$

As shown in Section 8.1.1.2, radial matrix elements can always be calculated easiest by using harmonic oscillator wave functions. For a first calculation we therefore make use of this type of radial wave functions. For the model of the uniformly charged sphere the Coulomb potential inside the nucleus acting on the last proton is given by

$$V_c = \frac{\alpha Z}{R} \left\{ \frac{3}{2} - \frac{1}{2} \left(\frac{r}{R} \right)^2 \right\} \tag{14.291}$$

where Z is the atomic number of the daughter nucleus of the β^+-decay. Then, the neutron and proton oscillator parameter b_n and b_p, respectively, (see Section 8.1.1.2) can be written as

$$\frac{1}{b_p^4} = \frac{1}{b_n^4} \left\{ 1 - \frac{\alpha Z M_N b_n^4}{R^3} \right\} = \frac{1}{b_n^4} \{1 - \Delta_b\}. \tag{14.292}$$

Exact formulae for harmonic oscillator overlap integrals can be found in the articles by Bell and Warsop (1967) and Talman (1970).

We have (see Section 8.1.1.2)

$$\Delta_b = \frac{2^{14/3}}{3^{2/3} 5^2} \frac{\alpha Z M_N R}{A^{2/3}} = 0{\cdot}02034 \left(\frac{r_0'}{1{\cdot}2} \right) \frac{Z}{A^{1/3}} \tag{14.293}$$

if we assume $R = r_0' A^{1/3}$ and r_0' to be given in fm. Since for our problem $\Delta_b \ll 1$, it is enough to consider the exact overlap integral in lowest order of Δ_b. Then we get

$$\Omega_j = 1 - \nu(\nu + l + \tfrac{1}{2}) \frac{\Delta_b^2}{32} + \ldots \tag{14.294}$$

and

$$\varepsilon_1^0 = 0{\cdot}2586 \times 10^{-4} \left(\frac{r_0'}{1{\cdot}2} \right)^2 \frac{Z^2}{A^{1/3}} \nu(\nu + l + \tfrac{1}{2}). \tag{14.295}$$

This result was first derived by Damgaard (1969) in first order perturbation theory for $r_0' = 1{\cdot}2$ fm. Later on Towner *et al.* (1977), however, stated that it is important always to use the correct radius parameter r_0' which especially for light nuclei deviates remarkably from the value $1{\cdot}2$ fm.

The approach where only the one-body average Coulomb force is considered is usually called the 'giant monopole' approach. It represents the tendency of the Coulomb repulsion between the protons to make the size of the nucleus increase with Z along the isospin multiplet (Wilkinson 1978).

Equivalent calculations of ε_1^0 have been carried out by Lane and Mekjian (1973) and by Fayans (1971) (the latter author used the Migdal approach to renormalize the effective Coulomb field). Besides, it suggests itself to take more refined radial nuclear wave functions, e.g. solutions of the Saxon–Woods potential (including spin orbit and Coulomb terms). In that case, first we have to determine the required parameters of this potential. While the radius and diffuseness parameters can be deduced from elecron scattering results (unfortunately not unambiguously) the potential depth V_0 is usually determined by requiring that the calculated binding energy of the last occupied proton orbit matches the experimental proton separation energy (see Towner *et al.* 1977).

If we now, however, decouple one particle from the other $N-1$ particles in a completely antisymmetric way, i.e. if we expand the A-particle wave function into $(A-1)$-particle wave functions and a single-particle wave function, we obtain not only one parent $(A-1)$-particle wave function, but many different ones. Thus we end up with many different binding energies of the last proton depending on which parent state is left after separation of this proton. So we have to consider many different overlap integrals Ω_π. M_F' is therefore now a function of all separate overlaps given by (Wilkinson 1976)

$$(1-\varepsilon_1^\pi)^{1/2} = \frac{1}{2}\left\{\frac{1}{T}\sum_{\pi<} S_\pi \Omega_\pi - \frac{1}{T+1}\sum_{\pi>} S_\pi \Omega_\pi\right\}. \qquad (14.297)$$

The S_π are the spectroscopic factors (generalized fractional parentage coefficients, see Brussard and Glaudemanns 1977) representing the overlap of the A- and $(A-1)$-particle wave functions. $\pi<$ belongs to $T_1 = T-\frac{1}{2}$ and $\pi>$ to $T_1 = T+\frac{1}{2}$ (see eqn (14.286)). Two calculations along this more refined method have been carried out, that of Wilkinson (1976) and that of Towner *et al.* (1977).

In addition to the one-body Coulomb effects we have contributions from the two-body Coulomb force. Because of the residual interactions the configurations are generally mixed (see eqn (8.78)). The two-body Coulomb interaction causes an additional mixing of these states which differs, however, for the different members of the isospin multiplet. Let us

elucidate that point with some simple examples by assuming we have two particles or two holes outside a double magic nucleus (in our formalism described in Sections 8.1.1.3 and 8.1.1.4 the treatment of holes is equivalent to that of particles) e.g. ^{14}O, ^{14}N, ^{14}C; ^{18}O, ^{18}F, ^{18}Ne; ^{38}Ca, ^{38m}K, ^{38}Ar.

Making use of the SU(s)–SU(3) coupling scheme which was explained in Section 8.1.1.4, i.e. of the L–S coupling scheme we then have for the initial and final nuclear wave functions

$$|0^+, 1\rangle_i = \alpha_i \phi_{L=0,S=0} + \beta_i \phi_{L=1,S=1} \tag{14.298}$$

$$|0^+, 1\rangle_f = \alpha_f \phi_{L=0,S=0} + \beta_f \phi_{L=1,S=1} \tag{14.299}$$

with $\alpha_i^2 + \beta_i^2 = 1$ and $\alpha_f^2 + \beta_f^2 = 1$. It should be noticed that the state $\phi_{L=0,S=0}$ has spin–isospin symmetry [11] (orbital symmetry [2]) and the state $\phi_{L=1,S=1}$ spin–isospin symmetry [2] (orbital symmetry [11]). Thus the above configuration corresponds to a mixture of two different symmetries.

Applying eqn (8.136) and the relations of Table 8.4 we then obtain for the p-shell, e.g. for the $A = 14$ triplet

$$M_F' = M_F\left(1 - \frac{(\alpha_i - \alpha_f)^2 + (\beta_i - \beta_f)^2}{2}\right)\Omega_p \tag{14.300}$$

and for the s–d shell, e.g. for the $A = 18$ and $A = 38$ triplet

$$M_F' = M_F\left\{1 - \frac{(\alpha_i - \alpha_f)^2 + (\beta_i - \beta_f)^2}{2}\Omega_d + \tfrac{5}{9}\alpha_i\alpha_f(\Omega_s - \Omega_d)\right\}. \tag{14.301}$$

Ω_p, Ω_s, and Ω_d are the corresponding overlap integrals in the 1p-shell, 2s-shell, and 1d-shell. We recognize that we get a second contribution ε_2 *to* δ_c which essentially reads as

$$\varepsilon_2 = (\alpha_i - \alpha_f)^2 + (\beta_i - \beta_f)^2. \tag{14.302}$$

It should, however, be noticed that in principle ε_1 and ε_2 should not be considered separately as it is clearly demonstrated by eqn (14.301) ($\Omega_s \neq \Omega_d$). Nevertheless, that has been done and an explicit microscopic shell model calculation of ε_2 has been carried out by Towner and Hardy (1973). The overall isospin impurity correction is then

$$\delta_c = \varepsilon_1 + \varepsilon_2. \tag{14.303}$$

The results of the different calculations (for the most important transitions) are summarized in Table 14.4 which has been taken from Wilkinson (1978).

Looking at Table 14.4 we see that the three estimates ε_1^0 of the one-body contribution using the giant monopole approach differ from the

TABLE 14.4 *The isospin impurity correction in percent according to the various approaches (from Wilkinson 1978)*

Decaying body	ε_2	ε_1^0 (LM)	ε_1^0 (THH)	ε_1^0 (F)	ε_1^π (W)	ε_1^π (THH)
^{14}O	0·05	0·04	0·05	0·03	0·41	0·28
$^{26}Al^m$	0·07	0·11	0·13	0·08	0·35	0·27
^{34}Cl	0·23	0·18	0·20	0·13	0·57	0·62
$^{38}K^m$	0·16	0·21	0·24	(0·18)	0·34	0·54
^{42}Sc	0·13	0·25	0·37	0·23	0·31	0·35
^{46}V	0·04	0·29	0·40	0·26	0·31	0·36
^{50}Mn	0·03	0·33	0·43	0·31	0·56	0·40
^{54}Co	0·04	0·38	0·47	0·34	0·67	0·56

ε_2 (Towner and Hardy 1973)
ε_1^0 (LM) (Lane and Mekjian 1973)
ε_1^0 (THH) (Towner *et al.* 1977)
ε_1^0 (F) (Fayans 1971)
ε_1^π (W) (Wilkinson 1976)
ε_1^π (THH) (Towner *et al.* 1977)

two other values ε_1^π calculated by applying the microscopic one-body approach of summing over parent states. It should, however, be noted that the overall trend as function of Z for ε_1^0 is in agreement with the BSAG theorem, but in the case of light nuclei for ε_1^π in disagreement with the latter theorem. This might be an indication that the summing over parent states method is not entirely correct (even if shell effects are expected). This inconsistency could have its origin in the fact that the whole procedure of determining the parameters for the Saxon–Woods potential is not the suitable one. Another way should perhaps be preferred, which is as follows.

First the parameters of the Saxon–Woods potential should be chosen such that the splitting of the single particle levels for one nucleon outside a double magnic core are reproduced (for the s–d shell O^{17}). Then the residual interaction (including two-body Coulomb interaction) should be switched on and all level and binding energies for the nuclei in question should be calculated. If the residual interaction is suitably chosen the binding energies of the last nucleons for the different parent states should come out more or less correctly. Thus no further adjustment of parameters is necessary and the isospin impurity correction δ_c can directly be calculated applying the methods described above. Of course one-body and two-body contributions ε_1 and ε_2 cannot here be separated.

Probably, the shortcomings of the approach which is used for the summing over parent states up to now are therefore caused by an improper separation of one-body and two-body Coulomb effects.

Following the above discussion we would now suggest using the BSAG

TABLE 14.5 *Decay properties of the eight accurately measured superallowed Fermi beta-decays. The t values are the half-lives corrected for electron capture, and, in the case* ^{14}O, *for branching.* f^R *is the phase space factor derived and adjusted for the 'outer' radiative correction as described in the text. The figures in parentheses in the t and* f^R *columns are the respective errors in percent (from Wilkinson et al. 1978)*

Decay	E_0 (keV)	t (ms)	f^R	$f^R t$ (s)
^{14}O–^{14}N	1808·62±0·35	71146 (0·028)	43·382 (0·083)	3086·5±2·7
$^{26}Al^m$–^{26}Mg	3210·79±0·37	6349·2 (0·044)	486·10 (0·052)	3086·3±2·1
^{34}Cl–^{34}S	4469·93±0·32	1526·6 (0·066)	2030·70 (0·030)	3100·1±2·2
$^{38}K^m$–^{38}Ar	5020·86±0·82	922·63 (0·061)	3351·3 (0·074)	3092·0±3·0
^{42}Sc–^{42}Ca	5401·64±0·39	681·64 (0·091)	4545·6 (0·034)	3098·4±3·0
^{46}V–^{46}Ti	6028·60±0·56	422·74 (0·049)	7340·3 (0·044)	3103·0±2·0
^{50}Mn–^{50}Cr	6609·90±0·38	283·04 (0·071)	10943·3 (0·027)	3097·4±2·3
^{54}Co–^{54}Fe	7219·53±0·56	193·46 (0·073)	16068·0 (0·039)	3108·5±2·6

and giant monopole approach for a computation of δ_c. At present 18 superallowed 0^+–0^+ transitions within a $T=1$ triplet have been investigated exactly. The eight most accurately measured transitions, i.e. those cases where the end point energy W_0 ($E_0 = W_0 - 1$) and the half-life t have been determined with highest precision, are listed in Table 14.5. Other decays, which are not listed in Table 14.5 and where the accuracy is not so high, are $^{10}C \rightarrow {}^{10}B$, $^{18}Ne \rightarrow {}^{18}F$, $^{22}Mg \rightarrow {}^{22}Na$, $^{26}Si \rightarrow {}^{26m}Al$, $^{30}S \rightarrow {}^{30}P$, $^{34}Ar \rightarrow {}^{34}Cl$, $^{38}Ca \rightarrow {}^{38m}K$, $^{42}Ti \rightarrow {}^{42}Sc$, $^{46}Cr \rightarrow {}^{46}V$ and $^{62}Ga \rightarrow {}^{62}Zn$. Details about these can be found in the compilations of Raman *et al.* (1975) and Hardy and Towner (1975).

It should be noticed that the accuracy of the *ft*-values of eight decays listed in Table 14.5 has reached a precision of $(0{\cdot}7\text{–}1{\cdot}0)10^{-3}$ which requires also the highest precision for the calculation of the integrated Fermi function† and the inclusion of effects like, for example, nuclear recoil, atomic excitation corrections‡ etc. which are usually neglected.

In Table 14.6 we have additionally collected the average *ft*-values (including all corrections, i.e. $f^R(1-\delta_c)t$) together with the values of the coupling constants determined from these by applying eqn (14.268). They have been taken from the different surveys or data compilations published in the last few years.

† Here, the limit is reached where the influence of the form of the nuclear charge distribution on the integrated Fermi function f could be of some importance (see Behrens and Bühring 1970, 1972).

‡ Atomic excitation corrections to Q-values (atomic cloud effects caused by the sudden change of the nuclear charge Z) in superallowed transitions are comparable to current experimental precision (see Feagin *et al.* 1979).

TABLE 14.6 *Average ft-values for the superallowed 0^+-0^+ transitions in seconds and effective vector coupling constants G'_β in natural units*

ft-value	G'_β	Reference
3088·6±2·1	$(2{\cdot}9940(11))\times10^{-12}$	Raman *et al.* (1975)
3081·7±1·9	$(2{\cdot}9972(\ 8))\times10^{-12}$	Hardy and Towner (1975)
3087·9±3·5	$(2{\cdot}9942(17))\times10^{-12}$	Wilkinson (1975)
3082·5±2·0	$(2{\cdot}9967(11))\times10^{-12}$	Towner *et al.* (1977)
3085·2±2·0	$(2{\cdot}9955(10))\times10^{-12}$	Wilkinson (1977)
3087·9±3·8	$(2{\cdot}9942(16))\times10^{-12}$	Vonach *et al.* (1977)
3084·1±1·9	$(2{\cdot}9961(\ 9))\times10^{-12}$	Wilkinson *et al.* (1978)

Inspection of Table 14.6 shows that the results for the average *ft*-values remained remarkably stable over the last years although a number of new, partly more precise, measurements was recently reported. As we have mentioned earlier the equality of the *ft*-values for the 0^+–0^+ superallowed transitions is required by the CVC theory which excludes a renormalization of the vector coupling constant. From the compilations listed in Table 14.6 we can now also derive that the form factor coefficient ${}^{V}F_{000}^{(0)}$ has the same value irrespective of which special superallowed 0^+–0^+ transition is considered, i.e.

$$\frac{{}^{V}F_{000}^{(0)}}{\sqrt{\{2(1-\delta_c)\}}} = 1\pm0{\cdot}8\times10^{-3}. \tag{14.304}$$

That means it also holds for the vector coupling constant f_V

$$f_V = 1\pm0{\cdot}8\times10^{-3} \tag{14.305}$$

and no renormalization occurs. It therefore comes out that CVC is valid, at least, in its weak form. By comparing the above results for the coupling constants G'_β with that determined from the muon lifetime (see Wilkinson *et al.* 1978) we can derive a value for the inner radiative correction $\Delta_R^{(V)}$, which is described in Chapter 13 in detail (or exactly for the difference between that for nucleon and muon). From the last compilation mentioned in Table 14.6 (Wilkinson *et al.* 1978) we then obtain

$$\Delta_R^{(V)} = (2{\cdot}12\pm0{\cdot}16)\%. \tag{14.306}$$

Essentially it is this inner radiative correction $\Delta_R^{(V)}$ which relates low energy nuclear beta-decay to the basic models of the weak interaction theory (Ryder 1977). As shown in Chapter 13, $\Delta_R^{(V)}$ depends on the mean charge $\bar{Q}$ of the quarks, which are believed to be the constituents of the nucleons, and on the mass of the intermediate Z-vector boson.

$\bar{Q}$ is defined as the average of the quarks u and d_c in the fundamental

doublets (see Chapter 12). In the Gell-Mann–Zweig model of fractionally charged quarks we have

$$\bar{Q} = \tfrac{1}{2}(\tfrac{2}{3} - \tfrac{1}{3}) = \tfrac{1}{6}. \tag{14.307}$$

From the results for $\Delta_R^{(V)}$ listed above we then obtain† (see Wilkinson 1978; Wilkinson *et al.* 1978)

$$\bar{Q} = 0{\cdot}17 \pm 0{\cdot}06 \tag{14.308}$$

in agreement with $\bar{Q} = \frac{1}{6}$.

14.2.2.2. J^π–J^π transitions

In the case of superallowed J^π–J^π transitions ($J \neq 0, \Delta T = 0$), i.e. transitions which take place between members of a common isospin multiplet, both the vector form factor coefficient ${}^{V}F_{000}^{(0)}$ and the axial vector one ${}^{A}F_{101}^{(0)}$ contribute. As shown before the vector form factor coefficient ${}^{V}F_{000}^{(0)}$ is related to the Fermi matrix element (see eqn (10.40)) and can be evaluated in a model independent way in this case (see eqn (14.262)). The corrections which are caused by the Coulomb interaction have also been discussed in the foregoing chapter in detail. The axial form factor coefficient ${}^{A}F_{101}^{(0)}$ is related to the Gamow–Teller matrix element (if exchange effects are neglected) by

$${}^{A}F_{101}^{(0)} = \lambda\ {}^{A}\mathfrak{M}_{101}^{(0)} = \frac{\lambda}{\sqrt{(2J_i+1)}} \langle J_f T_f T_{3f} \Big|\Big| \sum_{n=1}^{A} \{\boldsymbol{\sigma} t_+\}_n \Big|\Big| J_i T_i T_{3i} \rangle \tag{14.309}$$

(for β^+-decay, since β^+-decay is only energetically possible).

Our initial and final nuclear states are isobaric analogue states and therefore also members of the same supermultiplet. Thus we can make use of the formalism discussed in Section 8.1.1.4 and apply eqn (8.142). Evaluating this latter equation further we then obtain‡

$${}^{A}\mathfrak{M}_{101}^{(0)} = \frac{J(J+1) + S(S+1) - L(L+1)}{2\{J(J+1)\}^{1/2}} \frac{P''}{PP'} \sqrt{\{(T - T_{3i})(T + T_{3i} + 1)\}} \tag{14.310}.$$

The quantum numbers P, P', and P'' which characterize the irreducible representation of SU(4) can be taken from Table 8.6. The total angular

† Assuming that $74{\cdot}6 < M_Z < 100$ GeV.

‡ Note that this formula is only valid if

$$PP' = ST$$

(see Section 8.1.1.4).

momentum L, which is not a good quantum number, has to be determined by nuclear structure calculations (see, for example, Feldmeier *et al.* 1973). For $S=\frac{1}{2}$ two values for L are always possible, namely $L=J-\frac{1}{2}$ and $L=J+\frac{1}{2}$, for $S=1$ three values $L=J+1$, $L=J$ and $L=J-1$. Often, some of the two or three possible L values are, however, excluded because they are not compatible with the corresponding SU(s)–SU(3) structure (see, for example, Table 8.5).

In Table 14.7 we have applied eqn (14.310) for mirror transitions with $T=\frac{1}{2}$ and compared the theoretically calculated ft-values with experimental results (see also Petrauskas and Vanagas 1968).

Whenever two L values are possible the theoretical results for the higher L value are closer to the experiment than those for the lower. If we assume that the nuclear wave function for two mixed L values is written as a superposition

$$\phi = a\phi(L_1)+b\phi(L_2) \tag{14.311}$$

$(a^2+b^2=1)$
we get

$$^{A}\mathfrak{M}^{(0)}_{101} = a^2\{^{A}\mathfrak{M}^{(0)}_{101}\}_{L_1}+(1-a^2)\{^{A}\mathfrak{M}^{(0)}_{101}\}_{L_2}. \tag{14.312}$$

TABLE 14.7 *Comparison of theoretical (eqn (14.310)) and experimental ft-value for mirror transistions* ($T=\frac{1}{2}$, $S=\frac{1}{2}$, P P' $P''=\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}$)

Decay	J^{π}	L_1	L_2	$\lvert^{A}\mathfrak{M}^{(0)}_{101}\rvert$	$\log ft_{th}$	$\log ft_{exp}$
$^{3}H \rightarrow {}^{3}He$	$1/2^+$	0	—	$\sqrt{3}$	3·04	3·06
$^{7}Be \rightarrow {}^{7}Li$	$3/2^-$	1	—	$\sqrt{\frac{5}{3}}$	3·23	3·30
$^{11}C \rightarrow {}^{11}B$	$3/2^-$	2	1	$\sqrt{\frac{3}{5}}\,(\sqrt{\frac{5}{3}})$	3·50 (3·23)	3·60
$^{13}N \rightarrow {}^{13}C$	$1/2^-$	1	—	$\frac{1}{\sqrt{3}}$	3·61	3·67
$^{15}O \rightarrow {}^{15}N$	$1/2^-$	1	—	$\frac{1}{\sqrt{3}}$	3·61	3·65
$^{17}F \rightarrow {}^{17}O$	$5/2^+$	2	—	$\sqrt{\frac{7}{5}}$	3·29	3·37
$^{19}Ne \rightarrow {}^{19}F$	$1/2^+$	0	—	$\sqrt{3}$	3·04	3·24
$^{21}Na \rightarrow {}^{21}Ne$	$3/2^+$	2	1	$\sqrt{\frac{3}{5}}\,(\sqrt{\frac{5}{3}})$	3·50 (3·23)	3·62
$^{25}Al \rightarrow {}^{25}Mg$	$5/2^+$	3	2	$\sqrt{\frac{5}{7}}\,(\sqrt{\frac{7}{5}})$	3·47 (3·29)	3·58
$^{29}P \rightarrow {}^{29}Si$	$1/2^+$	1	0	$\frac{1}{\sqrt{3}}\,(\sqrt{3})$	3·61 (3·04)	3·69
$^{31}S \rightarrow {}^{31}P$	$1/2^+$	1	0	$\frac{1}{\sqrt{3}}\,(\sqrt{3})$	3·61 (3·04)	3·69
$^{35}Ar \rightarrow {}^{35}Cl$	$3/2^+$	2	1	$\sqrt{\frac{3}{5}}\,(\sqrt{\frac{5}{3}})$	3·50 (3·23)	3·76
$^{37}K \rightarrow {}^{37}Ar$	$3/2^+$	2	—	$\sqrt{\frac{3}{5}}$	3·50	3·67
$^{39}Ca \rightarrow {}^{39}K$	$3/2^+$	2	—	$\sqrt{\frac{3}{5}}$	3·50	3·64
$^{41}Sc \rightarrow {}^{41}Ca$	$7/2^-$	3	—	$\sqrt{\frac{9}{7}}$	3·31	3·47

TABLE 14.8 *Comparison of the Gamow–Teller matrix element* $^{A}\mathfrak{M}^{(0)}_{101}$ *calculated by using the simple formula (14.310) with that calculated by using sophisticated shell model methods (Raman et al. 1978)*

Decay	$^{A}\mathfrak{M}^{(0)}_{101}$	
	eqn (14.310)	shell model
$^{3}H \rightarrow {}^{3}He$	1·732	1·732
$^{13}N \rightarrow {}^{13}C$	0·577	0·535
$^{19}Ne \rightarrow {}^{19}F$	1·732	1·684
$^{37}K \rightarrow {}^{37}Ar$	0·775	0·675

Since $\{^{A}\mathfrak{M}^{(0)}_{101}\}_{L_1}$ and $\{^{A}\mathfrak{M}^{(0)}_{101}\}_{L_2}$ always differ in sign a suitable value for the mixing coefficient a can bring the theoretical ft-value closer to the experimental ones. If two L values L_1 and L_2 are allowed, the corresponding values of $^{A}\mathfrak{M}^{(0)}_{101}$ and $\log ft_{\text{th}}$ for L_2 are given in brackets. The experimental values have been taken from Raman *et al.* (1978).

Nevertheless, we observe that the overall agreement between theoretical and experimental results is good. Perfect agreement was not expected for two reasons. First, the supermultiplet quantum numbers are not entirely good ones, i.e. we have a certain configuration mixing between different supermultiplets. Secondly, the axial vector constant λ is renormalized in different nuclei in a different way (see Sections 8.2 and 10.3). Therefore the true λ will deviate from the free neutron value $\lambda = 1{\cdot}25$ used for the above calculations.

By inspection of Table 14.8 we see that the agreement between experiment and theory can be improved by applying sophisticated shell model calculations† (see, for example, Brown *et al.* 1978; Raman *et al.* 1978). It should, however, be mentioned that especially in the p-shell the agreement between the results obtained by our simple formula (14.310) and results of refined shell model calculations (see Richter and de Kock 1977; Desgrolard and Guichon 1979) is very good. The results of our simple formula for $^{A}\mathfrak{M}^{(0)}_{101}$ lie sometimes 10 to 20% higher than those obtained by using good shell model wave functions (Raman *et al.* 1978).

As explained in Sections 8.2 and 10.3 the axial vector coupling λ is renormalized by the strong interactions, and that in a magnitude which

† Of course, for those cases where we have only one particle or hole outside a double magic core the agreement between both methods is perfect. That fact is, however, trivial. Therefore these cases have been omitted in Table 14.8.

differs from nucleus to nucleus in principle. The mixed superallowed transitions and here especially the mirror transitions offer now the best possibility for an empirical determination of

$$R_e = \frac{\lambda_{Ae}}{\lambda} \tag{14.313}$$

where λ_{Ae} is the true axial vector coupling constant within a nucleus and λ the free neutron decay value. Doing so we have two choices.

The first method consists of a comparison of the $\lambda_{Ae}\,{}^{A}\mathfrak{M}^{(0)}_{101}$ determined from experiments (ft-values) with the $\lambda\,{}^{A}\mathfrak{M}^{(0)}_{101}$ calculated with best available shell model wave functions. Transitions within one supermultiplet, i.e. superallowed transitions like, for instance, mirror transitions, are especially suitable for that approach because they react, as we have seen before, not very sensitively to details of the shell model used (kind of residual interactions, configuration space, etc.). Corresponding calculations for the p-shell and s–d shell have been carried out by Wilkinson (1973) and by Brown *et al.* (1978) with the result

$$R_e = 0{\cdot}900 \pm 0{\cdot}05. \tag{14.314}$$

This value indicates that in nuclei indeed axial transitions are significantly slower than in the case of the free neutron.

The second method makes use of a more model independent approach which avoids major reliance on theoretical wave functions. The magnetic moment μ of a nuclear state is defined by (in nuclear magneton units)

$$\mu = \langle J, M = J, TT_3 | \hat{\mu}_3 | J, M = J, TT_3 \rangle \tag{14.315}$$

with the operator

$$\hat{\mu}_3 = \tfrac{1}{2} \sum_{i=1}^{A} \{(1-\tau_3)(l_3 + \sigma_3 \mu_p) + (1+\tau_3)\sigma_3 \mu_n\}_i. \tag{14.316}$$

μ_p and μ_n are the magnetic moments of the free proton and neutron, respectively ($\mu_p = 1 + \kappa_p$ and $\mu_n = \kappa_n$, where $\kappa_p = 1{\cdot}793$ and $\kappa_n = -1{\cdot}913$ are the anomalous magnetic moments used before).

The isoscalar magnetic moment μ_0 is then given by

$$2\mu_0 = \langle J, M = J, TT_3 | \sum_{i=1}^{A} \{l_3 + (\mu_p + \mu_n)\sigma_3\}_i | J, M = J, TT_3 \rangle. \tag{14.317}$$

Since

$$j_3 = l_3 + \tfrac{1}{2}\sigma_3 \tag{14.318}$$

the above equation can be transformed to

$$\begin{aligned} 2\mu_0 = \langle J, M=J, TT_3| \sum_{i=1}^{A} [\{j_3\}_i + (\mu_p + \mu_n - \tfrac{1}{2})\{\sigma_3\}_i] \\ \times |J, M=J, TT_3\rangle = J + (\mu_p + \mu_n - \tfrac{1}{2}) \\ \times \langle J, M=J, TT_3| \sum_{i=1}^{A} \{\sigma_3\}_i |J, M=J, TT_3\rangle \\ = J + (\mu_p + \mu_n - \tfrac{1}{2})\langle\sigma_3\rangle. \end{aligned} \tag{14.319}$$

That means we can deduce experimental values $\langle\sigma_3\rangle$ from the isoscalar magnetic moments μ_0 which can be obtained from the two members of a mirror pair as follows:

$$2\mu_0 = \mu(A, T_3) + \mu(A, -T_3). \tag{14.320}$$

Now we have for a transition between analogue states

$$\langle\sigma_3\tau_3\rangle = \left\{\frac{J}{J+1}\right\}^{1/2} 2T_{3i}\{(T-T_{3i})(T+T_{3i}+1)\}^{-1/2}\, {}^{A}\mathfrak{M}^{(0)}_{101}. \tag{14.321}$$

For one valence nucleon outside a double magic core we have on the other hand

$$\langle\sigma_3(1\pm\tau_3)\rangle = 0, \tag{14.322}$$

i.e. the Gamow–Teller matrix element ${}^{A}\mathfrak{M}^{(0)}_{101}$ for mirror transitions can be determined from the experimental μ_0. What if we now have more valence nucleons? By making use of the SU(s)–SU(3) coupling scheme, i.e. of the L–S coupling scheme, which has also been applied before in this chapter, we can easily derive (see Section 8.1.1.4 and especially eqn (8.141)) that

$$\langle\sigma_3\rangle = \frac{J(J+1)+S(S+1)-L(L+1)}{(J+1)} \tag{14.323}$$

and

$$\langle\sigma_3(1\pm\tau_3)\rangle = \frac{J(J+1)+S(S+1)-L(L+1)}{(J+1)}\left[1 \pm T_3\frac{P''}{PP'}\right]. \tag{14.324}$$

For mirror transitions with $T=\frac{1}{2}$, $T_3=\pm\frac{1}{2}$ and with the supermultiplet quantum numbers $PP'P''=\frac{1}{2},\frac{1}{2},\pm\frac{1}{2}$ the quantity $\langle\sigma_3(1\pm\tau_3)\rangle$ therefore also vanishes.† Then we obtain

$${}^{A}\mathfrak{M}^{(0)}_{101} = \frac{2\mu_0 - J}{\mu_p + \mu_n - \frac{1}{2}}\left\{\frac{J+1}{J}\right\}^{1/2}\{(T-T_{3i})(T+T_{3i}+1)\}^{1/2}\frac{1}{2}\frac{P''}{PP'} \tag{14.325}$$

† It should be noted that for A = even nuclei $\langle\sigma_3(1\pm\tau_3)\rangle$ does often not vanish. However, for $PP'P''=1,1,1$ and $T=1$, $T_3=\pm1$ $\langle\sigma_3(1\pm\tau_3)\rangle$ is also zero.

and for $T=\frac{1}{2}$, $T_{3i}=-\frac{1}{2}$ and P, P', $P''=\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}$

$$^{A}\mathfrak{M}^{(0)}_{101} = \text{sign}(P'')\frac{2\mu_0 - J}{\mu_p + \mu_n - \frac{1}{2}}\left\{\frac{J+1}{J}\right\}^{1/2}. \tag{14.326}$$

It should, however, be pointed out that the above equation is only able to make model independent predictions of $^{A}\mathfrak{M}^{(0)}_{101}$ if we have no meson exchange,† spin orbit force, core polarization and relativistic contributions to the isoscalar magnetic moment μ_0 and to the Gamow–Teller matrix element $^{A}\mathfrak{M}^{(0)}_{101}$ (or if, at least, these contributions are small).

All these effects have been discussed in detail by Wilkinson (1973*b*, 1974*a*), Raman *et al.* (1978) and especially for the magnetic moments by Yamazaki (1979), Arima and Hyuga (1979) and Wildenthal and Chung (1979). We shall not repeat this discussion here, but refer the reader to the above publications.

The usual procedure for taking care of these effects is to correct the isoscalar magnetic moments explicitly for exchange contributions (which are small). Then, to proceed further we have two possibilities. Firstly, we can make use of the best available shell model estimates for $\langle\sigma_3(1\pm\tau_3)\rangle$ and assume that this difference, which is small, is more or less model independent. If one adopts this approach, one can deduce from the experimentally determined $\langle\sigma_3\rangle$ a model independent value for $^{A}\mathfrak{M}^{(0)}_{101}$ and hence a value for R_c (see Wilkinson 1974*a*). This strategy has, however, the disadvantage that the usual shell model calculations do not contain corrections for core polarization. Secondly, because of these shortcomings one can choose another more empirical method and introduce a scale factor G such that

$$\langle\sigma_3\rangle_{\text{expt}} = \langle\sigma_3\rangle_{\text{SM}} + G\,\delta\langle\sigma_3\rangle_{2\hbar\omega}. \tag{14.327}$$

This factor is so adjusted that the shell model value $\langle\sigma_3\rangle_{\text{SM}}$ reproduces the experimental value $\langle\sigma_3\rangle_{\text{expt}}\cdot\delta\langle\sigma_3\rangle_{2\hbar\omega}$ is the core polarization correction obtained theoretically. Then we can assume that

$$\langle\sigma_3(1\pm\tau_3)\rangle = \langle\sigma_3(1\pm\tau_3)\rangle_{\text{SM}} + G\delta\langle\sigma_3(1\pm\tau_3)\rangle_{2\hbar\omega} \tag{14.328}$$

is scaled by the same amount. If the scale factor G is not outside reasonable limits we can expect to obtain reliable values for $\langle\sigma_3(1\pm\tau_3)\rangle$ (see Raman *et al.* 1978). The further steps are afterwards the same as before. Corresponding results of the second method can be found in the comprehensive compilation by Raman *et al.* (1978) from where the values shown in Table 14.9 have been taken. The experimental results‡ of this

† One pion exchange contributions to μ_0 are identically zero.

‡ In principle, the ratio of the Fermi to Gamow–Teller matrix element can be determined from a measurement of electron asymmetry in the decay of oriented nuclei. Two β-transitions between $T=\frac{1}{2}$ analogue states have been investigated in that context, ^{19}Ne and ^{35}Ar. However, in the case of ^{35}Ar the experimental results for ft-value and asymmetry a_1 cannot be explained by the theory. It is often speculated that the Cabibbo angle vanishes in this special nuclear transition (see Hardy and Towner 1975; Szybisz and Rao 1976; Lee and Khanna 1977; Hagberg *et al.* 1979).

TABLE 14.9 *Effective axial-vector coupling constant,* $1+\delta=\lambda_{AE}/\lambda=R_e$ *deduced from* β*-decay and magnetic moment data (Raman et al. 1978)*

Mass number A	$\lvert\lambda_{AE}\langle\sigma_3\tau_3\rangle\rvert$ (β-decay)	$\lvert\langle\sigma_3\tau_3\rangle\rvert$†	$\lvert\langle\sigma_3\tau_3\rangle\rvert_{shell}$	$\langle\sigma_3\rangle$ (magnetic moment)	$\langle\sigma_3(1\pm\tau_3)\rangle$ (estimate)	λ_{AE} (deduced)	δ (%) (deduced)	δ_s (%)‡ (deduced)	(a) δ (%)$_{theory}$	(b) δ_s (%)$_{theory}$	(c) δ_s (%)$_{theory}$
3	1·205±0·002	0·975±0·007	1·000	0·884±0·008	−0·050±0·004	1·29±0·01	+4·3± 0·9	−2·5±0·7	+(5·3–11·4)	−2·6	·
15	0·365±0·002	0·295±0·002	0·333	−0·216±0·010	0·045±0·004	1·40±0·03	+13·1± 2·7	−11·5±0·6	·	−9·4	−(12·4–15·5)
17	1·086±0·002	0·878±0·006	1·000	0·946±0·016	0·064±0·019	1·23±0·05	−0·4± 4·1	−12·2±0·6	·	−13·7	−(13·1–14·8)
19	0·923±0·003	0·746±0·006	0·972	0·732±0·019	−0·077±0·010	1·14±0·02	−7·9± 1·7	−23·3±0·6	·	·	·
35	0·216±0·004	0·175±0·004	0·217	−0·158±0·009	−0·022±0·005	1·60±0·06	+29·0± 5·0	−19·4±1·8	·	·	·
39	0·502±0·007	0·406±0·006	0·600	−0·279±0·010	0·086±0·003	1·37±0·03	+10·9± 2·8	−32·3±1·0	·	−24·0	−(21·7–24·5)
41	0·930±0·003	0·752±0·006	1·000	0·948±0·054	0·054±0·057	1·04±0·13	−15·9±10·4	−24·8±0·6	·	−18·3	−(19·0–20·9)

† Assuming $\lambda = 1{\cdot}237 \pm 0{\cdot}008$

‡ $\delta_s = \lvert\langle\sigma_3\tau_3\rangle\rvert/\lvert\langle\sigma_3\tau_3\rangle\rvert_{shell} - 1$

(a) Jaus (1976)

(b) Towner and Khanna (1979)

(c) Oset and Rho (1979)

table are also compared with recent theoretical estimations of exchange effects. Although one has to sum over many different exchange and other (e.g. relativistic) contributions (see Section 8.2), where a high degree of cancellation is involved, the agreement between theory and experiment is surprisingly good.

These exchange effects can be regarded as an additional renormalization of the free axial vector coupling constant λ according to eqn (14.313). As we expected, the quenching of the allowed Gamow–Teller transitions differs from nucleus to nucleus. In some cases the amount of quenching is remarkable and cannot be neglected.

14.2.3. Usual allowed transitions

14.2.3.1. Weak magnetism and second class currents

In the foregoing sections of this chapter we have explicitly shown that nearly all observables for allowed transitions depend on the weak magnetism coupling constant f_M and on the induced tensor coupling constant f_T to a larger or smaller extent if the relevant formulae are finally evaluated in impulse approximation. Both the validity of the CVC theory and the possible existence of second class currents can therefore be tested if suitable allowed transitions are looked for. In the following we will examine the points:

(i) Can the existing experimental data be explained by assuming strong validity of the CVC theory?
(ii) What is the result of the experiments concerning the possible existence of second class currents?
(iii) How model independently can these experiments be interpreted?
(iv) What is the influence of the nuclear structure?

In order to provide answers to the above questions we restrict ourselves to a detailed consideration of two isospin triplets only, i.e.

(i) the lowest $T=1$ triplet in the $A=12$ system, here the $\Delta T=1$ ground-state–ground-state transition ($1^+ \rightarrow 0^+$) of the decay ^{12}B to ^{12}C and of the decay ^{12}N to ^{12}C (for the decay scheme see Fig. 14.3);
(ii) the lowest $T=1$ triplet in the $A=20$ system, here the $\Delta T=1$ ($2^+ \rightarrow 2^+$) transition of the decay ^{20}F to the 1·63 MeV level in ^{20}Ne and of the decay ^{20}Na to the 1·63 MeV level in ^{20}Ne (for the decay scheme see Fig. 14.4).

Other transitions which have also been investigated with respect to the topic of this chapter will be treated in summary form later on.

Considering first the $1^+ \rightarrow 0^+$ transitions, i.e. the $\beta^{\pm}$ decays of $^{12}\text{N} \rightarrow {}^{12}\text{C}$ and $^{12}\text{B} \rightarrow {}^{12}\text{C}$, respectively, we obtain† for the shape factor (see eqns

† Note that in this case form factor coefficients of tensor rank $K=1$ are only different from zero.

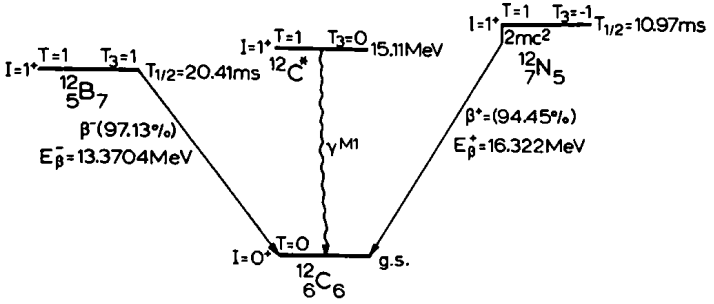


FIG. 14.3. Decay scheme of the $A = 12$ system (taken from Behrens *et al.* 1978).

(14.116), (14.117), (14.118), and (14.119))

$$C(W_e) = 1 + aW_e + \mu_1\gamma_1 b/W_e + cW_e^2 \tag{14.329}$$

with

$$a = 2R\frac{C_1}{C_0} \tag{14.330}$$

$$b = -2R\frac{D_0}{C_0} \tag{14.331}$$

$$c = R^2\left[\left(\frac{C_1}{C_0}\right)^2 + 2\frac{C_2}{C_0}\right], \tag{14.332}$$

for the ft-value

$$ft = \frac{2\pi^3 \ln 2}{G_\beta'^2\{C_0^2 + 2C_0C_1\langle W_e R\rangle\}}, \tag{14.333}$$

for the distribution of electrons emitted from oriented nuclei (see eqns

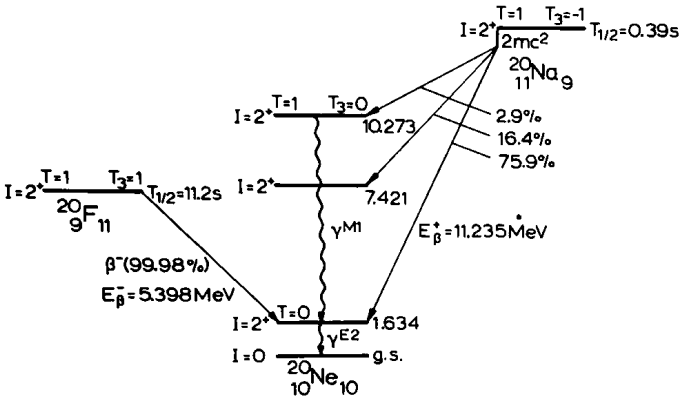


FIG. 14.4. Decay scheme of the $A = 20$ system (taken from Behrens *et al.* 1978).

(14.147), (14.148), and (14.149))

$$a_1 = A_1^0 f_1(J) \frac{p_e}{W_e}(1 + \alpha_\beta W_e) \tag{14.334}$$

with

$$A_1^0 = \mp \Lambda_1 \tag{14.335}$$

$$\alpha_\beta = \frac{\sqrt{2}}{3} R \eta_{12} \frac{F_0}{C_0}, \tag{14.336}$$

for the anisotropy of electrons emitted from oriented nuclei (see eqns (14.170), (14.171), and (14.172))

$$a_2 = f_2(J) \frac{p_e^2}{W_e} A_2^0 (1 + \hat{b} W_e) \tag{14.337}$$

with

$$A_2^0 = \sqrt{2}\, \nu_{12} R \frac{F_0}{C_0} \tag{14.338}$$

$$\hat{b} = R \frac{F_1}{F_0} \tag{14.339}$$

(upper sign β^--decay, lower sign β^+-decay).

For the orientation parameters $f_1(J)$ and $f_2(J)$ see eqns (14.151) and (14.174). The functions μ_1, η_{12}, and ν_{12} are the special Coulomb functions treated in Chapter 4 (see eqns (4.11), (4.13), and (4.15)).

The quantities $C_0, C_1, C_2, D_0, F_0, F_1$, which are linear combinations of the form factor coefficients, have to be taken from eqns (14.69), (14.70), (14.71), (14.72), (14.77), and (14.78), respectively.

For $|\alpha Z| \ll 1$ we have (since $|Z| = 6$, it is $\alpha\,|Z| = 0{\cdot}0438$)

$$\mu_1 = 1$$
$$\nu_{12} = 1$$
$$\eta_{12} = 1$$
$$\Lambda_1 = 1$$

(see eqn (4.29), and also Behrens and Jänecke (1969) and Jänecke (1969)). Inspecting eqns (14.336) and (14.338) we see that then†

$$A_2^0 = 3\alpha_\beta. \tag{14.340}$$

Further observables like, for instance, electron–neutrino correlation, electron–neutrino triple correlation, electron polarization etc. which are also sensitive to weak magnetism and second class current effects (see

† It should be pointed out that due to a different definition of the orientation parameter $f_2(J)$ in the work of Morita (1976) our value corresponds to $A_2^0 = 3A_{2_{\text{Morita}}}^0$.

Sections 14.2.1.3.4, 14.2.1.3.5, and 14.2.1.3.7) are not considered since corresponding experiments for them are difficult and have not been carried out up to now.

The eqns (14.329) to (14.339) are exact and include Coulomb effects. By inserting the quantities C_0, C_1 etc. into these equations we would obtain expressions in terms of from factor coefficients containing a larger number of different members. For a discussion of the essential features we will, therefore, first assume for simplicity that the Coulomb interaction is switched off and that isospin is therefore a good quantum number. In addition we will also omit some smaller not very important terms. Then we get:

(i) shape factor

$$a \simeq -2R\left[-\frac{2}{3}\sqrt{\left(\frac{2}{3}\right)}\frac{{}^{V}F^{(0)}_{111}}{{}^{A}F^{(0)}_{101}} + \tfrac{2}{27}W_0R\frac{-5\,{}^{A}F^{(1)}_{101} + \sqrt{2}\,{}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}}\right] \tag{14.341}$$

$$b \simeq -2R\left\{\frac{1}{3}\frac{\sqrt{\tfrac{1}{3}}\,{}^{A}F^{(0)}_{110} + \sqrt{\tfrac{2}{3}}\,{}^{V}F^{(0)}_{111}}{{}^{A}F^{(0)}_{101}} - \tfrac{1}{27}W_0R\frac{-{}^{A}F^{(1)}_{101} + 2\sqrt{2}\,{}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}}\right\} \tag{14.342}$$

$$c \simeq -2R^2\frac{\frac{10}{27}\,{}^{A}F^{(1)}_{101} - \frac{2}{27}\sqrt{2}\,{}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}}; \tag{14.343}$$

(ii) ft-value

$$ft = \frac{2\pi^3 \ln 2}{G_\beta'^2\{({}^{A}F^{(0)}_{101})^2 + \frac{2}{3}\sqrt{\tfrac{1}{3}}(W_0R)\,{}^{A}F^{(0)}_{101}\,{}^{A}F^{(0)}_{110}\}}; \tag{14.344}$$

(iii) electron distribution emitted from oriented nuclei

$$A_2^0 \simeq -\sqrt{2}\,R\left\{\frac{\sqrt{\tfrac{2}{3}}\,{}^{A}F^{(0)}_{110} - \sqrt{\tfrac{1}{3}}\,{}^{V}F^{(0)}_{111}}{{}^{A}F^{(0)}_{101}} - \tfrac{1}{9}(W_0R)\frac{2\sqrt{2}\,{}^{A}F^{(1)}_{101} + {}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}}\right\} \tag{14.345}$$

$$\alpha_\beta = \tfrac{1}{3}A_2^0 \tag{14.346}$$

$$A_2^0 b \simeq -\sqrt{2}\,R^2\frac{\frac{2}{9}\sqrt{2}\,{}^{A}F^{(1)}_{101} - \frac{4}{45}\,{}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}}. \tag{14.347}$$

The terms multiplied by W_0R which are usually not taken into account†

† In many publications the linear slope a and the anisotropy coefficient $A_2^0 = 3\alpha_\beta$ are simply written as

$$a = \frac{4}{3}\sqrt{\frac{2}{3}}R\frac{{}^{V}F^{(0)}_{111}}{{}^{A}F^{(0)}_{101}}$$

and

$$A_2^0 = -\sqrt{2}\,R\frac{\sqrt{\tfrac{2}{3}}\,{}^{A}F^{(0)}_{110} - \sqrt{\tfrac{1}{3}}\,{}^{V}F^{(0)}_{111}}{{}^{A}F^{(0)}_{101}}$$

while b, c, and $\tilde{b}$ are set equal to zero. In the case of the $^{12}N \rightarrow {}^{12}C$ and $^{12}B \rightarrow {}^{12}C$ transitions this approximation is, however, poor.

play the role of some correction terms which, because of the large transition energy can, however, not be neglected in our special case (we have $W_0R = 0{\cdot}24$).

Generally, the form factors can be split off into a contribution originating from first class currents (I) and into another one originating from second class currents (II) (see eqns (10.115*a–b*), (10.116*a–b*))

$$^{V}F_{KLs}(q^2) = {}^{V}F^{I}_{KLs}(q^2) + {}^{V}F^{II}_{KLs}(q^2) \tag{14.348a}$$

$$^{A}F_{KLs}(q^2) = {}^{A}F^{I}_{KLs}(q^2) + {}^{A}F^{II}_{KLs}(q^2). \tag{14.348b}$$

Now we always have two analogous transitions from each outer member of an isospin triplet to an isospin singlet (one β^--transition and one β^+-transition). Thus we are able to combine the corresponding observables for these two transitions. As derived in Section 10.4 the form factors of the two mirror transitions are related as follows (see eqns (10.133*a*) and (10.133*b*)):

$$T_{3i} = 1 \to T_{3f} = 0 \qquad T_{3i} = -1 \to T_{3f} = 0$$

$$^{V}F^{I}_{KLs}(q^2)_{\beta-} = -{}^{V}F^{I}_{KLs}(q^2)_{\beta+} \tag{14.349a}$$

$$^{V}F^{II}_{KLs}(q^2)_{\beta-} = {}^{V}F^{II}_{KLs}(q^2)_{\beta+} \tag{14.349b}$$

$$^{A}F^{I}_{KLs}(q^2)_{\beta-} = {}^{A}F^{I}_{KLs}(q^2)_{\beta+} \tag{14.349c}$$

$$^{A}F^{II}_{KLs}(q^2)_{\beta-} = -{}^{A}F^{II}_{KLs}(q^2)_{\beta+}. \tag{14.349d}$$

Two useful combinations of each observable can then be constructed, namely the difference and the sum, whereby in most cases the difference is only of importance.

From the near equality of the *ft*-values ($\delta = (ft)_{\beta+}/(ft)_{\beta-} - 1 \simeq 0{\cdot}11$) for β^-- and β^+-decay we can conclude that

$$\frac{^{A}F^{(0)II}_{101}}{^{A}F^{(0)I}_{101}} \ll 1 \tag{14.350}$$

$$\frac{^{A}F^{(0)II}_{121}}{^{A}F^{(0)I}_{121}} \ll 1. \tag{14.351}$$

By inspection of eqn (14.102) we see that at least in impulse approximation this latter assumption is justified.

Additionally, we set

$$^{V}F^{(0)II}_{111} = 0. \tag{14.352}$$

If the CVC theory is valid this latter condition is automatically fulfilled, but even if this is not the case, we can see from eqn (14.98) that this relation is at least valid in impulse approximation.

For the difference of the two mirror shape factors we then obtain†

$$C(W_e)_{\beta-} - C(W_e)_{\beta+} = [a(\beta^-) - a(\beta^+)]W_e + \gamma_1 \frac{b(\beta^-) - b(\beta^+)}{W_e}. \tag{14.353}$$

In the case of the beta-transitions under consideration measurements of the beta-spectrum have always been carried out in the upper half of the possible beta-energy range, i.e. for $W_e > 10$. For our further discussion the terms b/W_e are therefore completely meaningless and will not be treated in more detail. Then, we get

$$\Delta a = a(\beta^-) - a(\beta^+) = \tfrac{8}{3}\sqrt{\tfrac{2}{3}}\, R \frac{{}^{V}F^{(0)\prime}_{111}}{{}^{A}F^{(0)\prime}_{101}} \tag{14.354}$$

$$\Delta c = c(\beta^-) - c(\beta^+) \simeq 0. \tag{14.355}$$

For the normalized difference of the two mirror ft-values we get

$$\delta = \frac{(ft)_{\beta+}}{(ft)_{\beta-}} - 1 = 4\frac{{}^{A}F^{(0)\prime\prime}_{101}}{{}^{A}F^{(0)\prime}_{101}} + \tfrac{4}{3}\sqrt{\tfrac{1}{3}}\,(W_0 R)\frac{{}^{A}F^{(0)\prime\prime}_{110}}{{}^{A}F^{(0)\prime}_{101}}. \tag{14.356}$$

In addition, for the difference of the two mirror anisotropies of electrons emitted from oriented nuclei we obtain

$$\Delta a_2 = a_2(\beta^-) - a_2(\beta^+) = f_2(J)[A_2^0(\beta^-) - A_2^0(\beta^+)]\frac{p_e^2}{W_e}$$
$$+ f_2(J)[A_2^0(\beta^-)\hat{b}(\beta^-) - A_2^0(\beta^+)\hat{b}(\beta^+)]p_e^2 \tag{14.357a}$$

with

$$\Delta A_2^0 = 3\Delta\alpha_\beta = 3[\alpha(\beta^-) - \alpha(\beta^+)] = 2R\left[-2\sqrt{\left(\frac{1}{3}\right)}\frac{{}^{A}F^{(0)\prime\prime}_{110}}{{}^{A}F^{(0)\prime}_{101}} + \sqrt{\left(\frac{2}{3}\right)}\frac{{}^{V}F^{(0)\prime}_{111}}{{}^{A}F^{(0)\prime}_{101}}\right] \tag{14.357b}$$

$$A_2^0(\beta^-)\hat{b}(\beta^-) - A_2^0(\beta^+)\hat{b}(\beta^+) \simeq 0. \tag{14.358}$$

As shown in Section 10.2 the CVC theory relates model independently the form factor coefficient ${}^{V}F^{(0)}_{111}$ to the radiative width $\Gamma_\gamma(M1)$ of the

† We can also construct the sum and obtain

$$a(\beta^-) + a(\beta^+) = -\tfrac{8}{27}W_0R^2\frac{-5\,{}^{A}F^{(1)\prime}_{101} + \sqrt{2}\,{}^{A}F^{(0)\prime}_{121}}{{}^{A}F^{(0)\prime}_{101}}$$

$$A_2^0(\beta^-) + A_2^0(\beta^+) = -2\sqrt{2}\,R\left\{\sqrt{\left(\frac{2}{3}\right)}\frac{{}^{A}F^{(0)\prime}_{110}}{{}^{A}F^{(0)\prime}_{101}} - \tfrac{1}{5}(W_0R)\cdot\frac{2\sqrt{2}\,{}^{A}F^{(1)\prime}_{101} + {}^{A}F^{(0)\prime}_{121}}{{}^{A}F^{(0)\prime}_{101}}\right\}.$$

These combinations do not give information about CVC or second class currents. They can, however, be used to test theoretical calculations of some axial vector form factor coefficients.

isospin analogous γ-transition via (see eqns (10.77) and (10.90))

$$\{{}^{V}F^{(0)}_{111}\}^2 = \frac{9\Gamma_\gamma(M1)}{\alpha R^2 E_\gamma^3} \tag{14.359}$$

or if E_γ is expressed in MeV and $\Gamma_\gamma(M1)$ in eV via

$$\{{}^{V}F^{(0)}_{111}\}^2 = \frac{9\Gamma_\gamma(M1)(0{\cdot}511)^2}{\alpha R^2 E_\gamma^3} \times 10^{-6}. \tag{14.360}$$

Additionally, the absolute value of the form factor coefficient ${}^{A}F^{(0)!}_{101}$ can be model independently derived from (see eqns (14.35) and (14.344))

$$\tfrac{1}{2}\{ft(\beta^-) + ft(\beta^+)\} \simeq \frac{6173}{\{{}^{A}F^{(0)!}_{101}\}^2}. \tag{14.361}$$

Thus, as long as Coulomb effects and other correction terms are neglected the CVC theory can be tested in a completely model independent way by measuring the shape factor (linear shape) and the analogous radiative width $\Gamma_\gamma(M1)$. If the CVC theory is valid both observables are related by eqns (14.354) and (14.359), whereby eqn (14.361) has also to be taken into account.† The question of a possible existence of second class currents can be answered by either investigating the difference of the two mirror ft-values, i.e. δ or the difference of the two mirror anisotropies ΔA_2^0 of electrons emitted from oriented nuclei. The former observable is zero (see eqn (14.356) and the latter (see eqn (14.357)) assumes the value predicted by CVC if second class currents are absent.

By going back to the impulse approximation we can make allowance of further, but not model independent, test possibilities for the CVC theory and for the possible existence of second class currents, respectively. Then, we obtain for the difference:

(i) of the linear slope of the shape factor (see eqns (14.130) and (14.131))

$$\Delta a \simeq \frac{16}{3}\left\{\frac{1}{2M_N\lambda}\left[1 + \frac{\int \mathbf{l}}{\int \boldsymbol{\sigma}}\right] + \frac{f_M}{\lambda}\right\}; \tag{14.362}$$

(ii) of the ft-values (see eqns (14.121) and (14.126))

$$\delta \simeq -\tfrac{8}{3} W_0 \frac{f_T}{\lambda}; \tag{14.363}$$

† $$\Delta a = 3{\cdot}907 \times 10^{-2} \sqrt{\left\{\frac{\Gamma(M1)ft}{2E_\gamma^3 ft(0^+{-}0^+)}\right\}}$$

where $\Gamma(M1)$ in eV; E_γ in MeV; $2ft(0^+{-}0^+) = 6173$ s.

(iii) of the anisotropy of electrons emitted from oriented nuclei

$$\Delta a_2 = 3\,\Delta\alpha_\beta \simeq 4\left\{\frac{1}{2M_N\lambda}\left[1+\frac{\int \mathbf{l}}{\int \boldsymbol{\sigma}}\right]+\frac{f_M-f_T}{\lambda}\right\}. \tag{14.364}$$

It is the matrix element ratio of the angular momentum operator to the spin operator $\int \mathbf{l}/\int \boldsymbol{\sigma}$ which is sensitive to the nuclear structure. As demonstrated before (see eqns (14.133) to (14.135)) this influence becomes more important with increasing orbital angular momentum l of the valence nucleons.

The case of smallest influence happens when $j_i = j_f \pm 1$ (see eqn (14.135)). Since for the $A = 12$ system the valence nucleons belong to the p-shell we have the relatively small value $l = 1$. Furthermore, the transition is mainly of the type $p_{3/2} \rightarrow p_{1/2}$ such that the ratio $\int \mathbf{l}/\int \boldsymbol{\sigma}$ is small and the nuclear structure influence can be neglected in a first approximation. The latter fact distinguishes these transitions from other suitable ones.

As discussed in Section 8.2 we have off-mass shell effects and exchange contributions. Thus a further improvement of the simple impulse approximation treatment can be achieved if allowance is made for these effects, at least, as far as the axial form factor coefficients are concerned. Corresponding formulae have been derived in Section 8.2.3 by making use of the KDR model (Kubodera, Delorme, and Rho 1973). By inserting eqns (8.346) and (8.348) into eqns (14.356) and (14.357) we get therefore for the difference:†

(i) of the ft-values

$$\delta = -\frac{8}{3}\frac{\zeta}{\lambda}W_0+\frac{4}{3}\frac{\lambda_s L}{\lambda}W_0-4\frac{\lambda_s J}{\lambda}; \tag{14.365}$$

(ii) of the anisotropies of electrons emitted from oriented nuclei

$$\Delta A_2^0 = 3\,\Delta\alpha_\beta = 4\left\{\frac{1}{2M_N\lambda}\left[1+\frac{\int \mathbf{l}}{\int \boldsymbol{\sigma}}\right]+\frac{f_M}{\lambda}-\frac{\zeta+\lambda_s L}{\lambda}\right\}. \tag{14.366}$$

In the above considerations we have up to now assumed the Coulomb interaction to be absent. Since, of course, the Coulomb interaction cannot be switched off and isospin is therefore violated the following three aspects become important:

(i) terms multiplied by αZ occur which do not cancel in the difference of a or A_2^0;

† In principle, we have also to consider the different renormalization of the axial vector coupling constant λ in different nuclei (see Section 8.2.3).

(ii) terms multiplied by W_0R do not cancel, since $W_0(\beta^+) \neq W_0(\beta^-)$,† and this difference becomes even more important with increasing Z

$$(W_0(\beta^+) \simeq E_\gamma - (M_N - M_P) + \tfrac{6}{5}\alpha Z/R,$$
$$W_0(\beta^-) \simeq E_\gamma + (M_N - M_P) - \tfrac{6}{5}\alpha Z/R);$$

(iii) the form factor coefficients F_{KLs} have to be replaced by $F_{KLs}(1 \pm \delta_{bind})$, where δ_{bind} takes care of the isospin violation.

The effects (i) and (ii) are the most important ones for shape factor and electron anisotropy, the effect (iii) influences the *ft*-value.

Thus, the model independence of the tests for CVC and for the existence of second class currents is weakened and only approximately valid. In that context the low Z of the transitions under consideration ($|Z| = 6$, $|\alpha Z| = 0{\cdot}04378$) is of great advantage.

Experimental and theoretical results for the ground-state–ground-state transition in the $A = 12$ system are compared in Table 14.10. It turns out that all experimental results can be explained completely by the absence of second class currents (SCC).

In the case of the anisotropy of electrons emitted from oriented nuclei this is clearly evident. In the case of the *ft*-value‡ difference δ_{exp} the experimental result can be totally accounted for by isospin violation effects δ_{bind} because of the Coulomb interaction (see also the discussion in Section 14.2.2.1, especially eqns (14.284) to (14.303)), i.e. $\delta_{sec} = \delta_{exp} - \delta_{bind} \simeq 0$. In our case the meaning of δ_{bind} can be qualitatively understood because the beta-decaying proton in ^{12}N is less strongly bound than the corresponding decaying neutron in ^{12}B (binding energies 0·60 MeV and 3·37 MeV, respectively). Thus, the wave function of the proton has a broader tail than that of the neutron. We expect, therefore, intuitively the $p \rightarrow n$ overlap in the β^+-transition to be poorer than the $n \rightarrow p$ overlap in the β^--transition (neutron and proton binding energies in ^{12}C are 18·7 MeV and 16·0 MeV, respectively) such that $(ft)_{\beta+} > (ft)_{\beta-}$.

All results shown in Table 14.10 are also in complete agreement with the assumption that CVC theory is valid. In that context it should be clearly stated that especially the comparison of theory and experiment in the $A = 12$ system gives one of the best confirmations of the CVC theory and represents, therefore, something like a key experiment.

Furthermore it should be noted that the theoretical shape differences Δa given in Table 14.10 have been calculated by using the complete

† In eqns (14.363) and (14.365) W_0 has to be replaced by

$$\tfrac{1}{2}[W_0(\beta^+) + W_0(\beta^-)].$$

‡ The beta-decay of ^{12}B and ^{12}N to the first excited state of ^{12}C has also been investigated. In that case an *ft*-asymmetry δ of $-0{\cdot}001 \pm 0{\cdot}014$ was obtained (Kaina *et al.* 1981).

TABLE 14.10 *Difference of the shape factors, ft-values and of the beta-anisotropy coefficients between the β^-- and β^+-decay of the $A = 12$ system (ground state transitions)*

Experiment:	$\Delta a = (0{\cdot}50 \pm 0{\cdot}05)\%/m_e c^2$ [a]
	$\Delta a = (0{\cdot}43 \pm 0{\cdot}12)\%/m_e c^2$ [b]
$\Gamma_\gamma(M1)$:	$\Delta a = (0{\cdot}57 \pm 0{\cdot}01)\%/m_e c^2$ [c]
Theoretical calculation:	$\Delta a = (0{\cdot}478 \pm 0{\cdot}004)\%/m_e c^2$ [d]
Experiment:	$\delta = 0{\cdot}129 \pm 0{\cdot}008$ [e]
Theory:	$\delta_{\text{bind}} = 0{\cdot}08 - 0{\cdot}17$ [f]
(isospin violation effects)	
Experiment:	$\Delta\alpha_\beta = (0{\cdot}141 \pm 0{\cdot}015)\%/m_e c^2$ [g]
Theoretical calculation:	$\Delta\alpha_\beta = (0{\cdot}130 \pm 0{\cdot}002)\%/m_e c^2$ [h]
(assuming no second class currents exist)	
$\Gamma_\gamma(M1)$:	$\Delta\alpha_\beta = (0{\cdot}144 \pm 0{\cdot}002)\%/m_e c^2$ [i]
(assuming no second class currents exist)	

(a) Kaina *et al.* (1977).
(b) Wu *et al.* (1977).
(c) $\Gamma_\gamma(M1) = 37{\cdot}0 \pm 1{\cdot}1$ eV from F. Ajzenberg-Selove (1980). Here, the formula

$$\Delta a = 3{\cdot}907 \times 10^{-2} \sqrt{\left\{\frac{\Gamma_\gamma(M1)ft}{2E_\gamma^3 ft(0^+ - 0^+)}\right\}}$$

has been used (Γ_γ in eV, E_γ in MeV, $2ft(0^+ - 0^+) = 6173$ s).
(d) Behrens *et al.* (1978). Here, the complete formulae given in eqn (14.127) including the terms multiplied by $(W_0 R)$ and (αZ) have been used. The uncertainty is caused by nuclear structure effects (see also Koshigiri *et al.* 1979).
(e) Alburger and Nathan (1978).
(f) Laverne and Do Dang (1971), Wilkinson (1971*b*), Blomqvist (1971), Towner (1973), McDonald *et al.* (1974), Wilkinson (1974*b*), Behrens and Szybisz (1975), Greenland (1975).
(g) Brändle *et al.* (1978*a*), Lebrun *et al.* (1978), Brändle *et al.* (1978*b*), Sugimoto *et al.* (1978), Masuda *et al.* (1979).
(h) Behrens *et al.* (1978), Szybisz and Behrens (1980). Here, the complete formulae given in eqns (14.178*a*) and (*b*) including the terms multiplied by $W_0 R$ and αZ have been used. The uncertainty is caused by nuclear structure effects.
(i) $\Gamma_\gamma(M1) = 37{\cdot}0 \pm 1{\cdot}1$ eV from F. Ajzenberg-Selove (1980). Here, the formula

$$\Delta\alpha_\beta = 9{\cdot}768 \times 10^{-3} \sqrt{\left\{\frac{\Gamma_\gamma(M1)ft}{2E_\gamma^3 ft(0^+ - 0^+)}\right\}}$$

has been used ($\Gamma_\gamma(M1)$ in eV, E_γ in MeV, $2ft(0^+ - 0^+) = 6173$ s).

expressions of eqn (14.127) including the Coulomb interaction effects of (i) and (ii). If the same shape difference is calculated by using the simple version of eqn (14.354) a value which is 13% larger than the result of the complete calculation is obtained (Behrens *et al.* 1978). This indicates that a model independent test using eqns (14.354) and (14.359) can only be accurate within 13% which is the effect of the higher order terms. As for the difference of the slopes Δa we again calculated $\Delta\alpha_\beta$ from the complete expressions (14.336) or (14.338), respectively. If we take the simple eqn (14.357), a result is obtained that is 4% larger than $\Delta\alpha_\beta$ listed in Table 14.10 (Behrens *et al.* 1978). Thus, the effect of higher order

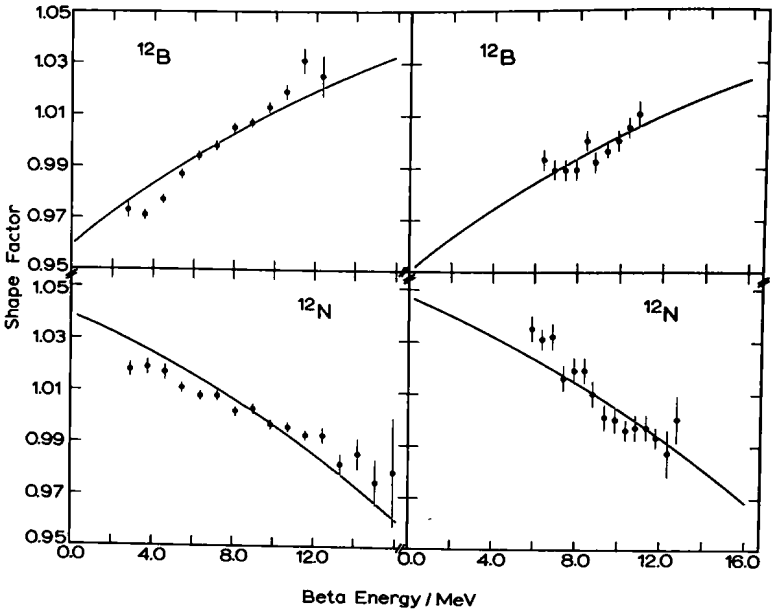


FIG. 14.5. Experimental shape factor for the $A = 12$ system (Kaina *et al.* 1977). The curves are the theoretical calculations (taken from Behrens *et al.* 1978).

FIG. 14.6. Experimental shape factor for the $A = 12$ system (Wu *et al.* 1977). The curves are the theoretical calculations (taken from Behrens *et al.* 1978).

terms is smaller for $\Delta\alpha_\beta$ than in the case of the shape difference Δa and a model independent test more accurate (within 4%).

As shown in Table 14.10 the overall agreement between theory and experiment for the difference of slopes and electron anisotropies is very good (assuming CVC to be valid and second class currents to be absent).

On the other hand, every observable for each transition (β^-- and β^+-decay) can also be considered separately. This is shown in Figs. 14.5, 14.6, and 14.7. By inspection of these figures we then, however, recognize that as far as the shape factor is concerned only the experiment of Wu *et al.* (see Lee *et al.* 1963; Wu 1964; Wu *et al.* 1977) performed some time ago,† but reanalysed later on, is in agreement with the theory. In contrast,

† There also exists two other older experiments for the shape factor (see Mayer-Kuckuk and Michel 1962; Glass and Peterson 1963).

the more recent one by Kaina *et al.* (1977) disagrees with the theoretical prediction. As far as the electron anisotropy is concerned we have nevertheless complete agreement with the calculations for both the β^-- and the β^+-decay. The whole problem can therefore not be considered as completely solved at the moment.

As a second example we will discuss the $2^+ \xrightarrow{\beta} 2^+ \xrightarrow{\gamma} 0^+$ transition in the decays of $^{20}F \rightarrow {}^{20}Ne$ and of $^{20}Na \rightarrow {}^{20}Ne$ in detail. The main difference between this system and that one discussed before is that now in addition to the form factor coefficients of tensor rank 1 other ones of tensor rank 0, 2, 3, and 4 can occur.

Thus all expressions are more complicated in principle. As shown in Section 14.2.1.2 (see eqns (14.110) to (14.114)) we have, however,

$$A_0 \simeq 0$$
$$A_1 \simeq 0$$
$$A_2 \simeq 0$$
$$B_0 \simeq 0$$
$$B_1 \simeq 0$$

because our transition is of the $\Delta T = 1$ type. All terms containing tensor rank 0 form factor coefficients are therefore neglected in the following.

In the case of the shape factor, from an inspection of eqns (14.117 to 14.119) it becomes apparent that the terms containing form factor coefficients of rank 2 (and of higher rank) can also be omitted (note that for ^{20}F we have $|E_0/C_0| \leq 0{\cdot}5$, $|G_0/C_0| \leq 2 \times 10^{-3}$, $W_0 R = 0{\cdot}1123$, and for ^{20}Na $|E_0/C_0| \leq 0{\cdot}1$, $|G_0/C_0| \leq 0{\cdot}15$, $W_0 R = 0{\cdot}2232$; see Behrens *et al.* 1978). Thus, for the shape we are dealing with the same formulae, i.e. the eqns (14.330), (14.331), and (14.332) as used before for the transitions $^{12}B \rightarrow {}^{12}C$ and $^{12}N \rightarrow {}^{12}C$.

On the other hand, the corresponding coefficients for the asymmetry and anisotropy of electrons emitted from oriented nuclei are more complicated and now read as (see eqns (14.157), (14.171), and (14.172))

$$A_1^0 = \mp \tfrac{1}{3} \Lambda_1 \tag{14.367}$$

$$\alpha_\beta = \frac{\sqrt{2}}{3} R \eta_{12} \left[\frac{F_0}{C_0} + \sqrt{(21)} \frac{H_0}{C_0} \right] \tag{14.368}$$

and

$$A_2^0 = -\tfrac{2}{3}\sqrt{2}\, R \nu_{12} \left[\frac{F_0}{C_0} - \sqrt{\left(\frac{3}{7}\right)} \frac{H_0}{C_0} \right] \tag{14.369}$$

$$\hat{b} = R \frac{F_1 - \sqrt{\frac{3}{7}}\, H_1 - \frac{1}{5} \sqrt{\left(\frac{2}{7}\right)} \frac{\eta_{13}}{\nu_{12}} X_0 - \frac{2}{5} \sqrt{\left(\frac{2}{7}\right)} \frac{\eta_{13}}{\nu_{12}} Y_0}{F_0 - \sqrt{\frac{3}{7}}\, H_0}. \tag{14.370}$$

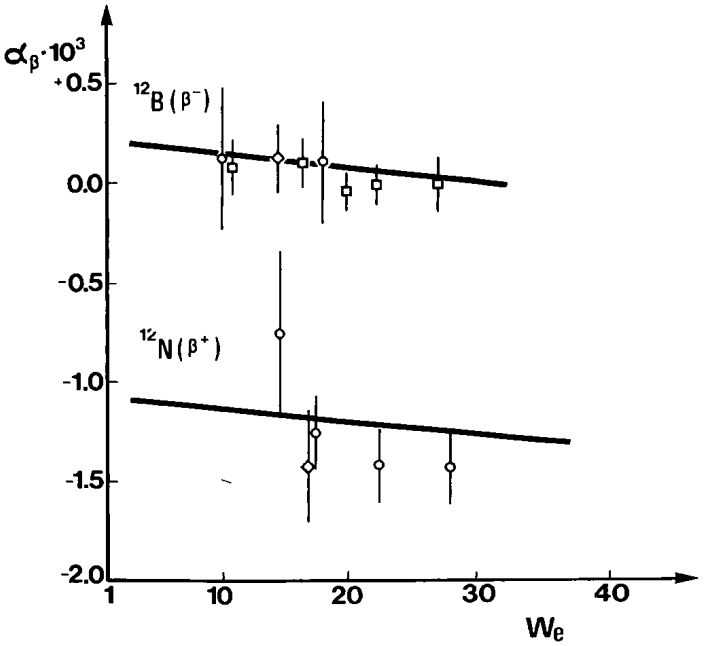


FIG. 14.7. Experimental asymmetries of electrons emitted from oriented nuclei for the $A = 12$ system. ◇ Sugimoto *et al.* (1978), □ Lebrun *et al.* (1978), ○ Brändle *et al.* (1978*a* and *b*). The curves are theoretical ones (taken from Szybisz and Behrens 1980).

It should be noted that A_2^0 is not equal to α_β times a constant factor, in contrast to the 1–0 transition considered before. Because of the γ-transition following the corresponding β-transitions we now have the additional possibility of observing $\beta-\gamma$ (CP) correlation and $\beta-\gamma$ angular correlation. For the $\beta-\gamma$ (CP) correlation it then follows (see eqns (14.179), (14.180), and (14.181))

$$\tilde{A} = \tilde{A}_0(1+\tilde{\alpha}W_e) \tag{14.371}$$

with

$$\tilde{A}_0 = \pm\tfrac{1}{6}\Lambda_1$$

$$\tilde{\alpha} = \frac{\sqrt{2}}{3} R\eta_{12}\left[\frac{F_0}{C_0} - \sqrt{(21)}\,\frac{H_0}{C_0}\right] \tag{14.372}$$

and for the $\beta-\gamma$ angular correlation† (see eqns (14.179), (14.189), and (14.190))

$$\varepsilon = a_{\beta\gamma}\frac{p_e^2}{W_e}(1+b_{\beta\gamma}W_e) \tag{14.373}$$

with

$$a_{\beta\gamma} = \frac{\sqrt{2}}{6}\nu_{12}R\left[\frac{F_0}{C_0}+\sqrt{\left(\frac{3}{7}\right)}\frac{H_0}{C_0}\right] \tag{14.374}$$

$$b_{\beta\gamma} = R\frac{F_1+\sqrt{\frac{3}{7}}H_1+\frac{1}{5}\sqrt{\left(\frac{2}{7}\right)}\frac{\eta_{13}}{\nu_{12}}X_0-\frac{2}{5}\sqrt{\left(\frac{2}{7}\right)}\frac{\eta_{13}}{\nu_{12}}Y_0}{F_0+\sqrt{\frac{3}{7}}H_0}. \tag{14.375}$$

Because of the low Z ($|Z|=10$) and of the relatively high beta-energies involved the necessary Coulomb functions can be assumed in good approximation to be

$$\begin{aligned}\mu_1&=1\\ \nu_{12}&=1\\ \eta_{12}&=1\\ \eta_{13}&=1\\ \Lambda_1&=1\end{aligned} \tag{14.376}$$

(see Behrens and Jänecke 1969).

If we, as before, for the moment switch off the Coulomb interaction, i.e. omit the terms multiplied by αZ we get from eqns (14.69), (14.77),

† It should be noted that we have also a term connected with $P_1(\cos\theta_{\beta\gamma})$ which is of purely kinematical origin

$$\omega(W_e,\theta_{\beta\gamma}) = 1+\tilde{R}\frac{p_e}{W_e}\cos\theta_{\beta\gamma}$$

with

$$\tilde{R}(\beta^+) = (0{\cdot}7-0{\cdot}57W_e)10^{-4}$$

$$\tilde{R}(\beta^-) = (0{\cdot}4-0{\cdot}57W_e)10^{-4}$$

(see Calaprice *et al.* 1977).

and (14.83)

$$\frac{F_0}{C_0} = -\sqrt{\left(\frac{2}{3}\right)}\frac{{}^{A}F^{(0)}_{110}}{{}^{A}F^{(0)}_{101}} + \sqrt{\left(\frac{1}{3}\right)}\frac{{}^{V}F^{(0)}_{111}}{{}^{A}F^{(0)}_{101}} + \tfrac{1}{9}(W_0R)\frac{2\sqrt{2}\,{}^{A}F^{(1)}_{101} + {}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}} \tag{14.377}$$

$$\frac{H_0}{C_0} = \frac{{}^{V}F^{(0)}_{211}}{{}^{A}F^{(0)}_{101}} + \tfrac{1}{3}(W_0R)\frac{\sqrt{\frac{2}{5}}\,{}^{V}F^{(0)}_{220} - \sqrt{\frac{3}{5}}\,{}^{A}F^{(0)}_{221}}{{}^{A}F^{(0)}_{101}} \tag{14.378}$$

and

$$A^0_2\hat{b} = \tfrac{2}{3}\sqrt{2}\,R^2\left[\frac{2}{9}\sqrt{2}\frac{{}^{A}F^{(1)}_{101}}{{}^{A}F^{(0)}_{101}} - \frac{4}{45}\frac{{}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}} - \frac{1}{3}\sqrt{\left(\frac{6}{35}\right)}\frac{{}^{V}F^{(0)}_{220}}{{}^{A}F^{(0)}_{101}} + 2\sqrt{\left(\frac{1}{35}\right)}\frac{{}^{A}F^{(0)}_{221}}{{}^{A}F^{(0)}_{101}} + \frac{2}{5}\sqrt{\left(\frac{2}{7}\right)}\frac{{}^{A}F^{(0)}_{321}}{{}^{A}F^{(0)}_{101}}\right] \tag{14.379}$$

$$a_{\beta\gamma}b_{\beta\gamma} = -\frac{\sqrt{2}}{6}R^2\left[\frac{2}{9}\sqrt{2}\frac{{}^{A}F^{(1)}_{101}}{{}^{A}F^{(0)}_{101}} - \frac{4}{45}\frac{{}^{A}F^{(0)}_{121}}{{}^{A}F^{(0)}_{101}} + \frac{1}{3}\sqrt{\left(\frac{6}{35}\right)}\frac{{}^{V}F^{(0)}_{220}}{{}^{A}F^{(0)}_{101}} - 2\sqrt{\left(\frac{1}{35}\right)}\frac{{}^{A}F^{(0)}_{221}}{{}^{A}F^{(0)}_{101}} + \frac{2}{5}\sqrt{\left(\frac{2}{7}\right)}\frac{{}^{A}F^{(0)}_{321}}{{}^{A}F^{(0)}_{101}}\right]. \tag{14.380}$$

Now, we can construct the differences and the sums of the two corresponding observables for β^-- and β^+-decay. In the case of the shape factor we get the same result as obtained before in the $A = 12$ system, i.e. that given in eqn (14.354). In the case of the other observables the following relations are of importance for the differences

$$\frac{F_0}{C_0}(\beta^-) - \frac{F_0}{C_0}(\beta^+) = -2\sqrt{\left(\frac{2}{3}\right)}\frac{{}^{A}F^{(0)\prime\prime}_{110}}{{}^{A}F^{(0)\prime}_{101}} + 2\sqrt{\left(\frac{1}{3}\right)}\frac{{}^{V}F^{(0)\prime}_{111}}{{}^{A}F^{(0)\prime}_{101}} \tag{14.381}$$

$$\frac{H_0}{C_0}(\beta^-) - \frac{H_0}{C_0}(\beta^+) = 2\frac{{}^{V}F^{(0)\prime}_{211}}{{}^{A}F^{(0)\prime}_{101}} + \tfrac{2}{3}\sqrt{\tfrac{2}{5}}\,(W_0R)\frac{{}^{V}F^{(0)\prime}_{220}}{{}^{A}F^{(0)\prime}_{101}}. \tag{14.382}$$

Here, it has been assumed that

$$\frac{{}^{V}F^{(0)\prime\prime}_{211}}{{}^{V}F^{(0)\prime}_{211}} \ll 1 \tag{14.383a}$$

$$\frac{{}^{V}F^{(0)\prime\prime}_{220}}{{}^{V}F^{(0)\prime}_{220}} \ll 1 \tag{14.383b}$$

$$\frac{{}^{A}F^{(0)\prime\prime}_{221}}{{}^{A}F^{(0)\prime}_{221}} \ll 1. \tag{14.383c}$$

It has already been shown (see eqn (14.359)) how the absolute value of the form factor coefficient ${}^{V}F^{(0)}_{111}$ can be related to the radiative width

$\Gamma_\gamma(M1)$. If CVC is valid, it is also possible to derive the form factor coefficients ${}^{V}F_{211}^{(0)}$ and ${}^{V}F_{220}^{(0)}$ from the radiative width $\Gamma_\gamma(E2)$. Based on the CVC theory (see eqns (10.77) and (10.89)), one obtains†

$$|{}^{V}F_{211}^{(0)}|^2 = \frac{15}{\alpha}\frac{\Gamma_\gamma(E2)}{E_\gamma^3 R^2} \tag{14.384}$$

$$|{}^{V}F_{220}^{(0)}|^2 = \frac{150}{\alpha}\frac{\Gamma_\gamma(E2)}{E_\gamma^5 R^4} \tag{14.385}$$

or if $\Gamma_\gamma(E2)$ is expressed in eV and E_γ in MeV

$$|{}^{V}F_{211}|^2 = \frac{15}{\alpha}\Gamma_\gamma(E2)\frac{(0{\cdot}511)^2 10^{-6}}{E_\gamma^3 R^2} \tag{14.386}$$

$$|{}^{V}F_{220}|^2 = \frac{150}{\alpha}\Gamma_\gamma(E2)\frac{(0{\cdot}511)^4}{E_\gamma^5}\frac{10^{-6}}{R^4}. \tag{14.387}$$

The sign between ${}^{V}F_{111}^{(0)}$ and ${}^{V}F_{220}^{(0)}$ (or ${}^{V}F_{211}^{(0)}$) can be obtained from the mixing ratio of the γ-transition. Then, we have

$$\left|\frac{H_0}{C_0}(\beta^-) - \frac{H_0}{C_0}(\beta^+)\right| = \frac{2}{3}\left|\frac{{}^{V}F_{211}^{(0)}}{{}^{A}F_{101}^{(0)}}\right| = \frac{2}{3}\left\{\frac{15\Gamma_\gamma(E2)ft}{\alpha E_\gamma^3 R^2 2ft(0\to 0)}\right\}^{1/2}. \tag{14.388}$$

By inserting the corresponding values $ft = 9{\cdot}62\times10^4$ s (see Genz *et al.* 1976), $\Gamma_\gamma(E2) = 0{\cdot}0016 \pm 0{\cdot}0028$ eV (see Fifield *et al.* 1977), $R = 9{\cdot}71\times10^{-3}$ (see Behrens *et al.* 1978) and $E_\gamma = 8{\cdot}6363 \pm 0{\cdot}0032$ MeV (see Fifield *et al.* 1977) one obtains

$$\left|\frac{H_0}{C_0}(\beta^-) - \frac{H_0}{C_0}(\beta^+)\right| \approx 0{\cdot}01. \tag{14.389}$$

Assuming second class currents to be absent for the moment we can also make use of eqn (14.359) and get

$$\left|\frac{F_0}{C_0}(\beta^-) - \frac{F_0}{C_0}(\beta^+)\right| = 2\sqrt{\left(\frac{1}{3}\right)}\left|\frac{{}^{V}F_{111}^{(0)}}{{}^{A}F_{101}^{(0)}}\right| = 2\left\{\frac{3\Gamma_\gamma(M1)ft}{\alpha R^2 E_\gamma^3 2ft(0\to 0)}\right\}^{1/2}. \tag{14.390}$$

By inserting $\Gamma_\gamma(M1) = 4{\cdot}26 \pm 0{\cdot}23$ eV (see Fifield *et al.* 1977) one obtains

$$\left|\frac{F_0}{C_0}(\beta^-) - \frac{F_0}{C_0}(\beta^+)\right| = 0{\cdot}685. \tag{14.391}$$

For the $A = 20$ system we have a configuration of 4 valence nucleons in the s–d shell and an inert ^{16}O core. Thus we are in a very good situation

† Note that on the basis of CVC ${}^{V}F_{211}^{(0)}$ and ${}^{V}F_{220}^{(0)}$ are related by (see eqn (10.67))

$${}^{V}F_{211}^{(0)} = -\frac{1}{\sqrt{(10)}}[(W_0 \mp 2{\cdot}5)R \pm \tfrac{6}{5}\alpha Z]\,{}^{V}F_{220}^{(0)} = -\frac{1}{\sqrt{(10)}}E_\gamma R\,{}^{V}F_{220}^{(0)}.$$

as far as theoretical shell model calculations are concerned. Those calculations for the corresponding beta-decay observables have been carried out by some authors (see Calaprice *et al.* 1977; Fifield *et al.* 1977; Behrens *et al.* 1978). As a result the ratios of ${}^{V}F_{211}^{(0)}/{}^{A}F_{101}^{(0)}$ and ${}^{V}F_{220}^{(0)}/{}^{A}F_{101}^{(0)}$ derived from CVC are also essentially confirmed by these explicit calculations.

In a model where the Coulomb interaction is considered to be switched off the contributions of the terms connected with form factor coefficients of rank 2 can therefore completely be neglected in the case of the $A = 20$ system. It should, however, be mentioned at this point that this fact is only valid in the $A = 20$ system, but in other systems like, for instance, the $A = 8$ system, the vector form factor coefficients of rank 2 contribute remarkably and have to be taken into account. As long as the Coulomb interaction is assumed to be absent the structure of the formulae relevant for the differences of the distributions of electrons emitted from oriented nuclei remains therefore the same as discussed before in the $A = 12$ system.†

However, it should be noted that in contrast to the $A = 12$ system the differences Δa and ΔA_2^0 are remarkably nuclear structure dependent. The reason for this fact will be immediately apparent if we go back to eqns (14.362) and (14.364). If we apply the simple independent single-particle model all valence nucleons are in the $d_{5/2}$ shell and one obtains (see eqn (14.133))

$$\frac{\int \mathbf{l}}{\int \boldsymbol{\sigma}} = l = 2. \tag{14.392}$$

Beyond that, a calculation in the complete $(sd)^4$ space yields a value which is not drastically different from this simple estimation, but depends on details of the residual interaction used by an amount of $\simeq \pm 10\%$ (see Calaprice *et al.* 1977; Behrens *et al.* 1978). From this number it is evident that the term $\int \mathbf{l} / \int \boldsymbol{\sigma}$ which reflects the influence of the nuclear structure cannot be neglected against 1 and contributes about 30% to the differences of the observables under discussion (and even more if Coulomb terms are additionally taken into account).

The assumption of a vanishing Coulomb interaction is for the $A = 20$ system more unrealistic than in the $A = 12$ system because of the larger Z ($|Z| = 10$). The switching on of the Coulomb interaction leads to the

† With exception of the differences $A_2 b(\beta^-) - A_2 b(\beta^+)$ and $a_{\beta\gamma} b_{\beta\gamma}(\beta^-) - a_{\beta\gamma} b_{\beta\gamma}(\beta^+)$. Here we have now

$$A_2^0 b(\beta^-) - A_2^0 b(\beta^+) = 4[a_{\beta\gamma} b_{\beta\gamma}(\beta^-) - a_{\beta\gamma} b_{\beta\gamma}(\beta^+)] = -\frac{4}{9}\sqrt{\frac{12}{35}} R^2 \frac{{}^{V}F_{220}^{(0)}}{{}^{A}F_{101}^{(0)}} \simeq -7 \times 10^{-6}.$$

same consequences as discussed for the $A = 12$ system, but in an even more amplified form. In the case of the difference of the two linear shape factor slopes Δa the inclusion of the terms (see eqn (14.127))

$$\Delta a = -2R\Big\{\ldots + \tfrac{2}{27}\{[W_0(\beta^-) - W_0(\beta^+)]R\}\frac{-5\,{}^{\mathrm{A}}F^{(0)}_{101} + \sqrt{2}\,{}^{\mathrm{A}}F^{(0)}_{121}}{{}^{\mathrm{A}}F^{(0)}_{101}} + \tfrac{2}{27}\alpha|Z| \times \frac{{}^{\mathrm{A}}F^{(1)}_{101}(1,1,1,1) - 2\sqrt{2}\,{}^{\mathrm{A}}F^{(0)}_{121}(1,1,1,1) + 9\,{}^{\mathrm{A}}F^{(1)}_{101}(1,2,2,1)}{{}^{\mathrm{A}}F^{(0)}_{101}}\Big\} \tag{14.393}$$

lowers the result for Δa by about 28% compared with the expression (14.354) which has been derived under the assumption of no Coulomb interaction (see Behrens *et al.* 1978). A model independent test of CVC by determining Δa from $\Gamma(M1)$ can therefore only be carried out in a very rough approximation in the $A = 20$ system.

The most important effects in the context of correlations are caused by the occurring of the following additional terms in the differences

$$\frac{F_0}{C_0}(\beta^-) - \frac{F_0}{C_0}(\beta^+) = \ldots + \tfrac{1}{9}\{[W_0(\beta^-) - W_0(\beta^+)]R\} \times \frac{2\sqrt{2}\,{}^{\mathrm{A}}F^{(1)}_{101} + {}^{\mathrm{A}}F^{(0)}_{121}}{{}^{\mathrm{A}}F^{(0)}_{101}} \tag{14.394}$$

$$\frac{H_0}{C_0}(\beta^-) - \frac{H_0}{C_0}(\beta^+) = \ldots - \tfrac{1}{3}\{[W_0(\beta^-) - W_0(\beta^+)]R\}\sqrt{\left(\frac{3}{5}\right)}\frac{{}^{\mathrm{A}}F^{(0)}_{221}}{{}^{\mathrm{A}}F^{(0)}_{101}} + \tfrac{2}{5}\alpha|Z|\sqrt{\left(\frac{3}{5}\right)}\frac{{}^{\mathrm{A}}F^{(0)}_{221}(2,1,1,1)}{{}^{\mathrm{A}}F^{(0)}_{101}} \tag{14.395}$$

where

$$[W_0(\beta^-) - W_0(\beta^+)]R \simeq 2[(M_n - M_p)R - \tfrac{6}{5}\alpha Z]. \tag{14.396}$$

Of special interest now is the situation for the difference of H_0/C_0. By inserting the corresponding values ($W_0R(\beta^-) = 0{\cdot}1123$, $W_0R(\beta^+) = 0{\cdot}2232$ (see Genz *et al.* 1976); ${}^{\mathrm{A}}F^{(0)}_{221}/{}^{\mathrm{A}}F^{(0)}_{101} = -1.8$ (see Behrens *et al.* 1978)) we get (without the term multiplied by αZ)

$$\frac{H_0}{C_0}(\beta^-) - \frac{H_0}{C_0}(\beta^+) \simeq -0{\cdot}05 \tag{14.397}$$

and including all terms (see Behrens *et al.* 1978)

$$\frac{H_0}{C_0}(\beta^-) - \frac{H_0}{C_0}(\beta^+) = -0{\cdot}076. \tag{14.398}$$

Thus now the influence of terms connected with form factor coefficients of rank 2 is of some importance and cannot be disregarded any more.

In Table 14.11 the experimental results for the transitions $^{20}\mathrm{F} \rightarrow {}^{20}\mathrm{Ne}$

TABLE 14.11 *Difference of the shape factors, ft-values and of the beta-anisotropy coefficients between the β^-- and β^+-decay of the $A = 20$ system*

Experiment:		Δa not measured
	^{20}F:	$a = (0{\cdot}40 \pm 0{\cdot}22)\%/m_e c^2$ [a]
		$a = (0{\cdot}20 \pm 0{\cdot}25)\%/m_e c^2$ [b]
$\Gamma_\gamma(M1)$:		$\Delta a = (1{\cdot}25 \pm 0{\cdot}04)\%/m_e c^2$ [c]
Theoretical calculation:		$\Delta a = (0{\cdot}81 \pm 0{\cdot}07)\%/m_e c^2$ [d]
	^{20}F:	$a = (0{\cdot}53 \pm 0{\cdot}04)\%/m_e c^2$ [d]
Experiment:		$\delta = 0{\cdot}025 \pm 0{\cdot}013$ [e]
Theory (isospin violation):		$\delta_{\text{bind}} = 0{\cdot}029 \pm 0{\cdot}025$ [f]
Experiment:		$\Delta a_{\beta\gamma} = (0{\cdot}114 \pm 0{\cdot}013)\%/m_e c^2$ [g]
Theoretical calculation: (assuming no second class currents exist)		$\Delta a_{\beta\gamma} = (0{\cdot}104 \pm 0{\cdot}04)\%/m_e c^2$ [h]
$\Gamma_\gamma(M1)$: (assuming no second class currents exist)		$\Delta a_{\beta\gamma} = (0{\cdot}156 \pm 0{\cdot}05)\%/m_e c^2$ [i]

(a) Genz *et al.* (1976).
(b) Calaprice and Alburger (1978).
(c) $\Gamma_\gamma(M1) = 4{\cdot}26 \pm 0{\cdot}23$ eV from Fifield *et al.* (1977). Here, the formula

$$\Delta a = 3{\cdot}907 \times 10^{-2} \sqrt{\left\{\frac{\Gamma_\gamma(M1)ft}{2E_\gamma^3 ft(0^+ - 0^+)}\right\}}$$

has been used ($\Gamma_\gamma(M1)$ in eV, E_γ in MeV, $2ft(0^+ - 0^+) = 6173$ s).
(d) Calaprice *et al.* (1977), Behrens *et al.* (1978). Here, the complete formulae given in eqn (14.127) including the terms multiplied by $W_0 R$ and αZ have been used. The uncertainty is caused by nuclear structure effects.
(e) Genz *et al.* (1976).
(f) Wilkinson (1974*b*).
(g) Dupois-Rolin *et al.* (1978) (see also Tribble and May 1978; Tribble *et al.* 1981).
(h) Behrens *et al.* (1978).
(i) $\Gamma_\gamma(M1) = 4{\cdot}26 \pm 0{\cdot}23$ eV from Fifield *et al.* (1977). Here, the formula

$$\Delta a_{\beta\gamma} = 4{\cdot}884 \times 10^{-3} \sqrt{\left\{\frac{\Gamma_\gamma(M1)ft}{2E_\gamma^3 ft(0^+ - 0^+)}\right\}}$$

has been used ($\Gamma_\gamma(M1)$ in eV, E_γ in MeV, $2ft(0^+ - 0^+) = 6173$ s).

and $^{20}\text{Na} \rightarrow {}^{20}\text{Ne}$ are compared with the corresponding theoretical ones. Since the addition of the correction terms to the simple eqns (14.354) and (14.381) decreases the values for Δa and $\Delta a_{\beta\gamma}$ by 28% and 50%, respectively, it is not surprising that the determination of Δa and $\Delta a_{\beta\gamma}$ from $\Gamma_\gamma(M1)$ gives no agreement with the results directly measured. By taking into consideration the percentage numbers just given, on the other hand, it becomes clear that no real discrepancy exists. We must, therefore, conclude that the model independent test by simply deriving Δa and $\Delta a_{\beta\gamma}$ from $\Gamma_\gamma(M1)$ on the basis of CVC cannot be applied any more.

Theoretical calculations of the correction terms connected with the form factor coefficient ratios ${}^{A}F^{(1)}_{101}/{}^{A}F^{(0)}_{101}$, ${}^{A}F^{(0)}_{121}/{}^{A}F^{(0)}_{101}$, ${}^{A}F^{(1)}_{101}(1,1,1,1)/{}^{A}F^{(0)}_{101}$, ${}^{A}F^{(0)}_{121}(1,1,1,1)/{}^{A}F^{(0)}_{101}$, ${}^{A}F^{(1)}_{101}(1,2,2,1)/{}^{A}F^{(0)}_{101}$, ${}^{A}F^{(0)}_{221}/{}^{A}F^{(0)}_{101}$, ${}^{A}F^{(0)}_{221}(2,1,1,1)/{}^{A}F^{(0)}_{101}$ have in every case to be carried out. By an inspection of the values listed in Table 14.11 we see also that the agreement between theory and experiment is very good if the CVC theory is assumed to be valid and second class currents are assumed to be absent.

As mentioned before we can also build up the sum $a(\beta^-)+a(\beta^+)$ and $a_{\beta\gamma}(\beta^-)+a_{\beta\gamma}(\beta^+)$, which gives, however, no information about CVC and second class currents, but can serve as a different test possibility of the quality of theoretical calculations.† As in the case of the $A=12$ system no definite conclusion can be drawn from δ. The theoretical shape factor of the β^--transition from ^{20}F (that from ^{20}Na has not been measured up to now) is in perfect agreement with the experimental one (see Fig. 14.8).

There are also several experiments in other nuclear systems. The $A=8$ system should be mentioned first in this context, which is very sensitive to the validity of CVC and to the possible existence of second class currents. In particular, the β–α correlation (see McKeown *et al.* 1980) and the *ft*-value difference δ (see Wilkinson and Alburger 1971) have been investigated. The $A=8$ system plays a unique role in a certain sense because the first excited state of ^{8}Be which is the final state of the β and γ decay is a broad 2^+ resonance decaying into two α-particles. Thus all form factors are a function of the corresponding excitation energy E_x (see Bowles and Garvey 1978; Tomoda and Kubodera 1978). This is an additional complication compared with the cases discussed before. A second system which has been considered for testing the validity of CVC and the possible existence of second class currents was the $A=6$ system (decay $^{6}\mathrm{He}\rightarrow{}^{6}\mathrm{Li}+e^-+\bar{\nu}_e$). Here, the electron–neutrino correlation which has been measured many years ago (see Johnson *et al.* 1963) has been discussed with respect to the above header (see Calaprice 1975; Kleppinger *et al.* 1977; Sato *et al.* 1977). As a third system the $A=19$ system has been investigated (decay $^{19}\mathrm{Ne}\rightarrow{}^{19}\mathrm{F}+e^++\nu_e$). That is, by the way, a $\Delta T=0$ mirror transition. In this case the asymmetry coefficient of

† In the $A=20$ system we have

$$[a_{\beta\gamma}(\beta^-)+a_{\beta\gamma}(\beta^+)]_{\mathrm{exp}}=-(0{\cdot}086\pm0{\cdot}013)\%/m_ec^2$$

(Dupois-Rolin *et al.* 1978)

$$[a_{\beta\gamma}(\beta^-)+a_{\beta\gamma}(\beta^+)]_{\mathrm{theory}}=-(0{\cdot}038\pm0{\cdot}09)\%/m_ec^2.$$

(Behrens *et al.* 1978).
Here, some inadequacies of the theoretical calculations (impulse approximation) are shown up which may be a consequence of exchange effects.

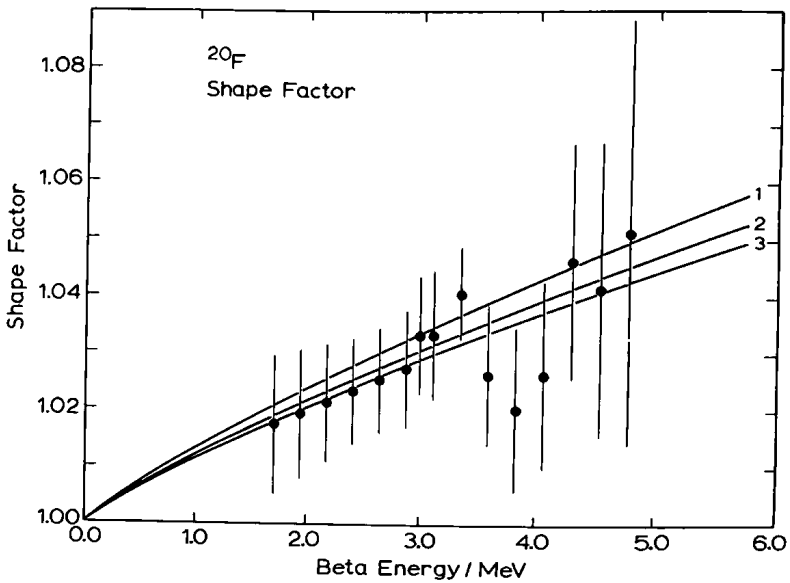


FIG. 14.8. Experimental shape factor for the decay of ^{20}F according to Genz *et al.* (1976). The labelled curves are theoretical calculations based on different residual interactions, namely (1) Kuo; (2) PW (2); (3) KB-F3 (taken from Behrens *et al.* 1978).

the electron distribution emitted from oriented nuclei has been investigated (see Calaprice *et al.* 1975; Kleppinger *et al.* 1978). Additionally, other δ-data have also been evaluated (see Wilkinson 1978).

Attention should also be drawn to the $A = 14$ system, i.e. to the processes $^{14}\mathrm{C}(^{14}\mathrm{O}) \rightarrow {}^{14}\mathrm{N} + e^-(e^+) + \bar{\nu}_e(\nu_e)$. Here, the spins, parities, and isospins involved are J^π, $T = 0^+$, $1 \rightarrow 1^+$, 0. These transitions are unusually inhibited (^{14}C: $\log ft = 9{\cdot}05$; ^{14}O: $\log ft = 7{\cdot}3$) such that large deviations from the allowed form of the observables occur (for the shape factor of ^{14}C see, for example, Sonntag *et al.* 1970). For that reason an analysis of these transitions is of special interest (see Goulard *et al.* 1977; Lorazo and Goulard 1980).

The appropriate existing experimental material of the systems mentioned above has been analysed by Oka and Kubodera (1980) as a whole. In their analysis off-mass shell and exchange effects have been included by making use of the KDR model (see Section 8.2.3). As a result they obtained satisfactorily model independently

$$|\zeta| \lesssim 0{\cdot}3 \times 10^{-3} \ (1\sigma^2 \text{ level}). \tag{14.399}$$

This implies for on-mass shell nucleons

$$|f_T| \lesssim 0{\cdot}3 \times 10^{-3} \, (1\sigma^2 \text{ level}) \tag{14.400}$$

(remember for comparison that $f_M = 1{\cdot}008 \times 10^{-3}$).

For the exchange second class strength the above authors quoted also

$$|\lambda L| \lesssim 0{\cdot}3 \times 10^{-3} \, (1\sigma^2 \text{ level}). \tag{14.401}$$

In conclusion we can therefore state that CVC is valid and no second class currents exist. It should, however, be remarked that the accuracy should certainly be improved. Namely, even if second class currents exist we would expect values for ζ or f_T in the range of f_M, i.e. in the range 10^{-4} to 10^{-3}.

A comprehensive compilation of the existing experimental data on the shape factors for allowed transitions can be found in a compilation by Behrens and Szybisz (1976). Other allowed transitions (with very few exceptions, see Müller 1977; Behrens *et al.* 1978) have, however, not been analysed with respect to CVC and SCC up to now (the $A = 24$ system might also be of interest).

Before closing this section we would like to mention some other newer publications which have treated the topic under consideration in a more review type of manner, i.e. the articles by Wu (1977), Behrens *et al.* (1978), Calaprice (1978), Wilkinson (1978), Kim and Primakoff (1979), and Telegdi (1979).

14.2.3.2. Isospin forbidden Fermi transitions and isospin impurity

In the case where the Coulomb interaction (and other charge dependent nucleon–nucleon interactions) would be absent, isospin would be a good quantum number. If the CVC theory is valid the form factor coefficient ${}^{V}F_{000}^{(0)}$, is, on the other hand, related to the Fermi matrix element M_F by (see eqns (10.40) to (10.43))

$${}^{V}F_{000}^{(0)} = {}^{V}\mathfrak{M}_{000}^{(0)} = M_F. \tag{14.402}$$

For a beta-transition between states which differ in isospin it then holds exactly (see eqn (8.140)).

$${}^{V}F_{000}^{(0)} = 0.$$

Conversely, a finite Fermi matrix element found experimentally is an indication that isospin is violated as a result of charge dependent interactions. As shown below the size of the Fermi matrix element is therefore directly related to the order of magnitude of isospin impurities (see, for example, Schopper 1966; Blin-Stoyle 1969; Blin-Stoyle 1973; and Raman *et al.* 1975). Let us now consider two different cases:

(i) the special case of an isospin triplet (see Fig. 14.9). The middle

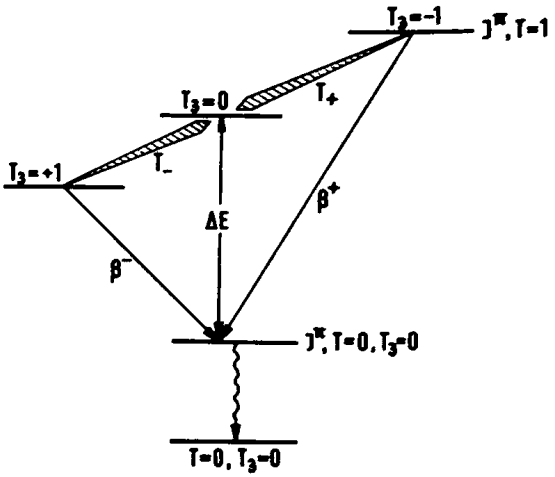


FIG. 14.9. Decay scheme for isospin-forbidden mirror transitions (Raman *et al.* 1975).

member of the isospin triplet $|J^\pi, 1, 0\rangle$ can be generated by applying either the ladder operator T_- to the $|J^\pi, 1, +1\rangle$ state or the ladder operator T_+ to the $|J^\pi, 1, -1\rangle$ state. On the other hand, the β^-- or β^+-transition under consideration proceeds to the $|J^\pi, 0, 0\rangle$ state, but, if charge dependent effects are present, this state (and other states also) does not have pure isospin. Neglecting terms of second order in the isospin admixture amplitudes we have

$$\begin{aligned} |i\rangle_{\beta^-} &= |J^\pi, 1, 1\rangle \qquad |i\rangle_{\beta^+} = |J^\pi, 1, -1\rangle \\ |f\rangle &= |J^\pi, 0, 0\rangle + \alpha_T |J^\pi, 1, 0\rangle \end{aligned} \tag{14.403}$$

whereby α_T denotes the amplitude by which the state $|J^\pi, 1, 0\rangle$ being the isobaric analogue state to the β-decay parent state is admixed into the β-decay daughter state $|J^\pi, 0, 0\rangle$. From eqn (8.140) we then obtain

$$M_F(\beta^-) = M_F(\beta^+) = \sqrt{2}\,\alpha_T; \tag{14.404}$$

(ii) the general situation which is characterized by the fact that $T_3 > 0$ (see Fig. 14.10). Then, the Fermi matrix element for either the β^-- or the β^+-transition is given by (see eqn (8.140))

$$M_F = \alpha_T \sqrt{\{(T + T_{3i})(T - T_{3i} + 1)\}} \tag{14.405}$$

where α_T is the admixture amplitude of the state A, which is the analogue state to the β-decay parent state P, into the β-decay daughter state T_ζ. In the β^--decay, the isospin impurity admixture

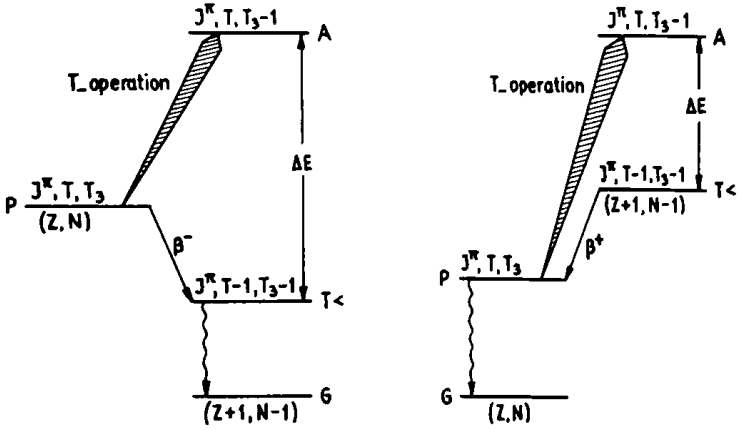


FIG. 14.10. Decay schemes for the general case ($T_3>0$) of isospin-forbidden transistions. β^--decay (left side) and β^+-decay (right side) (Raman *et al.* 1975).

α_T responsible for a finite M_F is that of the final state while in the β^+-decay the corresponding impurity α_T is that of the initial state. Note also that α_T is the admixture amplitude of one special state, namely the analogue state. Thus $|\alpha_T|^2$ should not be taken for the total isospin impurity of the state $T_<$ which is defined by the admixture of all $T \neq T_<$ states into the $T_<$ state.

Generally, for low lying states the third component T_3 takes its maximum value T. Then we have

$$M_F = \alpha_T \sqrt{(2T)}. \tag{14.406}$$

By introducing a charge dependent Hamiltonian V_{CD}, which contains the Coulomb interaction as well as other charge dependent nucleon–nucleon interactions, the admixture amplitude α_T can be calculated in first order perturbation theory

$$\alpha_T = -\frac{\langle T_< | V_{CD} | A \rangle}{\Delta E} = -\frac{\langle V_{CD} \rangle}{\Delta E} \tag{14.407}$$

where ΔE is the energy difference between the $T_<$ and A states given by

$$\Delta E = \Delta E_C + W_0(\beta^-) - (M_n - M_p) \tag{14.408a}$$

$$\Delta E = \Delta E_C - W_0(\beta^+) - (M_n - M_p) \tag{14.408b}$$

with ΔE_C equal to the Coulomb displacement energy (for ΔE_C see, for example, Jänecke 1969). Usually $\langle V_{CD} \rangle$ is called the effective Coulomb matrix element. In the past many attempts have been made to calculate

these effective Coulomb matrix elements (for summaries see Blin-Stoyle 1969; Bertsch and Mekjian 1972; Blin-Stoyle 1973). A comprehensive collection of all theoretical results can be found in the compilation by Raman *et al.* (1975) (for a more recent publication see Saw and Yap 1977). In the context of spherical shell model calculations we will, however, discuss one illuminating example first suggested by Damgaard (1967) for illustration. This is the β^--decay of the $J^\pi = 3/2^-$, $T = 9/2$ ground state of ^{49}Ca to the $J^\pi = 3/2^-$, $T = 7/2$ excited state of ^{49}Sc. Both the initial and final nucleus have one nucleon outside the double-closed shell of ^{48}Ca and are therefore easy to handle from a theoretical point of view. In the independent particle model the corresponding shell model configurations then read as (see, for example, Bertsch and Mekjian 1972)

$$P\colon |3/2^-, 9/2, 9/2\rangle = (\nu 2p_{3/2})^1(\nu 1f_{7/2})^8_{0^+} \tag{14.409a}$$

$$T_<\colon |3/2^-, 7/2, 7/2\rangle = \sqrt{\tfrac{8}{9}}(\pi 2p_{3/2})^1(\nu 1f_{7/2})^8_{0^+} - \sqrt{\tfrac{1}{9}}(\nu 2p_{3/2})^1 \times\{(\pi 1f_{7/2})^1(\nu 1f_{7/2})^7\}_{0^+} \tag{14.40bb}$$

$$A\colon |3/2^-, 9/2, 7/2\rangle = \sqrt{\tfrac{1}{9}}(\pi 2p_{3/2})^1(\nu 1f_{7/2})^8_{0^+} + \sqrt{\tfrac{8}{9}}(\nu 2p_{3/2})^1\{(\pi 1f_{7/2})^1(\nu 1f_{7/2})^7\}_{0^+}. \tag{14.409c}$$

Taking into account the Coulomb interaction only we then easily obtain

$$\langle V_{CD}\rangle = \langle T_<|\, V_{Coul}\,|A\rangle = \tfrac{1}{9}\sqrt{8}\,\{\langle 1f_{7/2}|\, V_{Coul}\,|1f_{7/2}\rangle - \langle 2p_{3/2}|\, V_{Coul}\,|2p_{3/2}\rangle\}. \tag{14.410}$$

Using the Coulomb potential of the uniform charge distribution

$$V_{Coul} = \frac{\alpha Z}{R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right] \tag{14.411}$$

one can calculate the Coulomb matrix elements for a proton in a $2p_{3/2}$ and $1f_{7/2}$ orbit. For proton wave functions of a Woods–Saxon potential one then gets† (see Bertsch and Mekjian 1972)

$$\langle 1f_{7/2}|\, V_{Coul}\,|1f_{7/2}\rangle - \langle 2p_{3/2}|\, V_{Coul}\,|2p_{3/2}\rangle = 121\ \text{keV}. \tag{14.412}$$

Since we have $\Delta E = 8{\cdot}472$ MeV we obtain finally

$$\langle V_{CD}\rangle = 38\ \text{keV} \tag{14.413a}$$

$$\alpha_T = -4{\cdot}5\times 10^{-3} \tag{14.413b}$$

$$|M_F| = 1{\cdot}5\times 10^{-2} \tag{14.413c}$$

† In the oscillator model we would obtain (see eqn (8.48*a*))

$$\langle \nu_1 l_1|\, V_{Coul}\,|\nu_1 l_1\rangle - \langle \nu_2 l_2|\, V_{Coul}\,|\nu_2 l_2\rangle = \frac{\alpha Z b^2}{2R^3}\{(2\nu_1 + l_1) - (2\nu_2 + l_2)\}.$$

For $1f_{7/2}$ ($\nu_1 = 1, l_1 = 3$) and $2p_{3/2}$ ($\nu_2 = 2, l_2 = 1$) we get zero.

(for comparison, the experimental result is $\langle V_{CD}\rangle = (1{\cdot}3^{+21}_{-1{\cdot}3}\,\text{keV}$, see Raman *et al.* 1975).

Because we have treated the $\Delta T = 1$ transition $^{20}\text{F} \rightarrow {}^{20}\text{Ne}$ $(2^+ \rightarrow 2^+)$ in detail before, we will also explicitly refer to the theoretical results for this transition for convenience of the reader. By making use of a sophisticated shell model calculation in the s–d shell, where the Coulomb interaction as well as other charge dependent effects have been included, Bertsch and Wildenthal (1973) obtained in this case

$$\langle V_{CD}\rangle = 16\,\text{keV} \tag{14.414a}$$

$$\alpha_T = -1{\cdot}8 \times 10^{-3} \tag{14.414b}$$

$$|M_F| = 2{\cdot}6 \times 10^{-3} \tag{14.414c}$$

(for comparison, the experimental result is $\langle V_{CD}\rangle = (14^{+29}_{-14})\,\text{keV}$, see Raman *et al.* (1975)). That means it was justified to neglect all terms connected with A_0 since $|A_0/C_0| \lesssim 10^{-2}$.

Experimentally, essentially three approaches exist to determine the size of isospin forbidden Fermi matrix elements:

(i) *ft*-values of isospin forbidden 0^+–0^+ transitions. In this case we have (see eqns (14.122) and (14.264))

$$|M_F| = \frac{\sqrt{K}}{G_\beta}\frac{1}{\sqrt{(ft)}} \tag{14.415}$$

or if $\sqrt{K}/G_\beta$ is expressed by the average *ft*-value of the superallowed 0^+–0^+ transitions (see eqn (14.264))

$$|M_F| = \sqrt{\left\{\frac{2ft(0^+\text{–}0^+)}{ft\ (\text{case under study})}\right\}} = \sqrt{\left\{\frac{6173}{ft\ (\text{case under study})}\right\}}. \tag{14.416}$$

If higher order terms become important we would, however, get instead

$$|A_0|\sqrt{\{\overline{C(W_e)}\}} = \sqrt{\left\{\frac{2ft\ (\text{superallowed})}{ft\ (\text{case under study})}\right\}} \tag{14.417}$$

where A_0 has to be taken from eqn (14.64) and $\overline{C(W_e)}$ from eqns (14.116) to (14.120). A model independent determination of M_F is then impossible.

(ii) Asymmetry coefficients of electrons emitted from oriented nuclei for $J^\pi \rightarrow J^\pi$ transitions. The asymmetry coefficient A_1^0 reads for

this case (see eqn (14.148))

$$A_0^1 = \mp \frac{1}{1+\left(\frac{A_0}{C_0}\right)^2} \Lambda_1 \left[\frac{1}{J+1} - 2\sqrt{\left(\frac{J}{J+1}\right)} \frac{A_0}{C_0}\right] \quad (14.418)$$

(for A_0 and C_0 see eqns (14.64) and (14.69)).

If higher order terms can be neglected, as is usually possible, this expression simplifies to

$$A_0^1 = \mp \frac{1}{1+y^2} \Lambda_1 \left[\frac{1}{J+1} + 2\sqrt{\left(\frac{J}{J+1}\right)} y\right] \quad (14.419)$$

with

$$y = \frac{{}^{V}F_{000}^{(0)}}{{}^{A}F_{101}^{(0)}} = \frac{M_F}{{}^{A}F_{101}^{(0)}} \quad (14.420)$$

or in impulse approximation with

$$y = \mp \frac{M_F}{\lambda M_{GT}}. \quad (14.421)$$

Thus, a measurement of A_0^1 allows us to determine y. By combining the above result for y with the corresponding expression for the ft-value we then find

$$|M_F| = \sqrt{\left\{\frac{2ft(0^+ - 0^+)}{ft \text{ (case under study)}}\right\}} \frac{|y|}{\sqrt{(1+y^2)}} \quad (14.422)$$

(the sign of M_F can only be determined relatively to M_{GT}).

(iii) Asymmetry coefficients of the β–γ (CP) correlation of $J^\pi \xrightarrow{\beta} J^\pi \xrightarrow{\gamma} J_{ff}$ transitions. The asymmetry coefficient $\tilde{A}_0$ for this case is written as (see eqn (14.181))

$$\tilde{A}_0 = \mp \frac{\sqrt{3}}{6} \frac{\Lambda_1 A_1(\gamma)}{1+\left(\frac{A_0}{C_0}\right)^2} \left[\frac{2}{\sqrt{\{J(J+1)\}}} + 4\frac{A_0}{C_0}\right] \quad (14.423)$$

or if higher order terms can be neglected

$$\tilde{A}_0 = \mp \frac{\sqrt{3}}{6} \frac{\Lambda_1 A_1(\gamma)}{1+y^2} \left[\frac{2}{\sqrt{\{J(J+1)\}}} - 4y\right]. \quad (14.424)$$

For $A_1(\gamma)$ see eqn (14.185). We have

$$A_1(\gamma) = \frac{F_1(LL\,J_{ff}\,J) + 2\delta F_1(LL+1\,J_{ff}\,J) + \delta^2 F_1(L+1\,L+1\,J_{ff}J)}{1+\delta^2} \quad (14.425)$$

with $\delta = \delta_{L+1}/\delta_L$ (upper sign β^--decay, lower β^+-decay).

In the case where the log ft value or the nuclear charge Z is large higher order corrections (so-called second forbidden terms) may have to be taken into account. This fact, however, manifests itself in a deviation of other observables from the allowed form, as for example, of the shape factor (see van Neste *et al.* 1966; Coussement and van Neste 1967). Thus, it can in principle be proved experimentally.

The necessary corrections themselves have, however, to be calculated by making use of specific nuclear models. Therefore, the incorporation of them depends on the special nucleus under consideration such that no attention is usually drawn to them. Above all, their influence is smaller than the experimental uncertainties in most cases.

A large number of experiments for various mixed allowed transitions have been carried out in the past (especially in the sixties) in order to determine the size of isospin forbidden Fermi matrix elements. A comprehensive critical compilation of all experimental data has been published by Raman *et al.* (1975).† The results of this compilation for α_T and $\langle V_{CD} \rangle$ are shown in Fig. 14.11. It was the aim of these investigations to explore details of the charge dependence of nuclear forces (see Henley 1969). It was attempted to achieve this goal by comparing experimental and theoretical values of M_F. The interactions chosen for these theoretical calculations were either the Coulomb interaction only or the Coulomb interaction plus an effective charge dependent internucleon potential. A comparison between calculated and experimental effective Coulomb matrix elements (see Raman *et al.* 1975) shows that the calculated values have the correct order of magnitude, but quantitative agreement exists for only few cases. The reason is that all calculations depend very sensitively on the nuclear wave functions used. The uncertainties are such that definite conclusions cannot be drawn about the charge dependence of nuclear forces. Other phenomena‡ such as nucleon–nucleon scattering (see, for example, Henley 1969; Sauer 1974; Wong *et al.* 1975), three-particle nuclear reactions (see, for example, Henley 1969; Kluge 1974; von Witsch *et al.* 1979) and Coulomb displacement energies (see, for example, Okamato and Pask 1971; Shlomo and Riska 1975; Sato 1976; Sherr 1977; Brown and Sherr 1979) are more suitable for an investigation of that problem. Results from these experiments show that charge symmetry holds within a few tenths of a percent and that charge independence is broken by a few percent.

† For some more recent results see Farman *et al.* (1976), Hung *et al.* (1976), Barr and Sapp (1977), Ingalls *et al.* (1977), Dickey (1978), Perlman *et al.* (1978), and Cohen *et al.* (1979).

‡ Departures from mirror symmetry in γ-transitions and Gamow–Teller β-decays can also be used as reasonably model independent but very insensitive tests for charge asymmetry effects of nuclear forces (see Chemtob 1974).

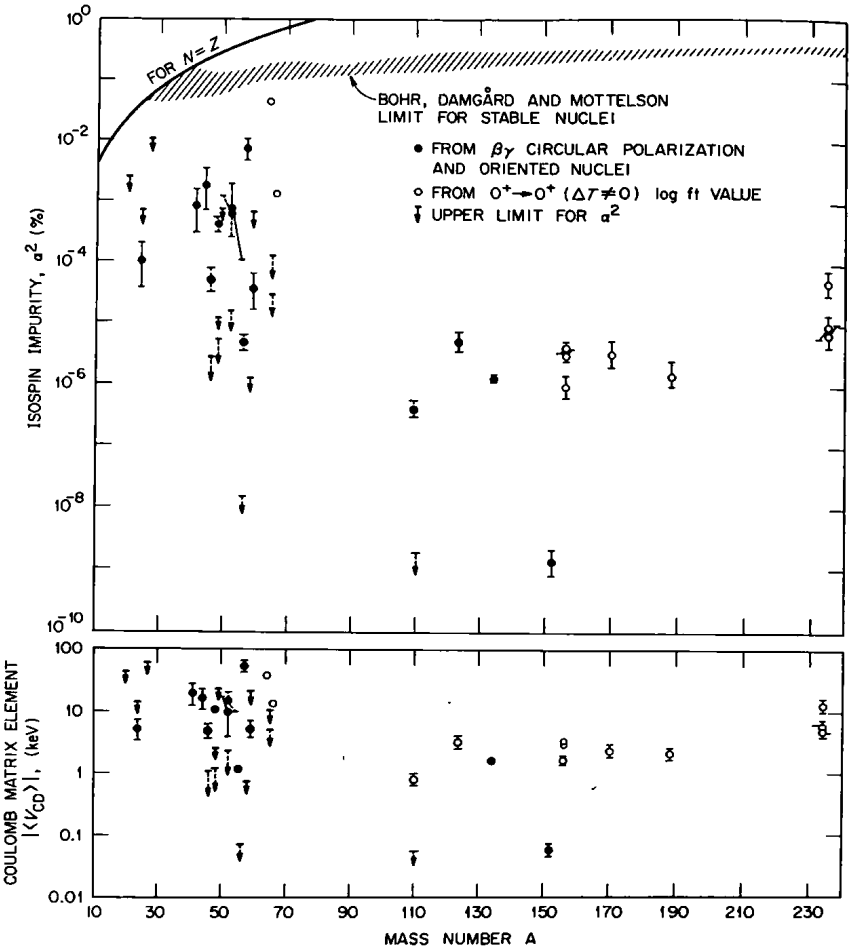


FIG. 14.11. Isospin impurities and effective Coulomb matrix elements deduced from beta-decay experiments (taken from Raman *et al.* 1975).

14.2.3.3. Gamow–Teller transitions and log ft values

In an independent particle model where the nucleons move within a pure oscillator potential we have the selection rule that Gamow–Teller transitions may proceed only between members of the same supermultiplet (see Section 8.1.1.4). Thus in such a model most of the real existing Gamow–Teller transitions are forbidden. Because of the residual interactions acting between the different nucleons the supermultiplet labels (P, P', P'') are, however, not good quantum numbers any more and

Gamow–Teller transitions can take place. Nevertheless, they are inhibited and the experimental data support this conclusion qualitatively. Namely, the experimental $\log ft$ values of the existing Gamow–Teller transitions are on average about 5·89 (see Fig. 14.1). That means the usual transitions are slower than the superallowed ones by about a factor of 100. Thus, it is very clear that they are sensitive to details of nuclear models if theoretical calculations are undertaken.

For usual allowed transitions we have (see eqns (14.35) and (14.122))

$$\log ft = 3{\cdot}790 - 2\log|{}^{A}F_{101}^{(0)}| \tag{14.426}$$

(note that the last term is positive in reality since $|{}^{A}F_{101}^{(0)}|<1$). Even for mixed transitions contributions of ${}^{V}F_{000}^{(0)}$ can be neglected since we are generally dealing with $\Delta T = 1$ transitions (see the discussion in the foregoing chapter).

In impulse approximation this corresponds to

$$\log ft = 3{\cdot}790 - 2\log|\lambda\, \mathfrak{M}_{101}^{(0)}|$$

$$= 3{\cdot}790 - 2\log\left|\lambda \int \boldsymbol{\sigma}\right|$$

where ${}^{A}\mathfrak{M}_{101}^{(0)} = \int \boldsymbol{\sigma}$ is the Gamow–Teller matrix element.

It should also be remarked that the ft-value is dependent on the absolute magnitude of the form factor coefficient ${}^{A}F_{101}^{(0)}$ or of the Gamow–Teller matrix element, respectively, while all the other observables depend on form factor coefficient ratios (or matrix element ratios) only.

In most cases, it is more difficult to describe the absolute magnitude of a matrix element correctly than some matrix element ratios so that ft-values are most sensitive to details of nuclear models. It is that reason why in many nuclear structure calculations theoretically calculated and experimentally† determined ft-values are compared. Without going into details some prominent examples are listed in the following:

p-shell:	Wilkinson (1973*b*)
	Richter and de Kock (1977)
	Desgrolard and Guichon (1979)
s–d-shell:	Wilkinson (1973*b*)
	Lanford and Wildenthal (1973)
	Timmer *et al.* (1978)
	Brown *et al.* (1978)
	Wilson *et al.* (1980)
deformed nuclei:	Gromov *et al.* (1976)
	Meijer (1976)
	Bogdan *et al.* (1979)
	Mikhajlov and Mikhajlova (1979)

† It should be noted that Gamow–Teller matrix elements can also be extracted from (p, n) reaction cross sections (see Goodman *et al.* 1980).

The results of the comparisons listed above can be summarized such that good qualitative accounting of the observed log ft-values is obtained for the shell model calculations (with a number of exceptions, of course). Evidently, the reproduction of relative log ft-value magnitudes within one nucleus is closer. Because of the reasons explained at the beginning of this section the agreement is, however, not as perfect as in the case of the superallowed transitions (where $\log ft \leqslant 3{\cdot}8$).

As far as transitions in deformed nuclei ($150 < A < 190$) are concerned, agreement between the theoretical and experimental ft-values cannot be obtained by using simple Nilsson wave functions (see Section 8.1.1.5.2). Pairing correlations and the corresponding blocking effects do play an important role. Further improvements can be achieved by making use of more refined models like, for instance, the asymmetric rotor model with a decoupled nucleon and a variable moment of inertia (see Bogdan *et al.* 1979), but as a whole the agreement cannot be considered as so good as in the case of the shell model calculations in the p- and s–d-shell.

Of course, for an exact comparison of theoretical and experimental transition strengths exchange effects have also to be taken into account (see Section 8.2 and 14.2.2.1).

14.2.4. Other aspects

In earlier years (1950 to 1970) many investigations of allowed transitions were directed to the confirmation of the general form of the weak interaction density (V–A interaction) and to the confirmation of the maximum violation of parity. The V–A form of the weak interaction is essentially based on electron–neutrino angular correlation experiments and on the electron asymmetry emitted from oriented neutrons.

The maximum non-conservation of parity has essentially been derived from measurements of the longitudinal electron polarization, of the asymmetry of electrons emitted from oriented nuclei in Gamow–Teller transitions,† of the β–γ (CP) correlation in Gamow–Teller transitions, of the asymmetry of neutrinos emitted from oriented nuclei and of the neutrino helicity. Detailed reviews about these topics can be found in various other books (see Schopper 1966; Wu and Moszkowski 1966; Blin-Stoyle 1973; Morita 1973). The present status of non-conservation of parity in nuclear beta-decay has been discussed in some newer publications (see Holstein and Treiman 1977; van Klinken *et al.* 1978).

Of course, some observables for Gamow–Teller transitions can also be applied for a determination of the gamma-transition mixing ratios, like, for example, the β–γ (CP) correlation (see Schopper 1966).

† For the last precision experiments for the β–γ (CP) correlation of ^{60}Co see Schopper *et al.* (1967) and for the asymmetry of electrons emitted from oriented ^{60}Co see Chirovsky *et al.* (1980).

As far as electron capture is concerned various capture ratios from different atomic shells (L_1/K, L_2/K, M_1/L_1 etc.) and the ratio of electron capture to β^+-decay have been experimentally determined. By making use of these results theoretical calculations of bound electron radial wave functions (see Sections 3.3 and 4.6) can be tested (see eqns (14.142) to (14.144)). Experimental and theoretical results for these ratios are compared and discussed in detail in a comprehensive review article published by Bambynek *et al.* (1977) whereby, in general, good agreement between theory and experiment has been found.

As demonstrated in many of the foregoing chapters, nuclear beta-decay and electron capture is strongly affected by the fact that the nucleus is charged. In the case of nuclear beta-decay the main influence of the Coulomb interaction is subsumed under the Fermi function (or exactly under F_0L_0) if the decay constant λ is considered. In the case of electron capture the transition probability is directly proportional to the electron density at the nucleus (square of the bound state atomic radial wave functions). On the other hand, the function F_0L_0 as well as the square of the bound state amplitude β_x^2 depend on the form and properties of the atomic cloud surrounding the nucleus. In the former case this cloud gives rise to a screening potential (see Sections 3.2.4 and 4.5.3) such that the function F_0L_0 is changed and therefore also the decay constant λ. In the latter case the electron density at the nucleus is strongly influenced by the presence of other atomic electrons (see Sections 3.3.5 and 4.6). Now, the form of the atomic electron distribution is not a unique one, but it depends on the chemical environment. Even other physical effects, like, for example, pressure, temperature, electric and magnetic fields etc. have an influence on the atomic cloud. Thus, we are not surprised that the decay constants λ of all beta-decay and electron capture processes are no constants in a strong sense, but a function of the chemical and physical environment. Generally, chemically induced half-life changes are, however, very small ($\Delta\lambda/\lambda \leqslant 10^{-3}$).

Excellent reviews and data compilations about that topic can be found in a number of articles (see Emery 1972; Hahn *et al.* 1976; Dostal *et al.* 1977; Daniel 1979).

In view of the fact that these effects are so small it is justified to neglect these effects in most cases. This is why we have not paid further attention to them in our text. We should, however, keep in mind that our beta-transitions always take place between atoms and not between nuclei.

14.3. First non-unique forbidden transitions

14.3.1. The quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$

General formulae for all kind of observables have been derived in Chapter 7. In that context the relevant expressions are expressed in terms

of the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$. As before, we shall later on give the electron (positron) energy dependence for the observables explicitly. Thus, we write the quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$ in terms of the quantities A_0, A_1, A_2, B_0, B_1, C_0, C_1, C_2, D_0, D_1, E_0, E_1, $\bar{E}_0$, F_0, F_1, $\bar{F}_0$, G_0, G_1, $\bar{G}_0$, H_0, H_1, $\bar{H}_0$, $\bar{M}_0$, N_0, X_0, Y_0 and in terms of p_eR, $p_\nu R$, W_eR, $(W_eR)^2$ identically as given before in eqns (14.48) to (14.63).

In the case of the first non-unique forbidden transitions the spin selection rule is the same as for the allowed transitions, i.e. (see eqn (14.30))

$$\Delta J = 0, 1$$

but the parity selection rule is different and reads now (see eqn (14.30))

$$\pi_i \pi_f = -1.$$

This latter fact leads to the consequence that we have to interchange:

Allowed transitions	First non-unique forbidden transitions
${}^{V}F^{(N)}_{KLs}(k_e, m, n, \rho)$	$\rightarrow {}^{A}F^{(N)}_{KLs}(k_e, m, n, \rho)$
${}^{A}F^{(N)}_{KLs}(k_e, m, n, \rho)$	$\rightarrow {}^{V}F^{(N)}_{KLs}(k_e, m, n, \rho)$

i.e. all vector form factor coefficients have to be replaced by the corresponding axial vector ones and vice versa. That means, however, that the non-relativistic type of form factor coefficients are replaced by relativistic ones (see eqns (14.5)) and inversely the relativistic ones by the non-relativistic ones. Although the general structure of the electron and neutrino energy independent linear combinations A_0, A_1, A_2, etc. of the form factor coefficients remains the same as before in the case of the allowed transitions, the order of magnitude of the different terms in these linear combinations is now completely changed compared with that of the allowed transitions. Thus, first we are confronted with the fact that the role of main and higher order contributions will now be played by other terms. Secondly, we have to take into account more additional terms which could be neglected in allowed transitions, but which are here of importance. From the general eqns (6.171) and (6.172) we then derive the following relations for first non-unique forbidden transitions (see also Behrens and Szybisz 1974; Behrens *et al.* 1975):

$$\begin{aligned} A_0 = {}&{}^{A}F^{(0)}_{000} - \tfrac{1}{3}W_0R\,{}^{A}F^{(0)}_{011} \mp \tfrac{1}{3}\alpha Z\,{}^{A}F^{(0)}_{011}(1,1,1,1) - \tfrac{1}{6}(W_0R)^2\,{}^{A}F^{(1)}_{000} \\ &\mp \tfrac{1}{9}\alpha Z[(W_0R)\,{}^{A}F^{(1)}_{000}(1,1,1,1) - \tfrac{1}{2}(W_0R)^2\,{}^{A}F^{(1)}_{011}(1,1,1,1)] \\ &- \tfrac{1}{6}(\alpha Z)^2[{}^{A}F^{(1)}_{000}(1,2,2,2) - \tfrac{1}{3}(W_0R)\,{}^{A}F^{(1)}_{011}(1,2,2,2)] \\ &\pm \tfrac{1}{30}(\alpha Z)^3\,{}^{A}F^{(1)}_{011}(1,3,3,3) \end{aligned} \qquad (14.427)$$

$$A_1 = \tfrac{2}{9}(W_0R)\,{}^{\wedge}F^{(1)}_{000} + \tfrac{1}{18}(W_0R)^2\,{}^{\wedge}F^{(1)}_{011}$$
$$\mp\tfrac{1}{3}\alpha Z\{{}^{\wedge}F^{(1)}_{000}(1,2,2,1) - \tfrac{1}{3}\,{}^{\wedge}F^{(1)}_{000}(1,1,1,1)$$
$$+\tfrac{1}{3}(W_0R)[{}^{\wedge}F^{(1)}_{011}(1,1,1,1) - {}^{\wedge}F^{(1)}_{011}(1,2,2,1)]\}$$
$$+\tfrac{1}{2}(\alpha Z)^2[\tfrac{1}{5}\,{}^{\wedge}F^{(1)}_{011}(1,3,3,2) - \tfrac{1}{9}\,{}^{\wedge}F^{(1)}_{011}(1,2,2,2)] \qquad (14.428)$$

$$A_2 = -\tfrac{2}{9}\,{}^{\wedge}F^{(1)}_{000} - \tfrac{1}{18}(W_0R)\,{}^{\wedge}F^{(1)}_{011}$$
$$\pm\tfrac{1}{2}(\alpha Z)[\tfrac{1}{5}\,{}^{\wedge}F^{(1)}_{011}(1,3,3,1) + \tfrac{1}{9}\,{}^{\wedge}F^{(1)}_{011}(1,1,1,1) - \tfrac{2}{9}\,{}^{\wedge}F^{(1)}_{011}(1,2,2,1)] \qquad (14.429)$$

$$A_3 = \tfrac{1}{30}\,{}^{\wedge}F^{(1)}_{011} \qquad (14.430)$$

$$B_0 = -\tfrac{1}{3}\,{}^{\wedge}F^{(0)}_{011} - \tfrac{1}{9}(W_0R)\,{}^{\wedge}F^{(1)}_{000} \pm \tfrac{1}{6}(\alpha Z)\,{}^{\wedge}F^{(1)}_{000}(1,2,1,1)$$
$$+\tfrac{1}{30}(\alpha Z)^2\,{}^{\wedge}F^{(1)}_{011}(1,3,2,2) \qquad (14.431)$$

$$B_1 = \tfrac{1}{9}\,{}^{\wedge}F^{(1)}_{000} \qquad (14.432)$$

$$C_0 = -{}^{\vee}F^{(0)}_{101} + \tfrac{1}{3}W_0R[-\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(0)}_{110} + \sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(0)}_{111}]$$
$$+\tfrac{1}{6}(W_0^2-1)R^2\,{}^{\vee}F^{(1)}_{101} + \tfrac{1}{6}W_0(\tfrac{1}{5}W_0^2 - \tfrac{1}{3})R^3[\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110} - \sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}]$$
$$\pm\tfrac{1}{3}\alpha Z\{-\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(0)}_{110}(1,1,1,1) - \sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(0)}_{111}(1,1,1,1)$$
$$+\tfrac{1}{9}W_0R[-{}^{\vee}F^{(1)}_{101}(1,1,1,1) + 2\sqrt{2}\,{}^{\vee}F^{(0)}_{121}(1,1,1,1)]$$
$$+\tfrac{1}{6}(W_0R)^2[\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,1,1,1) + \sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,1,1,1)]\}$$
$$+\tfrac{1}{6}(\alpha Z)^2\{{}^{\vee}F^{(1)}_{101}(1,2,2,2) + \tfrac{1}{3}W_0R[\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,2,2,2)$$
$$-\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,2,2,2)]\} \pm \tfrac{1}{30}(\alpha Z)^3\{\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,3,3,3)$$
$$+\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,3,3,3)\} \qquad (14.433)$$

$$C_1 = -\tfrac{2}{3}\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(0)}_{111} + \tfrac{2}{27}W_0R[-5\,{}^{\vee}F^{(1)}_{101} + \sqrt{2}\,{}^{\vee}F^{(0)}_{121}]$$
$$-\tfrac{1}{45}R^2[(2W_0^2-1)\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110} - (\tfrac{7}{2}W_0^2 - 4)\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}]$$
$$\pm\tfrac{1}{3}\alpha Z\{\tfrac{1}{9}\,{}^{\vee}F^{(1)}_{101}(1,1,1,1) - \tfrac{2}{9}\sqrt{2}\,{}^{\vee}F^{(0)}_{121}(1,1,1,1) + {}^{\vee}F^{(1)}_{101}(1,2,2,1)$$
$$+\tfrac{1}{3}W_0R[-\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,1,1,1) - \sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,1,1,1)$$
$$+\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,2,2,1) - \sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,2,2,1)]\}$$
$$+\tfrac{1}{2}(\alpha Z)^2[\tfrac{1}{5}\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,3,3,2) - \tfrac{1}{9}\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,2,2,2)$$
$$+\tfrac{1}{5}\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,3,3,2) + \tfrac{1}{9}\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,2,2,2)] \qquad (14.434)$$

$$C_2 = \tfrac{10}{27}\,{}^{\vee}F^{(1)}_{101} - \tfrac{2}{27}\sqrt{2}\,{}^{\vee}F^{(0)}_{121} + \tfrac{2}{15}W_0R[\tfrac{1}{3}\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110} - 2\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}]$$
$$\pm\alpha Z[\tfrac{1}{18}\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,1,1,1) + \tfrac{1}{18}\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,1,1,1)$$
$$+\tfrac{1}{10}\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,3,3,1) - \tfrac{1}{9}\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(1)}_{110}(1,2,2,1)$$
$$+\tfrac{1}{10}\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,3,3,1) + \tfrac{1}{9}\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111}(1,2,2,1)] \qquad (14.435)$$

$$C_3 = \tfrac{8}{45}\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(1)}_{111} \qquad (14.436)$$

$$D_0 = -\tfrac{1}{3}(1+\tfrac{1}{10}R^2)[\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(0)}_{110}+\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111}]$$
$$+\tfrac{1}{27}W_0R[-{}^{\mathrm{V}}F^{(1)}_{101}+2\sqrt{2}\,{}^{\mathrm{V}}F^{(0)}_{121}]+\tfrac{1}{6}\alpha Z\{{}^{\mathrm{V}}F^{(1)}_{101}(1,2,1,1)$$
$$+\tfrac{1}{3}W_0R[\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(1,2,1,1)-\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(1,2,1,1)]\}$$
$$+\tfrac{1}{30}(\alpha Z)^2[\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(1,3,2,2)+\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(1,3,2,2)] \qquad (14.437)$$

$$D_1 = \tfrac{1}{18}\{\tfrac{11}{3}\,{}^{\mathrm{V}}F^{(1)}_{101}-\tfrac{4}{3}\sqrt{2}\,{}^{\mathrm{V}}F^{(0)}_{121}+W_0R[\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}-\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}]\}$$
$$\pm\tfrac{1}{3}(\alpha Z)[\tfrac{1}{5}\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(1,3,2,1)-\tfrac{1}{6}\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(1,2,1,1)$$
$$+\tfrac{1}{5}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(1,3,2,1)+\tfrac{1}{6}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(1,2,1,1)] \qquad (14.438)$$

$$D_2 = \tfrac{1}{45}[-\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}+4\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}] \qquad (14.439)$$

$$E_0 = \sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(0)}_{110}+\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111}-\tfrac{1}{5}W_0R\,{}^{\mathrm{V}}F^{(0)}_{121}$$
$$-\tfrac{1}{2}(\tfrac{1}{5}W_0^2-\tfrac{1}{3})R^2[\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}+\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}]$$
$$\pm\tfrac{1}{3}\alpha Z\{-\tfrac{2}{3}\sqrt{2}\,{}^{\mathrm{V}}F^{(1)}_{101}(1,1,1,1)-\tfrac{1}{3}\,{}^{\mathrm{V}}F^{(0)}_{121}(1,1,1,1)$$
$$+\tfrac{1}{5}W_0R[-\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(1,1,1,1)+\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(1,1,1,1)]\}$$
$$+\tfrac{1}{6}(\alpha Z)^2[-\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(1,2,2,2)-\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(1,2,2,2)] \qquad (14.440)$$

$$E_1 = -\tfrac{2}{9}\sqrt{2}\,{}^{\mathrm{V}}F^{(1)}_{101}+\tfrac{4}{45}\,{}^{\mathrm{V}}F^{(0)}_{121}+\tfrac{2}{15}W_0R[\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}+2\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}]$$
$$\pm\tfrac{1}{3}\alpha Z[-\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(1,2,2,1)-\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(1,2,2,1)$$
$$+\tfrac{1}{5}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(1,1,1,1)-\tfrac{1}{5}\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(1,1,1,1)] \qquad (14.441)$$

$$E_2 = -\tfrac{1}{5}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}-\tfrac{1}{3}\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111} \qquad (14.442)$$

$$F_0 = \sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(0)}_{110}-\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111}-\tfrac{1}{9}W_0R[2\sqrt{2}\,{}^{\mathrm{V}}F^{(1)}_{101}+{}^{\mathrm{V}}F^{(0)}_{121}]$$
$$-\tfrac{1}{2}(\tfrac{1}{3}W_0^2-\tfrac{1}{5})R^2[\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}-\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}]$$
$$\mp\tfrac{1}{5}\alpha Z\{{}^{\mathrm{V}}F^{(0)}_{121}(2,1,1,1)+\tfrac{1}{3}W_0R[\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(2,1,1,1)$$
$$+\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(2,1,1,1)]\}+\tfrac{1}{10}(\alpha Z)^2[-\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(2,2,2,2)$$
$$+\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(2,2,2,2)] \qquad (14.443)$$

$$F_1 = \tfrac{2}{9}\sqrt{2}\,{}^{\mathrm{V}}F^{(1)}_{101}-\tfrac{4}{45}\,{}^{\mathrm{V}}F^{(0)}_{121}+\tfrac{2}{5}W_0R[\tfrac{2}{3}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}-\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}]$$
$$\pm\tfrac{1}{5}\alpha Z[\tfrac{1}{3}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(2,1,1,1)+\tfrac{1}{3}\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(2,1,1,1)$$
$$-\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}(2,2,2,1)+\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111}(2,2,2,1)] \qquad (14.444)$$

$$F_2 = -\tfrac{1}{5}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(1)}_{110}+\tfrac{1}{3}\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(1)}_{111} \qquad (14.445)$$

$$G_0 = -{}^{\mathrm{A}}F^{(0)}_{211}+\tfrac{1}{5}W_0R[-\sqrt{\tfrac{2}{5}}\,{}^{\mathrm{A}}F^{(0)}_{220}+\sqrt{\tfrac{3}{5}}\,{}^{\mathrm{V}}F^{(0)}_{221}]+\tfrac{1}{2}(\tfrac{1}{5}W_0^2-\tfrac{1}{3})R^2\,{}^{\mathrm{A}}F^{(1)}_{211}$$
$$\mp\tfrac{1}{3}\alpha Z\{\sqrt{\tfrac{2}{5}}\,{}^{\mathrm{A}}F^{(0)}_{220}(1,1,1,1)+\sqrt{\tfrac{3}{5}}\,{}^{\mathrm{V}}F^{(0)}_{221}(1,1,1,1)$$
$$+\tfrac{1}{25}W_0R[{}^{\mathrm{A}}F^{(1)}_{211}(1,1,1,1)-2\sqrt{6}\,{}^{\mathrm{A}}F^{(0)}_{231}(1,1,1,1)]\}$$
$$+\tfrac{1}{6}(\alpha Z)^2\,{}^{\mathrm{A}}F^{(1)}_{211}(1,2,2,2) \qquad (14.446)$$

$$G_1 = -\tfrac{2}{15}[\sqrt{\tfrac{2}{5}}\,{}^{\mathrm{A}}F_{220}^{(0)} + 4\sqrt{\tfrac{3}{5}}\,{}^{\mathrm{V}}F_{221}^{(0)}] + \tfrac{2}{75}W_0R[-8\,{}^{\mathrm{A}}F_{211}^{(1)} + \sqrt{6}\,{}^{\mathrm{A}}F_{231}^{(0)}]$$
$$\pm\tfrac{1}{3}\alpha Z\{{}^{\mathrm{A}}F_{211}^{(1)}(1,2,2,1)$$
$$+\tfrac{1}{25}[{}^{\mathrm{A}}F_{211}^{(1)}(1,1,1,1) - 2\sqrt{6}\,{}^{\mathrm{A}}F_{231}^{(0)}(1,1,1,1)]\} \qquad (14.447)$$

$$G_2 = \tfrac{1}{25}[7\,{}^{\mathrm{A}}F_{211}^{(1)} - \tfrac{2}{3}\sqrt{6}\,{}^{\mathrm{A}}F_{231}^{(0)}] \qquad (14.448)$$

$$H_0 = -{}^{\mathrm{A}}F_{211}^{(0)} + \tfrac{1}{3}W_0R[-\sqrt{\tfrac{2}{5}}\,{}^{\mathrm{A}}F_{220}^{(0)} + \sqrt{\tfrac{3}{5}}\,{}^{\mathrm{V}}F_{221}^{(0)}] + \tfrac{1}{2}(\tfrac{1}{3}W_0^2 - \tfrac{1}{5})R^2\,{}^{\mathrm{A}}F_{211}^{(1)}$$
$$\mp\tfrac{1}{5}\alpha Z\{\sqrt{\tfrac{2}{5}}\,{}^{\mathrm{A}}F_{220}^{(0)}(2,1,1,1) + \sqrt{\tfrac{3}{5}}\,{}^{\mathrm{V}}F_{221}^{(0)}(2,1,1,1)$$
$$+\tfrac{1}{15}W_0R[{}^{\mathrm{A}}F_{211}^{(1)}(2,1,1,1) - 2\sqrt{6}\,{}^{\mathrm{A}}F_{231}^{(0)}(2,1,1,1)]\}$$
$$+\tfrac{1}{10}(\alpha Z)^2\,{}^{\mathrm{A}}F_{211}^{(1)}(2,2,2,2) \qquad (14.449)$$

$$H_1 = \tfrac{2}{15}[\sqrt{\tfrac{2}{5}}\,{}^{\mathrm{A}}F_{220}^{(0)} - 4\sqrt{\tfrac{3}{5}}\,{}^{\mathrm{V}}F_{221}^{(0)}]$$
$$+\tfrac{2}{75}W_0R[-13\,{}^{\mathrm{A}}F_{211}^{(1)} + \sqrt{6}\,{}^{\mathrm{A}}F_{231}^{(0)}] \pm \tfrac{1}{5}\alpha Z\{{}^{\mathrm{A}}F_{211}^{(1)}(2,2,2,1)$$
$$+\tfrac{1}{15}[{}^{\mathrm{A}}F_{211}^{(1)}(2,1,1,1) - 2\sqrt{6}\,{}^{\mathrm{A}}F_{231}^{(0)}(2,1,1,1)]\} \qquad (14.450)$$

$$H_2 = \tfrac{1}{25}[7\,{}^{\mathrm{A}}F_{211}^{(1)} - \tfrac{2}{3}\sqrt{6}\,{}^{\mathrm{A}}F_{231}^{(0)}] \qquad (14.451)$$

(upper sign β^--decay, lower sign β^+-decay).

W_0 is, as before, the maximum beta-particle energy, Z the atomic number of the daughter nucleus and R the nuclear radius (equivalent charge radius).

The expressions given above look very complicated. In the overwhelming majority of first non-unique forbidden transitions it is, however, completely sufficient to take into account the dominant terms only. Then, the above formulae are remarkably simplified. They can be written, if these dominant terms only are included† (see eqns (14.10) and (14.11)), as

$$A_0 = {}^{\mathrm{A}}F_{000}^{(0)} \mp \tfrac{1}{3}\alpha Z\,{}^{\mathrm{A}}F_{011}^{(0)}(1,1,1,1) - \tfrac{1}{3}W_0R\,{}^{\mathrm{A}}F_{011}^{(0)} \qquad (14.452)$$

† For electron capture we have

$$A_0 = {}^{\mathrm{A}}F_{000}^{(0)} + \tfrac{1}{3}\alpha\bar{Z}\,{}^{\mathrm{A}}F_{011}^{(0)}(1,1,1,1) - \tfrac{1}{3}(W_0R)\,{}^{\mathrm{A}}F_{011}^{(0)}$$
$$B_0 = \tfrac{1}{3}\,{}^{\mathrm{A}}F_{011}^{(0)}$$
$$C_0 = -{}^{\mathrm{V}}F_{101}^{(0)} + \tfrac{1}{3}\alpha\bar{Z}\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F_{110}^{(0)}(1,1,1,1) - \tfrac{1}{3}(W_0R)\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F_{110}^{(0)}$$
$$-\tfrac{1}{3}\alpha\bar{Z}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F_{111}^{(0)}(1,1,1,1) - \tfrac{1}{3}(W_0R)\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F_{111}^{(0)}$$
$$C_1 = -\tfrac{2}{3}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F_{111}^{(0)}$$
$$D_0 = \tfrac{1}{3}[\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F_{110}^{(0)} - \sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F_{111}^{(0)}]$$
$$E_0 = -[\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F_{110}^{(0)} - \sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F_{111}^{(0)}]$$
$$F_0 = -[\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F_{110}^{(0)} + \sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F_{111}^{(0)}]$$
$$G_0 = -{}^{\mathrm{A}}F_{211}^{(0)}$$
$$H_0 = -{}^{\mathrm{A}}F_{211}^{(0)}.$$

Here, $\bar{Z}$ is the atomic number of the parent nucleus. The form factor coefficients are defined as for β^--decay but with the other changes required by eqns (8.254*a–f*).

$$B_0 = -\tfrac{1}{3}\,{}^{\mathrm{A}}F^{(0)}_{011} \tag{14.453}$$

$$C_0 = -{}^{\mathrm{V}}F^{(0)}_{101} \mp \tfrac{1}{3}\alpha Z\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(0)}_{110}(1,1,1,1) - \tfrac{1}{3}W_0R\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(0)}_{110} \mp \tfrac{1}{3}\alpha Z\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111}(1,1,1,1) + \tfrac{1}{3}W_0R\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111} \tag{14.454}$$

$$C_1 = -\tfrac{2}{3}\sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111} \tag{14.455}$$

$$D_0 = -\tfrac{1}{3}[\sqrt{\tfrac{1}{3}}\,{}^{\mathrm{V}}F^{(0)}_{110} + \sqrt{\tfrac{2}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111}] \tag{14.456}$$

$$E_0 = \sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(0)}_{110} + \sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111} \tag{14.457}$$

$$F_0 = \sqrt{\tfrac{2}{3}}\,{}^{\mathrm{V}}F^{(0)}_{110} - \sqrt{\tfrac{1}{3}}\,{}^{\mathrm{A}}F^{(0)}_{111} \tag{14.458}$$

$$G_0 = -{}^{\mathrm{A}}F^{(0)}_{211} \tag{14.459}$$

$$H_0 = -{}^{\mathrm{A}}F^{(0)}_{211}. \tag{14.460}$$

14.3.2. The form factor coefficients in terms of nuclear matrix elements

Every beta-decay observable can be expressed in terms of form factors or form factor coefficients, respectively, independently of details of nuclear structure and weak interaction theories. On the other hand, for a theoretical calculation of these observables a relation of the form factors or form factor coefficients, respectively, to nuclear matrix elements is required. The simplest approach in this context is the impulse approximation treatment discussed before in Section 8.1 (for exchange contributions see Section 8.2). Applying the general formulae, eqns (8.263*a–h*), to the special case of a first unique forbidden transition we obtain

$${}^{\mathrm{A}}F^{(0)}_{000} = \pm\lambda\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{000} \mp \frac{f_{\mathrm{P}}}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{\mathrm{D}}\mathfrak{N}^{(0)}_{000} \tag{14.461}$$

$${}^{\mathrm{A}}F^{(0)}_{011} = \mp\lambda\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{011} + \frac{f_{\mathrm{T}}}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{011} - 3\frac{f_{\mathrm{P}}}{R}\,{}^{\mathrm{D}}\mathfrak{N}^{(0)}_{000} \tag{14.462}$$

$${}^{\mathrm{A}}F^{(0)}_{011}(1,1,1,1) = \mp\lambda\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{011}(1,1,1,1) + \frac{f_{\mathrm{T}}}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{\mathrm{A}}\mathfrak{M}^{(0)}_{011}(1,1,1,1) \mp \frac{f_{\mathrm{P}}}{R}\int[3I(r) + rI'(r)]\beta\gamma_5 T_{000} \tag{14.463}$$

$${}^{\mathrm{V}}F^{(0)}_{101} = -{}^{\mathrm{V}}\mathfrak{M}^{(0)}_{101} - \frac{f_{\mathrm{M}}}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{\mathrm{C}}\mathfrak{N}^{(0)}_{101} \tag{14.464}$$

$${}^{\mathrm{V}}F^{(0)}_{110} = {}^{\mathrm{V}}\mathfrak{M}^{(0)}_{110} + \frac{f_{\mathrm{M}}}{R}\sqrt{3}\,{}^{\mathrm{C}}\mathfrak{N}^{(0)}_{101} \pm \frac{f_{\mathrm{S}}}{R}(W_0R \pm \tfrac{6}{5}\alpha Z)\,{}^{\mathrm{V}}\mathfrak{M}^{(0)}_{110} \tag{14.465}$$

$$
\begin{aligned}
{}^{V}F^{(0)}_{110}(1,1,1,1) = {}^{V}\mathfrak{M}^{(0)}_{110}(1,1,1,1) + \frac{f_M}{R}\Bigg\{\sqrt{\left(\frac{1}{3}\right)}\int [3I(r) + rI'(r)]\beta T_{101} \\
+ \sqrt{\left(\frac{2}{3}\right)}\int rI'(r)\beta T_{121}\Bigg\} \pm \frac{f_S}{R}(W_0 R \pm \tfrac{6}{5}\alpha Z) \\
\times {}^{V}\mathfrak{M}^{(0)}_{110}(1,1,1,1)
\end{aligned} \tag{14.466}
$$

$$
{}^{A}F^{(0)}_{111} = \pm\lambda\ {}^{A}\mathfrak{M}^{(0)}_{111} + \frac{f_T}{R}\{\sqrt{6}\ {}^{C}\mathfrak{N}^{(0)}_{101} - (W_0 R \pm \tfrac{6}{5}\alpha Z)\ {}^{A}\mathfrak{M}^{(0)}_{111}\} \tag{14.467}
$$

$$
\begin{aligned}
{}^{A}F^{(0)}_{111}(1,1,1,1) = \pm\lambda\ {}^{A}\mathfrak{M}^{(0)}_{111}(1,1,1,1) + \frac{f_T}{R}\Bigg\{\sqrt{\left(\frac{2}{3}\right)}\int [3I(r) + r'I(r)] \\
\times \beta T_{101} - \sqrt{\left(\frac{1}{3}\right)}\int rI'(r)\beta T_{121} - (W_0 R \pm \tfrac{6}{5}\alpha Z) \\
\times {}^{A}\mathfrak{M}^{(0)}_{111}(1,1,1,1)\Bigg\}
\end{aligned} \tag{14.468}
$$

$$
{}^{A}F^{(0)}_{211} = \mp\lambda\ {}^{A}\mathfrak{M}^{(0)}_{211} + \frac{f_T}{R}(W_0 R \pm \tfrac{6}{5}\alpha Z)\ {}^{A}\mathfrak{M}^{(0)}_{211} \tag{14.469}
$$

(upper sign β^--decay, lower sign β^+-decay).

Let us now consider the influence† of the induced terms, i.e. of those terms multiplied by f_M, f_S, f_T, and f_P. By inspection of the above expressions (14.461)–(14.469) we find that the influence of the induced pseudoscalar interaction, i.e. of terms connected with f_P, is of no relevance‡ since we have (see Table 14.12)

$$
{}^{D}\mathfrak{N}^{(0)}_{000} = 0 \tag{14.470}
$$

and since in eqn (14.463) the matrix element connected with f_P is of the relativistic type compared with the main matrix element ${}^{A}\mathfrak{M}^{(0)}_{011}(1,1,1,1)$ which is of the non-relativistic type. As far as the other induced terms are concerned we see that their influence on the form factor coefficients is of the following order of magnitude in all cases ($W_0^* = W_0 \pm \frac{6}{5}\alpha Z/R$):

$$
{}^{V}F^{(0)}_{KLs} \approx (-1)^{K-L}\ {}^{V}\mathfrak{M}^{(0)}_{KLs}[1 + O(f_M W_0^*) + O(f_S W_0^*)] \tag{14.471}
$$

$$
{}^{A}F^{(0)}_{KLs} \approx \pm(-1)^{K-L}\lambda\ {}^{A}\mathfrak{M}^{(0)}_{KLs}[1 + O(f_T W_0^*)] \tag{14.472}
$$

† For $\Delta j = 0$ transitions we have $|{}^{C}\mathfrak{N}^{(0)}_{101}| = c_0\,|{}^{V}\mathfrak{M}^{(0)}_{101}|$ where it is $c_0 = 2l_i$ or $c_0 = 2(l_i + 1)$, respectively (see Table 14.12). Because of this enhancement factor c_0, sometimes the terms within the vector form factor coefficients connected with f_M are not small compared with the usual transition matrix elements. Then, the influence of weak magnetism can often not be neglected.

‡ The $0^- \to 0^+$ transitions can be sensitive to the induced pseudoscalar interaction if strong cancellation effects exist in A_0 such that higher order terms are of importance. Then this type of induced interactions influences the observables, but especially via the higher order contributions (Bühring 1963*a*,*b*; Krmpotic and Tadic 1969; Eman *et al.* 1975).

TABLE 14.12 *Single-particle matrix elements (in j–j coupling) for first non-unique forbidden beta-transitions (j = spin and l = orbital angular momentum of the nucleon wave functions)*

$j_f = j_i \qquad j_f = l_f + \frac{1}{2} \qquad j_i = l_i - \frac{1}{2}$
$l_f = l_i - 1$

$$^{A}\mathfrak{M}^{(0)}_{000} = \frac{1}{\sqrt{(2J_i+1)}}\{2l_i\}^{1/2} R \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\frac{r}{R} g_i r^2\,dr$$

$$^{D}\mathfrak{N}^{(0)}_{000} = 0$$

$$^{A}\mathfrak{M}^{(0)}_{011} = -\frac{1}{\sqrt{(2J_i+1)}}\{2l_i\}^{1/2} \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{(0)}_{101} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{2l_i}{(2l_i-1)(2l_i+1)}\right\}^{1/2} R \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\frac{r}{R} g_i r^2\,dr$$

$$^{C}\mathfrak{N}^{(0)}_{101} = -2l_i\,{}^{V}\mathfrak{M}^{(0)}_{101}$$

$$^{A}\mathfrak{M}^{(0)}_{111} = -\frac{1}{\sqrt{(2J_i+1)}} 2l_i\left\{\frac{3l_i}{(2l_i-1)(2l_i+1)}\right\}^{1/2} \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{(0)}_{110} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{6l_i}{(2l_i-1)(2l_i+1)}\right\}^{1/2} \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2\,dr$$

$$^{A}\mathfrak{M}^{(0)}_{211} = -\frac{1}{\sqrt{(2J_i+1)}} 2\left\{\frac{(l_i+1)(l_i-1)l_i}{(2l_i-1)(2l_i+1)}\right\}^{1/2} \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2\,dr$$

$j_f = j_i \qquad j_f = l_f - \frac{1}{2} \qquad j_i = l_i + \frac{1}{2}$
$l_f = l_i + 1$

$$^{A}\mathfrak{M}^{(0)}_{000} = -\frac{1}{\sqrt{(2J_i+1)}}\{2(l_i+1)\}^{1/2} R \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\frac{r}{R} g_i r^2\,dr$$

$$^{D}\mathfrak{N}^{(0)}_{000} = 0$$

$$^{A}\mathfrak{M}^{(0)}_{011} = \frac{1}{\sqrt{(2J_i+1)}}\{2(l_i+1)\}^{1/2} \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{(0)}_{101} = -\frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{2(l_i+1)}{(2l_i+1)(2l_i+3)}\right\}^{1/2} R \int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\frac{r}{R} g_i r^2\,dr$$

$$^{C}\mathfrak{N}^{(0)}_{101} = 2(l_i+1)\,{}^{V}\mathfrak{M}^{(0)}_{101}$$

$$^{A}\mathfrak{M}^{(0)}_{111} = -\frac{1}{\sqrt{(2J_i+1)}} 2(l_i+1)\left\{\frac{3(l_i+1)}{(2l_i+1)(2l_i+3)}\right\}^{1/2} \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{(0)}_{110} = -\frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{6(l_i+1)}{(2l_i+1)(2l_i+3)}\right\}^{1/2} \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2\,dr$$

$$^{A}\mathfrak{M}^{(0)}_{211} = \frac{1}{\sqrt{(2J_i+1)}} 2\left\{\frac{l_i(l_i+1)(l_i+2)}{(2l_i+1)(2l_i+3)}\right\}^{1/2} \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2\,dr$$

TABLE 14.12 (*Continued*)

$j_i - j_f = 1 \qquad j_f = l_f - \frac{1}{2} \qquad j_i = l_i - \frac{1}{2}$
$\qquad\qquad\quad l_f = l_i - 1$

$^{A}\mathfrak{M}^{(0)}_{000} = 0$

$^{A}\mathfrak{M}^{(0)}_{011} = 0$

$^{D}\mathfrak{N}^{(0)}_{000} = 0$

$$^{V}\mathfrak{M}^{(0)}_{101} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{2l_i(l_i-1)}{2l_i-1}\right\}^{1/2} R\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\frac{r}{R}\,g_i r^2\,dr$$

$$^{C}\mathfrak{N}^{(0)}_{101} = -{}^{V}\mathfrak{M}^{(0)}_{101}$$

$$^{A}\mathfrak{M}^{(0)}_{111} = -\frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{3l_i(l_i-1)}{2l_i-1}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{(0)}_{110} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{6l_i(l_i-1)}{2l_i-1}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$$^{A}\mathfrak{M}^{(0)}_{211} = -\frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{3l_i(l_i-1)(2l_i-3)}{(2l_i-1)(2l_i+1)}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$j_i - j_f = -1 \qquad j_f = l_f - \frac{1}{2} \qquad j_i = l_i - \frac{1}{2}$
$\qquad\qquad\quad l_f = l_i + 1$

$^{A}\mathfrak{M}^{(0)}_{000} = 0$

$^{A}\mathfrak{M}^{(0)}_{011} = 0$

$^{D}\mathfrak{N}^{(0)}_{000} = 0$

$$^{V}\mathfrak{M}^{(0)}_{101} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{2l_i(l_i+1)}{2l_i+1}\right\}^{1/2} R\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\frac{r}{R}\,g_i r^2\,dr$$

$$^{C}\mathfrak{N}^{(0)}_{101} = {}^{V}\mathfrak{M}^{(0)}_{101}$$

$$^{A}\mathfrak{M}^{(0)}_{111} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{3l_i(l_i+1)}{2l_i+1}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{(0)}_{110} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{6l_i(l_i+1)}{2l_i+1}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$$^{A}\mathfrak{M}^{(0)}_{211} = -\frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{3l_i(l_i+1)(2l_i-1)}{(2l_i+1)(2l_i+3)}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$j_i - j_f = 1 \qquad j_f = l_f + \frac{1}{2} \qquad j_i = l_i + \frac{1}{2}$
$\qquad\qquad\quad l_f = l_i - 1$

$^{A}\mathfrak{M}^{(0)}_{000} = 0$

$^{A}\mathfrak{M}^{(0)}_{011} = 0$

$^{D}\mathfrak{N}^{(0)}_{000} = 0$

$$^{V}\mathfrak{M}^{(0)}_{101} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{2l_i(l_i+1)}{2l_i+1}\right\}^{1/2} R\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\frac{r}{R}\,g_i r^2\,dr$$

$$^{C}\mathfrak{N}^{(0)}_{101} = {}^{V}\mathfrak{M}^{(0)}_{101}$$

$$^{A}\mathfrak{M}^{(0)}_{111} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{3l_i(l_i+1)}{2l_i+1}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{(0)}_{110} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{6l_i(l_i+1)}{2l_i+1}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

TABLE 14.12 (*Continued*)

$j_i - j_f = 1 \qquad j_f = l_f + \tfrac{1}{2} \qquad j_i = l_i + \tfrac{1}{2}$
$l_f = l_i - 1$

$$^{A}\mathfrak{M}^{(0)}_{211} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{3l_i(l_i+1)(2l_i+3)}{(2l_i+1)(2l_i-1)}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$j_i - j_f = -1 \qquad j_f = l_f + \tfrac{1}{2} \qquad j_i = l_i + \tfrac{1}{2}$
$l_f = l_i + 1$

$$^{A}\mathfrak{M}^{(0)}_{000} = 0$$

$$^{A}\mathfrak{M}^{(0)}_{011} = 0$$

$$^{D}\mathfrak{M}^{(0)}_{000} = 0$$

$$^{V}\mathfrak{M}^{(0)}_{101} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{2(l_i+1)(l_i+2)}{2l_i+3}\right\}^{1/2} R\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\frac{r}{R}\,g_i r^2\,dr$$

$$^{C}\mathfrak{N}^{(0)}_{101} = -\,^{V}\mathfrak{M}^{(0)}_{101}$$

$$^{A}\mathfrak{M}^{(0)}_{111} = -\frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{3(l_i+1)(l_i+2)}{2l_i+3}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$$^{V}\mathfrak{M}^{(0)}_{110} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{6(l_i+1)(l_i+2)}{2l_i+3}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$$^{A}\mathfrak{M}^{(0)}_{211} = \frac{1}{\sqrt{(2J_i+1)}}\left\{\frac{3(l_i+1)(l_i+2)(2l_i+5)}{(2l_i+1)(2l_i+3)}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$j_i - j_f = 2 \qquad j_f = l_f - \tfrac{1}{2} \qquad j_i = l_i + \tfrac{1}{2}$
$l_f = l_i - 1$

$$^{A}\mathfrak{M}^{(0)}_{211} = \frac{1}{\sqrt{(2J_i+1)}}\,2\left\{\frac{6(l_i-1)l_i(l_i+1)}{(2l_i-1)(2l_i+1)}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr$$

$j_i - j_f = -2 \qquad j_f = l_f + \tfrac{1}{2} \qquad j_i = l_i - \tfrac{1}{2}$
$l_f = l_i + 1$

$$^{A}\mathfrak{M}^{(0)}_{211} = -\frac{1}{\sqrt{(2J_i+1)}}\,2\left\{\frac{6l_i(l_i+1)(l_i+2)}{(2l_i+1)(2l_i+3)}\right\}^{1/2}\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr.$$

In the case of the so-called non-relativistic nuclear matrix elements ($^{A}\mathfrak{M}^{(0)}_{011}$, $^{A}\mathfrak{M}^{(0)}_{111}$, $^{V}\mathfrak{M}^{(0)}_{110}$, $^{A}\mathfrak{M}^{(0)}_{211}$) the matrix elements

$$\mathfrak{M}^{(N)}_{KLs}(k_e, m, n, \rho)$$

can simply be calculated by replacing the radial integrals

$$\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,dr \to \int_0^\infty g_f\left(\frac{r}{R}\right)^{1+2N} I(k_e, m, n, \rho; r)\,g_i r^2\,dr.$$

In the expressions listed above g is defined as the solution of the Schrödinger radial equation (see eqn (8.27))

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2M_N[E - V(r)] - \frac{l(l+1)}{r^2}\right\}g(r) = 0.$$

E and $V(r)$ corresponds to the energy and potential in the initial and final bound states.

so that contrary to allowed transitions no enhancement of induced terms exists.

Since we have

$$f_M W_0^* \lesssim 4\times 10^{-2} \tag{14.473a}$$

$$f_T W_0^* \lesssim 4\times 10^{-2} \tag{14.473b}$$

$$f_S W_0^* = 0 \quad \text{(if CVC is valid)} \tag{14.473c}$$

the effect of the induced interactions on first non-unique forbidden transitions is very small ($<4\times 10^{-2}$) and can in most cases be completely neglected. In contrast to allowed transitions we will therefore not pay further attention to induced effects (CVC theory and second class currents) when we are considering the first non-unique forbidden transitions in more detail.

In Table 14.12 we have compiled the necessary single-particle matrix elements for the main contributions. They have been evaluated by making use of eqns (8.21*a–d*) and of eqns (8.30*a–b*). For the oscillator model the radial matrix elements can be taken from eqns (8.47*a–d*).

As discussed in Section 10.2 the application of the CVC theory offers us a relation between the form factor coefficients ${}^{V}F^{(0)}_{101}$ and ${}^{V}F^{(0)}_{110}$ which reads as (see eqn (10.67))

$${}^{V}F^{(0)}_{101} = -\frac{1}{\sqrt{3}}\{(W_0 \mp 2{\cdot}5)R \pm \tfrac{6}{5}\alpha Z\}\,{}^{V}F^{(0)}_{110}. \tag{14.474}$$

One should, however, be warned against applying simply the above CVC relation in theoretical calculations where use has been made of the impulse approximation treatment (see Hwang 1980). For example, in more refined shell model calculations for heavier nuclei, radial wave functions based on Woods–Saxon potentials with different well depths for neutrons and protons (symmetry potential) are generally used. Clearly, those potentials violate isospin symmetry so that in such a model the CVC relation in the above form is not valid. Then, the results for ${}^{V}F^{(0)}_{101}$ computed by applying the formulae of Table 14.12 disagree from those obtained by using the CVC relation. On the other hand, the other relativistic form factor coefficient ${}^{A}F^{(0)}_{000}$ has always to be evaluated by making use of the non-relativistic approximation treatment, i.e. by using the expressions in Table 14.12. In order to be consistent, both relativistic form factor coefficients should, however, be computed in the same way, i.e. by applying the formulae of Table 14.12. All the more, the CVC relation given above is generally not an exact one (see Fujita and Une 1976), especially not in impulse approximation (see Hwang 1980). In truth, it is itself dependent on nuclear structure model details.

14.3.3. Observables

General expressions for the various kinds of observables have been calculated and treated in Chapter 7. In this section special explicit formulae for first non-unique forbidden transitions derived on the basis of the relations given in the preceding sections and of the general formulae given in Chapter 7 are listed. By combining eqns (7.56), (14.48)–(14.63), and (14.452)–(14.460) we obtain for the spectrum shape factor

$$C(W_e) = k_n(1 + aW_e + \mu_1\gamma_1 b/W_e + cW_e^2) \tag{14.475}$$

with

$$k_n = A_0^2 + C_0^2 - 2\mu_1\gamma_1 R^2 C_1 D_0 + \tfrac{1}{9}(W_0 R)^2(E_0^2 + G_0^2) \tag{14.476}$$

$$a = R[2C_0C_1 - \tfrac{2}{9}(W_0R)(E_0^2 + G_0^2)]/k_n \tag{14.477}$$

$$b = -2R(A_0B_0 + C_0D_0)/k_n \tag{14.478}$$

$$c = R^2[C_1^2 + 2C_0C_2 + \tfrac{1}{9}(E_0^2 + \lambda_2F_0^2 + G_0^2 + \lambda_2H_0^2)]/k_n \tag{14.479}$$

$$\gamma_1 = \sqrt{\{1 - (\alpha Z)^2\}}.$$

Here, the dominant terms as given in eqns (14.452)–(14.460) have essentially been taken into account.† If higher order terms are additionally included the more general expressions, eqns (14.117)–(14.120), of a, b, and c in terms of the quantities A_0, A_1, A_2 etc. have then to be used. It should also be noted that the coefficients k_n, a, b, and c are no constants in the strict sense since they depend on the special Coulomb functions μ_1 and λ_2 which are β-energy dependent (see Behrens and Jänecke 1969). For β-energies of $W_e \geqslant 2$ (for low Z this limit is lower) the function μ_1 and λ_2 are, however, nearly β-energy independent. The coefficients k_n, a, b, and c can then be considered as constants, too.

The distribution of electrons emitted from oriented nuclei for transitions $J \rightarrow J$ and $J \rightarrow J \pm 1$ reads as (see eqn (7.38))

$$\omega(\theta_e, W_e) = 1 + a_1P_1(\cos\theta_e) + a_2P_2(\cos\theta_e) + a_3P_3(\cos\theta_e) \tag{14.480}$$

where θ_e is the angle between the orientation axis and the emitted beta-particle. By combining eqns (7.39), (14.48)–(14.63), (14.452)–(14.460) with the expressions for the particle parameters $b_{KK'}^{(k)}$ given in Table 7.2 we get for the coefficients a_1, a_2, and a_3

(i) asymmetry coefficient a_1

$$a_1 = \pm f_1(J)\frac{p_e}{W_e}\left\{\frac{3J}{J+1}\right\}^{1/2}[a_1^{(0)} + a_1^{(1)}(W_eR) + a_1^{(2)}(W_eR)^2] \tag{14.481a}$$

† For electron capture W_e has to be replaced by W_x which is the energy of the bound electron.

with†

$$a_1^{(n)} = [\tilde{\Gamma}_{01}(1)b_{01}^{(1)}(n) - \tilde{\Gamma}_{11}(1)b_{11}^{(1)}(n) + \tilde{\Gamma}_{12}(1)b_{12}^{(1)}(n) - \tilde{\Gamma}_{22}(1)b_{22}^{(1)}(n)]/C(W_e) \tag{14.481b}$$

where (see Table 7.2)

$$b_{01}^{(1)}(0) = 2\Lambda_1(A_0C_0 - R^2B_0D_0) + \hat{\eta}_{12}\frac{2\sqrt{2}}{3}R^2B_0F_0 \tag{14.482a}$$

$$b_{01}^{(1)}(1) = 2\Lambda_1A_0C_1 - \tfrac{2}{3}\sqrt{2}\,\eta_{12}A_0F_0 \tag{14.482b}$$

$$b_{01}^{(1)}(2) = 0 \tag{14.482c}$$

$$b_{11}^{(1)}(0) = \sqrt{2}\,\Lambda_1(C_0^2 - R^2D_0^2) - \tfrac{2}{3}\hat{\eta}_{12}R^2D_0F_0 - \frac{\sqrt{2}}{18}R^2(\Lambda_1W_0^2E_0^2 + \lambda_2\Lambda_2H_0^2) \tag{14.482d}$$

$$b_{11}^{(1)}(1) = 2\sqrt{2}\,\Lambda_1C_0C_1 + \tfrac{2}{3}\eta_{12}C_0F_0 + \frac{\sqrt{2}}{9}(W_0R)\Lambda_1E_0^2 \tag{14.482e}$$

$$b_{11}^{(1)}(2) = \sqrt{2}\,\Lambda_1C_1^2 + \tfrac{2}{3}\eta_{12}C_1F_0 - \frac{\sqrt{2}}{18}(\Lambda_1E_0^2 - \lambda_2\Lambda_2F_0^2) \tag{14.482f}$$

$$b_{12}^{(1)}(0) = -R^2\left[\frac{2\sqrt{5}}{3}\hat{\eta}_{12}D_0H_0 + \frac{\sqrt{10}}{9}\Lambda_1W_0^2E_0G_0 - \frac{\sqrt{10}}{45}\lambda_2\Lambda_2F_0H_0\right] \tag{14.482g}$$

$$b_{12}^{(1)}(1) = \frac{2\sqrt{5}}{3}\eta_{12}C_0H_0 + \frac{2\sqrt{10}}{9}(W_0R)\Lambda_1E_0G_0 \tag{14.482h}$$

$$b_{12}^{(1)}(2) = \frac{2\sqrt{5}}{3}\eta_{12}C_1H_0 - \frac{\sqrt{10}}{9}\Lambda_1E_0G_0 - \frac{\sqrt{10}}{45}\lambda_2\Lambda_2F_0H_0 \tag{14.482i}$$

$$b_{22}^{(1)}(0) = -\tfrac{1}{9}\sqrt{\tfrac{5}{2}}\,[\Lambda_1(W_0R)^2G_0^2 - \tfrac{3}{5}\lambda_2\Lambda_2R^2H_0^2] \tag{14.482j}$$

$$b_{22}^{(1)}(1) = \tfrac{2}{9}\sqrt{\tfrac{5}{2}}\,\Lambda_1(W_0R)G_0^2 \tag{14.482k}$$

$$b_{22}^{(1)}(2) = -\tfrac{1}{9}\sqrt{(\tfrac{5}{2})}[\Lambda_1G_0^2 + \tfrac{3}{5}\lambda_2\Lambda_2H_0^2]; \tag{14.482l}$$

(ii) anisotropy coefficient a_2

$$a_2 = f_2(J)J^2\left\{\frac{45}{J(J+1)(2J+3)(2J-1)}\right\}^{1/2} \times\left[a_2^{(0)}(\alpha Z)^2R^2 + a_2^{(1)}\frac{p_e^2}{W_e}R + a_2^{(2)}(p_eR)^2\right] \tag{14.483}$$

† The particle parameters $b_{KK'}^{(1)}$ of Table 7.2 have here been expressed as

$$b_{KK'}^{(1)} = \pm\frac{p_e}{W_e}[b_{KK'}^{(1)}(0) + b_{KK'}^{(1)}(1)(W_eR) + b_{KK'}^{(1)}(2)(W_eR)^2]L_0.$$

with†

$$a_2^{(n)} = [\tilde{\Gamma}_{02}(2)b_{02}^{(2)}(n) + \tilde{\Gamma}_{11}(2)b_{11}^{(2)}(n) - \tilde{\Gamma}_{12}(2)b_{12}^{(2)}(n) + \tilde{\Gamma}_{22}(2)b_{22}^{(2)}(n)]/C(W_e) \tag{14.484}$$

where

$$b_{02}^{(2)}(0) = -\tfrac{2}{3}\sqrt{2}\,\hat{\nu}_{12}B_0H_0 \tag{14.485a}$$

$$b_{02}^{(2)}(1) = -\tfrac{2}{3}\sqrt{2}\,\nu_{12}A_0H_0 \tag{14.485b}$$

$$b_{02}^{(2)}(2) = 0 \tag{14.485c}$$

$$b_{11}^{(2)}(0) = -\frac{2\sqrt{3}}{3}\,\hat{\nu}_{12}D_0F_0 \tag{14.485d}$$

$$b_{11}^{(2)}(1) = -\tfrac{2}{3}\sqrt{3}\,\nu_{12}C_0F_0 \tag{14.485e}$$

$$b_{11}^{(2)}(2) = -\tfrac{2}{3}\sqrt{3}\,\nu_{12}C_1F_0 + \tfrac{1}{9}\sqrt{\tfrac{3}{2}}\,\lambda_2F_0^2 \tag{14.485f}$$

$$b_{12}^{(2)}(0) = -\tfrac{2}{3}\sqrt{3}\,\hat{\nu}_{12}D_0H_0 \tag{14.485g}$$

$$b_{12}^{(2)}(1) = -\tfrac{2}{3}\sqrt{3}\,\nu_{12}C_0H_0 \tag{14.485h}$$

$$b_{12}^{(2)}(2) = -\tfrac{2}{3}\sqrt{3}\,\nu_{12}C_1H_0 - \frac{\sqrt{6}}{9}\,\lambda_2F_0H_0 \tag{14.485i}$$

$$b_{22}^{(2)}(0) = 0 \tag{14.485j}$$

$$b_{22}^{(2)}(1) = 0 \tag{14.485k}$$

$$b_{22}^{(2)}(2) = -\tfrac{1}{9}\sqrt{\tfrac{7}{2}}\,\lambda_2H_0^2; \tag{14.485l}$$

(iii) the asymmetry coefficient a_3

$$a_3 = \pm f_3(J)\,\frac{p_e}{W_e}\,(p_eR)^2J^3\left\{\frac{175}{J(J+1)(2J+3)(2J-1)(J-1)(J+2)}\right\}^{1/2}a_3^{(0)} \tag{14.486}$$

with

$$a_3^{(0)} = \frac{\sqrt{5}}{15}\,\lambda_2\Lambda_2[2\sqrt{3}\,\tilde{\Gamma}_{12}(3)F_0H_0 - \sqrt{2}\,\tilde{\Gamma}_{22}(3)H_0^2]/C(W_e). \tag{14.487}$$

The quantities $\tilde{\Gamma}_{KK'}(k)$ are the same as those in the corresponding allowed transitions. Thus, they can be taken from eqns (14.153)–(14.155) and (14.175)–(14.177) with the exception of $\tilde{\Gamma}_{12}(3)$ and $\tilde{\Gamma}_{22}(3)$. These latter ones have to be computed by making use of the general formula (7.40). Expressions for the orientation parameters $f_1(J)$ and $f_2(J)$ have also been derived before (see eqns

† The particle parameters $b_{KK'}^{(2)}$ of Table 7.2 have here been written as

$$b_{KK'}^{(2)} = \left[b_{KK'}^{(2)}(0)(\alpha Z)^2R^2 + b_{KK'}^{(2)}(1)\,\frac{p_e^2}{W_e}\,R + b_{KK'}^{(2)}(2)(p_eR)^2\right]L_0.$$

(14.151) and (14.174)). The orientation parameter $f_3(J)$ can be obtained by applying eqn (7.34)

$$f_3(J)=\frac{1}{J^3}\left[\sum_m m^3 a_m-\tfrac{1}{5}(3J^2+3J-1)\sum_m m a_m\right]. \quad (14.488)$$

Note that the coefficients $a_k^{(n)}$ depend on the electron (positron) energy for two reasons. Firstly, the special Coulomb functions η_{12}, $\hat{\eta}_{12}$, ν_{12}, $\hat{\nu}_{12}$, Λ_1, Λ_2, and λ_2 are functions of W_e. In the case $\alpha Z \ll 1$ or $W_e \gtrsim 2$ they can, however, be nearly considered as constants. Secondly, the spectrum shape factor $C(W_e)$ deviates more or less from the allowed form. Its electron (positron) energy dependence is, however, directly reflected in the corresponding behaviour of the asymmetry and anisotropy coefficients.

Expressions for the β–γ correlations can be derived from eqns (7.75) and (7.79). It is written as

$$\omega(W_e, \theta_{\beta\gamma})=1+\frac{v}{c}\tilde{A}P_1(\cos\theta_{\beta\gamma})+\varepsilon P_2(\cos\theta_{\beta\gamma}). \quad (14.489)$$

Here, $\theta_{\beta\gamma}$ is the angle between beta-particle and gamma-quantum. As mentioned before (see eqn (14.179)) the first term represents the β–γ (CP) correlation and the second the β–γ angular correlation. The corresponding asymmetry and anisotropy coefficients $\tilde{A}$ and ε, respectively, can be calculated by making use of eqns (7.75), (7.79), (14.482$a-l$), (14.85$a-l$). Then we get for:

(i) the asymmetry coefficient $\tilde{A}$ of the β–γ (CP) correlation

$$\tilde{A}=\mp\{A_1(\gamma)[\tilde{a}_1^{(0)}+\tilde{a}_1^{(1)}(W_eR)+\tilde{a}_1^{(2)}(W_eR)^2] + A_3(\gamma)(p_eR)^2[\tilde{a}_3^{(0)}(\tfrac{5}{2}\cos^2\theta_{\beta\gamma}-\tfrac{3}{2})]\}/[1+\varepsilon(\tfrac{3}{2}\cos^2\theta_{\beta\gamma}-\tfrac{1}{2})] \quad (14.490)$$

with

$$\tilde{a}_1^{(n)}=[\Gamma_{01}(1)b_{01}^{(1)}(n)+\Gamma_{11}(1)b_{11}^{(1)}(n)+\Gamma_{12}(1)b_{12}^{(1)}(n) + \Gamma_{22}(1)b_{22}^{(1)}(n)]/C(W_e) \quad (14.491)$$

$$\tilde{a}_3^{(0)}=\frac{\sqrt{5}}{15}\lambda_2\Lambda_2[2\sqrt{3}\,\Gamma_{12}(3)F_0H_0+\Gamma_{22}(3)\sqrt{2}\,H_0^2]/C(W_e); \quad (14.492)$$

(ii) anisotropy coefficient ε of the β–γ angular correlation

$$\varepsilon=A_2(\gamma)\left[\tilde{a}_2^{(0)}(\alpha Z)^2R^2+\tilde{a}_2^{(1)}\frac{p_e^2}{W_e}R+\tilde{a}_2^{(2)}(p_eR)^2\right] \quad (14.493)$$

with

$$\tilde{a}_2^{(n)} = [\Gamma_{02}(2)b_{02}^{(2)}(n) + \Gamma_{11}(2)b_{11}^{(2)}(n) + \Gamma_{12}(2)b_{12}^{(2)}(n) + \Gamma_{22}(2)b_{22}^{(2)}(n)]/C(W_e). \quad (14.494)$$

For β–α correlations $A_2(\gamma)$ has to be replaced by $A_2(\alpha)$. Of course, a β–α (CP) correlation does not exist.

The quantities $\Gamma_{KK'}(k)$ have also explicitly been derived before (see eqns (14.183)–(14.184) and eqns (14.191)–(14.192)). For $\Gamma_{12}(3)$ and $\Gamma_{22}(3)$ one has to go back to eqn (7.40), but with the requirement to interchange the spins J_i and J_f.

The $A_1(\gamma)$, $A_2(\gamma)$, and $A_3(\gamma)$ contain the information about the γ-transition and can be taken from eqn (7.80). For $A_2(\alpha)$, which replaces $A_2(\gamma)$ in the case of β–α correlations, see eqn (7.76).

The electron–neutrino angular correlation has been generally treated in Section 7.2.4.1. By making use of eqns (7.97) to (7.98) and of the relations for the particle parameters $d_K^{(k)}$ listed in Table 7.4 we are able to derive the following explicit expressions for the above correlation:

$$\omega(W_e, \theta_{e\nu}) = 1 + d_1 P_1(\cos\theta_{e\nu}) + d_2 P_2(\cos\theta_{e\nu}) \quad (14.495)$$

($\theta_{e\nu}$ is the angle between electron and neutrino).
The two correlation coefficients d_1 and d_2 read as

$$d_1 = \frac{p_e}{W_e}[d_1(0) + d_1(1)(W_eR) + d_1(2)(W_eR)^2] \quad (14.496)$$

with

$$d_1(0) = \{\Lambda_1(A_0^2 - R^2B_0^2) - \tfrac{1}{3}\Lambda_1(C_0^2 - R^2D_0^2) - \tfrac{4}{9}\sqrt{2}\,\Lambda_1(W_0R)C_0E_0 + \tfrac{4}{9}\sqrt{2}\,\hat{\eta}_{12}R^2D_0F_0 + \tfrac{1}{9}[\Lambda_1(W_0R)^2(\tfrac{1}{3}E_0^2 - \tfrac{1}{5}G_0^2) - \lambda_2\Lambda_2R^2(\tfrac{1}{3}F_0^2 - \tfrac{1}{5}H_0^2)]\}/C(W_e) \quad (14.497)$$

$$d_1(1) = \{-\tfrac{2}{3}\Lambda_1C_0C_1 + \tfrac{4}{9}\sqrt{2}\,\Lambda_1C_0E_0 - \tfrac{4}{9}\sqrt{2}\,\eta_{12}C_0F_0 - \tfrac{4}{9}\sqrt{2}\,\Lambda_1(W_0R)C_1E_0 + \tfrac{2}{27}\eta_{12}(W_0R)[E_0F_0 + 3G_0H_0] - \tfrac{2}{9}\Lambda_1(W_0R)(\tfrac{1}{3}E_0^2 - \tfrac{1}{5}G_0^2)\}/C(W_e) \quad (14.498)$$

$$d_1(2) = \{\tfrac{4}{9}\sqrt{2}\,\Lambda_1C_1E_0 - \tfrac{1}{3}\Lambda_1C_1^2 - \tfrac{2}{27}\eta_{12}[E_0F_0 + 3G_0H_0] + \tfrac{1}{9}[\Lambda_1(\tfrac{1}{3}E_0^2 - \tfrac{1}{5}G_0^2) + \lambda_2\Lambda_2(\tfrac{1}{3}F_0^2 - \tfrac{1}{5}H_0^2)]\}/C(W_e) \quad (14.499)$$

and

$$d_2 = \frac{2}{9}\frac{p_e^2}{W_e}(W_0 - W_e)\nu_{12}R^2(E_0F_0 - \tfrac{1}{5}G_0H_0)/C(W_e). \quad (14.500)$$

As before, the coefficients $d_1(0)$, $d_1(1)$, and $d_1(2)$ are constants only, if, on the one hand, the special Coulomb function Λ_1, Λ_2, λ_2, η_{12}, and $\hat{\eta}_{12}$ are

approximated by constant values. That, however, is only possible for higher energies ($W_e \geq 2$) or for low Z values (see Behrens and Jänecke 1969). On the other hand, the energy dependence of $C(W_e)$ should also be small such that it could be neglected.

The electron–neutrino triple correlation can be obtained by applying the formulae (7.112), (7.114), and (7.116). Then, it holds

$$\omega_J(\theta_e, \theta_\nu, W_e) = 1 + \tilde{d}\frac{\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)}{J p_e p_\nu} \tag{14.501}$$

with

$$\tilde{d} = \left\{\frac{2J}{(J+1)}\right\}^{1/2} f_1(J)\{sl_1[\tilde{d}(0) + \tilde{d}(1)(W_e R)] + se_{12}(p_e R) \times [\tilde{d}(2) + \tilde{d}(3)(W_e R)] + s\hat{e}_{12}(p_e R)\tilde{d}(4)\}/C(W_e) \tag{14.502}$$

where

$$\tilde{d}(0) = \sqrt{\tfrac{1}{3}}\, R\{-\tilde{\Gamma}_{01}(1)[3\sqrt{2}\,(C_0 B_0 - A_0 D_0) + (W_0 R) E_0 B_0] + \tfrac{3}{2}\sqrt{2}\,\tilde{\Gamma}_{11}(1)(W_0 R) E_0 D_0 - \sqrt{\tfrac{5}{2}}\tilde{\Gamma}_{12}(1)(W_0 R) G_0 D_0\} \tag{14.503}$$

$$\tilde{d}(1) = \sqrt{\tfrac{1}{3}}\, R\{\tilde{\Gamma}_{01}(1)[-3\sqrt{2}\, C_1 B_0 + E_0 B_0] - \tfrac{3}{2}\sqrt{2}\,\tilde{\Gamma}_{11}(1) E_0 D_0 + \sqrt{\tfrac{5}{2}}\,\tilde{\Gamma}_{12}(1) G_0 D_0\} \tag{14.504}$$

$$\tilde{d}(2) = \sqrt{\tfrac{1}{3}}\,[\tilde{\Gamma}_{01}(1) A_0 F_0 + \tfrac{3}{2}\sqrt{2}\,\tilde{\Gamma}_{11}(1) C_0 F_0 + \sqrt{\tfrac{5}{2}}\,\tilde{\Gamma}_{12}(1) C_0 H_0] \tag{14.505}$$

$$\tilde{d}(3) = \sqrt{\tfrac{1}{3}}\,[\tfrac{3}{2}\sqrt{2}\,\tilde{\Gamma}_{11}(1) C_1 F_0 + \sqrt{\tfrac{5}{2}}\,\tilde{\Gamma}_{12}(1) C_1 H_0] \tag{14.506}$$

$$\tilde{d}(4) = -\sqrt{\tfrac{1}{3}}\, R[\tilde{\Gamma}_{01}(1) F_0 B_0 + \tfrac{3}{2}\sqrt{2}\,\tilde{\Gamma}_{11}(1) F_0 D_0 + \sqrt{\tfrac{5}{2}}\,\tilde{\Gamma}_{12}(1) H_0 D_0]. \tag{14.507}$$

Contrary to the coefficients treated before the quantities $\tilde{d}(0)$, $\tilde{d}(1)$, $\tilde{d}(2)$, $\tilde{d}(3)$, and $\tilde{d}(4)$, which characterize the β-particle energy dependence, are real constants. It should also be noted that the special Coulomb functions sl_1, se_{12}, and $s\hat{e}_{12}$ (see eqns (4.48), (4.51), and (4.52)) depend strongly on the electron (positron) energy. For very low atomic numbers Z ($\alpha Z \ll 1$) they can be approximated to (see eqns (4.55), (4.58), and (4.59))

$$sl_1 \simeq \frac{\alpha Z}{W_e} \tag{14.508}$$

$$se_{12} \simeq \alpha Z\left(\frac{W_e}{p_e} - \frac{1}{4}\frac{p_e}{W_e}\right) \tag{14.509}$$

$$s\hat{e}_{12} \simeq \frac{\alpha Z}{p_e}. \tag{14.510}$$

The quantities $\tilde{\Gamma}_{KK'}(1)$ are explicitly listed in eqns (14.153)–(14.155) and should be taken from these. For $f_1(J)$ see eqn (14.151).

General expressions for the distribution of gamma radiation emitted from oriented nuclei without detecting the intermediate beta-decay have been derived in Section 7.2.5. By making use of eqns (7.54) and (7.132) we then obtain for the first non-unique forbidden transitions (if γ-polarization is not observed)

$$\omega_\gamma(\theta_\gamma) = 1 + \sum_{k=\text{even}} h_k P_k(\cos\theta_\gamma) \tag{14.511}$$

with

$$\begin{aligned} h_k = (-1)^{J_i+J_f}(2J_i+1)\sqrt{(2J_f+1)}\,A_k(\gamma)\binom{2k}{k}J_i^k\left[\frac{(2k+1)(2J_i-k)!}{(2J_i+k+1)!}\right]^{1/2} \\ \times f_k(J_i)\left[\begin{Bmatrix} J_i & J_i & k \\ J_f & J_f & 0 \end{Bmatrix}\overline{b_{00}} + \frac{1}{\sqrt{3}}\begin{Bmatrix} J_i & J_i & k \\ J_f & J_f & 1 \end{Bmatrix}\overline{b_{11}}\right. \\ \left. + \frac{1}{\sqrt{5}}\begin{Bmatrix} J_i & J_i & k \\ J_f & J_f & 2 \end{Bmatrix}\overline{b_{22}}\right] \Big/ L_0 C(W_e). \end{aligned} \tag{14.512}$$

The particle parameters $b_{KK}^{(0)}$ have to be averaged over the beta-spectrum (see eqn (7.123)) as denoted by the bar and read as

$$\overline{b_{00}^{(0)}}/L_0 = A_0^2 + R^2 B_0^2 - 2\gamma_1 R A_0 B_0 \overline{\left(\frac{\mu_1}{W_e}\right)} \tag{14.513}$$

$$\begin{aligned} \overline{b_{11}^{(0)}}/L_0 = -\sqrt{3}\left\{C_0^2 + R^2 D_0^2 - 2\bar{\mu}_1\gamma_1 R^2 C_1 D_0 + 2C_0 C_1 \overline{(W_e R)}\right. \\ \left. + C_1^2\overline{(W_e R)^2} - 2\gamma_1 R C_0 D_0 \overline{\left(\frac{\mu_1}{W_e}\right)} + \tfrac{1}{9}R^2[\overline{p_\nu^2}E_0^2 + \overline{(\lambda_2 p_e^2)}F_0^2]\right\} \end{aligned} \tag{14.514}$$

$$\overline{b_{22}^{(0)}}/L_0 = \frac{\sqrt{5}}{9} R^2\{\overline{p_\nu^2}G_0^2 + \overline{(\lambda_2 p_e^2)}H_0^2\}. \tag{14.515}$$

θ_γ is the angle between the gamma-quantum and the orientation axis of the initial nucleus. As mentioned earlier, the quantity $A_k(\gamma)$ is determined by the properties of the gamma-transition (see eqn (7.80)).†

The longitudinal polarization of the electron (positron) was discussed generally in Section 7.3. Specializing these results, as before (see eqn (7.151)), to first non-unique forbidden transitions we get (see eqns

† It should be mentioned that formulae for various recoil effects (for example angular correlation between the recoil nucleus and the following gamma-ray) of first forbidden transitions can be found in an article by Bouchiat (1958).

(14.48)–(14.63) and (14.452)–(14.460))

$$P_e = \mp \frac{p_e}{W_e} \Lambda_1 \{1 + g_0 + g_1/W_e + g_2(p_e R)^2\} \tag{14.516}$$

with

$$g_0 = 2R^2[-(B_0^2 + D_0^2) + \mu_1\gamma_1 C_1 D_0]/C(W_e) \tag{14.517}$$

$$g_1 = 2R\mu_1\gamma_1[A_0 B_0 + C_0 D_0]/C(W_e) \tag{14.518}$$

$$g_2 = \frac{1}{9}\left(\frac{\Lambda_2}{\Lambda_1} - 1\right)\lambda_2(F_0^2 + H_0^2)/C(W_e). \tag{14.519}$$

The neutrino shape factor $C(W_\nu)$ can directly be derived from eqn (7.86) and is obtained as (see also eqns (14.255) and (14.256))

$$C(W_\nu) = k_0[1 - (a + 2W_0 c)W_\nu + cW_\nu^2 + \mu_1\gamma_1 b/(W_0 - W_\nu)] \tag{14.520}$$

with

$$k_0 = A_0^2 + C_0^2 - 2\mu_1\gamma_1 R^2 C_1 D_0 + 2(W_0 R)C_0 C_1 + (W_0 R)^2 C_1^2 + \tfrac{1}{9}\lambda_2(W_0 R)^2(F_0^2 + H_0^2). \tag{14.521}$$

The quantities a, b, and c are identical with the corresponding coefficients of the beta-spectrum shape factor (see eqns (14.477), (14.478), and (14.479)).

Before concluding this section we should note that contrary to allowed transitions the anisotropy coefficients a_2, ε, d_2, and $\vec{d}$ are enhanced because usually $|F_0 D_0|/k_n \gg 1$, $|C_1 F_0|/k_n \gg 1$, $F_0^2/k_n \gg 1$, $|D_0 H_0|/k_n \gg 1$, $|C_1 H_0|/k_n \gg 1$, $|F_0 H_0|/k_n \gg 1$, $H_0^2/k_n \gg 1$, $|E_0 F_0|/k_n \gg 1$, $|G_0 H_0|/k_n \gg 1$ etc. and also $|A_0 H_0|/k_n > 1$, $|C_0 F_0|/k_n > 1$, $|C_0 B_0|/k_n > 1$ etc. Thus these distribution or correlation coefficients, respectively, are usually about two orders of magnitude larger than the corresponding ones of allowed transitions. It should be kept in mind that in the case of the first non-unique forbidden transitions the observables just mentioned have finite values even if only main contributions are taken into account. Large deviations of the shape factor $C(W_e)$ from the allowed form are now also possible (see, for example, the comprehensive compilation of experimental results by Behrens and Szybisz 1976). By the way, the dependence of all observables on the form factor coefficients can be characterized as a linear combination of products like $A_0 C_0$, $B_0 D_0$, $A_0 C_1$, $A_0 F_0$, $C_0 F_0$, $C_0 H_0$, etc. and squares like A_0^2, C_0^2, F_0^2, E_0^2, H_0^2 etc., which are always divided by the shape factor $C(W_e)$ (with exception of the shape factor itself, of course). Factors and signs connecting these different terms differ for the various kinds of observables. The quantities A_0, B_0, C_0, C_1, D_0 etc. are themselves linear combinations of the form factor coefficients and

constants for one special beta-transition. Thus the form of every special first non-unique forbidden beta-transition is essentially determined by these quantities and, of course, also by the spins of initial and final nuclear states.

For first non-unique forbidden transitions general formulae which are, however, expressed in terms of nuclear matrix elements (this formulation is nowadays a little obsolete) can also be found in the articles by Morita and Morita (1958), Kotani and Ross (1958), Lee-Whiting (1958), Kotani (1959), Weidenmüller (1961), Bühring (1963*a*,*b*), and in the books by Konopinski (1966) and by Morita (1973). Corresponding formulae in terms of form factors or form factor coefficients can, on the other hand, be also found in the books by Schopper (1966) and by Behrens and Jänecke (1969).

14.3.4. Examples

The expressions outlined in the foregoing section are lengthy and not very simple. In order to illustrate the general behaviour of first non-unique forbidden transitions a bit more we will therefore consider some special cases in detail.

Beta-transitions between nuclei with a double magic core plus or minus one or two nucleons outside offer the best possibilities for a simple

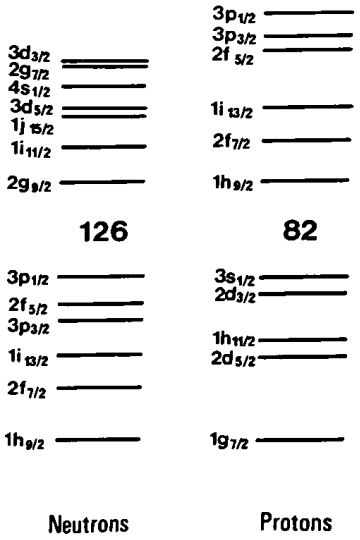


FIG. 14.12. Single particle levels in the lead region.

theoretical calculation of the corresponding observables. A particularly stable core is that of Pb^{208} with 82 protons and 126 neutrons. The single particle levels in the lead region are shown in Fig. 14.12. Thus let us first consider the $9/2^+ \rightarrow 9/2^-$ beta-transition of $^{209}Pb \rightarrow {}^{209}Bi$ and the $1/2^+ \rightarrow 1/2^-$ $(3/2^-)$ transition of $^{207}Tl \rightarrow {}^{207}Pb$ where we have one nucleon (hole) outside the double closed shell of ^{208}Pb (all experimental results belonging to these transitions and the participating nuclei have been compiled and evaluated by Schmorak (1977) and Martin (1977). Thus in the following use has always been made of these compilations).

In the case of the transition $^{209}Pb \rightarrow {}^{209}Bi$ we have a shell model transition $2g_{9/2} \rightarrow 1h_{9/2}$. Besides, it is a ground-state–ground-state β^--transition with a half-life of $t = 3{\cdot}253 \pm 0{\cdot}014$ h and a maximal electron energy $W_0 = 2{\cdot}261 \pm 0{\cdot}003$ (see Behrens *et al.* 1972). The decay scheme is shown in Fig. 14.13. By making use of eqns (14.461)–(14.469) and of the expression given in Table 14.12 we obtain for the form factor coefficients in impulse approximation (if induced terms are neglected)†

$$^{A}F^{(0)}_{000} = -\lambda\Omega_1 \tag{14.522a}$$

$$^{A}F^{(0)}_{011} = -\lambda\Omega_2 \tag{14.522b}$$

$$^{A}F^{(0)}_{011}(1,1,1,1) = -\lambda\Omega_3 \tag{14.522c}$$

$$^{V}F^{(0)}_{101} = \tfrac{1}{3}\sqrt{\tfrac{1}{11}}\,\Omega_1 \tag{14.522d}$$

$$^{A}F^{(0)}_{111} = -\lambda\sqrt{\tfrac{50}{33}}\,\Omega_2 \tag{14.522e}$$

$$^{A}F^{(0)}_{111}(1,1,1,1) = -\lambda\sqrt{\tfrac{50}{33}}\,\Omega_3 \tag{14.522f}$$

$$^{V}F^{(0)}_{110} = -\sqrt{\tfrac{1}{33}}\,\Omega_2 \tag{14.522g}$$

$$^{V}F^{(0)}_{110}(1,1,1,1) = -\sqrt{\tfrac{1}{33}}\,\Omega_3 \tag{14.522h}$$

$$^{A}F^{(0)}_{211} = -4\lambda\sqrt{\tfrac{1}{33}}\,\Omega_2 \tag{14.522i}$$

where Ω_1, Ω_2, and Ω_3 are the three different radial integrals which read as

$$\Omega_1 = R\int_0^\infty g_f\{E_i - E_f - [V_i(r) - V_f(r)]\}\left(\frac{r}{R}\right)g_i r^2\,dr \tag{14.523}$$

† As far as the vector form factor coefficients are concerned the neglection of the induced weak magnetism terms is really not entirely justified. In this special case we have, for example

$$^{V}F^{(0)}_{101} = \frac{1}{3}\sqrt{\left(\frac{1}{11}\right)}\Omega_1\left\{1 + \frac{10f_M}{R}(W_0R + \tfrac{5}{6}\alpha Z)\right\}$$

$$^{V}F^{(0)}_{110} = -\sqrt{\left(\frac{1}{33}\right)}\left\{\Omega_2 + \frac{10f_M}{R}\Omega_1\right\}.$$

If the transition is, however, considered as a whole the influence of these terms is of not so much importance. Thus, in order to keep the discussion of the first non-unique forbidden transitions more transparent we have omitted the induced terms in the following.

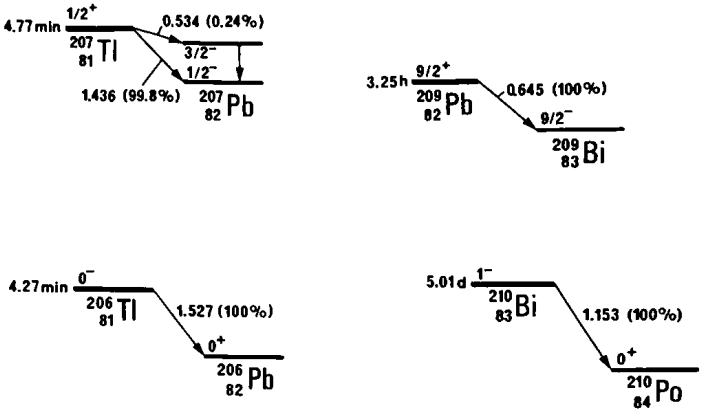


FIG. 14.13. Decay schemes of considered examples in the lead region.

$$\Omega_2 = \int_0^\infty g_f\left(\frac{r}{R}\right) g_i r^2 \, dr \tag{14.524}$$

$$\Omega_3 = \int_0^\infty g_f\left(\frac{r}{R}\right) I(1,1,1,1;r) g_i r^2 \, dr. \tag{14.525}$$

For the model of a uniform charge distribution we have (see Table 4.3)

$$I(1,1,1,1;r) = \begin{cases} \frac{3}{2} - \frac{3}{10}\left(\frac{r}{R}\right)^2 & \text{if } 0 \leq r \leq R \\ \frac{3}{2}\frac{R}{r} - \frac{3}{10}\left(\frac{R}{r}\right)^3 & \text{if } R \leq r. \end{cases} \tag{14.526}$$

The evaluation of these radial integrals has to be carried out by choosing suitable neutron and proton radial wave functions. At our disposal we have the simple harmonic oscillator (HO) or the more refined Woods–Saxon (WS) radial wave functions. For harmonic oscillator radial wave functions (see Section 8.1.1.2) the radial integrals Ω_1 and Ω_2 can be solved analytically. In all the other cases numerical integrations have to be performed. Then we obtain† (see eqn (8.47*c*)) for $\nu_i = 2$, $l_i = 4$ ($N_i = 6$) and $\nu_f = 1$, $l_f = 5$ ($N_f = 5$) the results shown in Table 14.13.

† For the harmonic oscillator potential we have (see eqn (8.36))

$$\{E_i - E_f - [V_i(r) - V_f(r)]\} = \omega = \frac{1}{M_N b^2}$$

and (see Section 8.1.1.2)

$$\frac{b}{R} = 2^{7/6} 3^{-1/6} 5^{-1/2} A^{-1/6}.$$

TABLE 14.13 *Radial integrals* Ω_1, Ω_2, *and* Ω_3 *by using different radial wave functions for the transition* $2g \rightarrow 1h$

	$\Omega_1 = -\frac{1}{M_N b} = -0{\cdot}0861$	(HO)
	$\Omega_1 = -0{\cdot}131$	(WS)†
and		
	$\Omega_2 = -\frac{b}{R} = -0{\cdot}343$	(HO)
	$\Omega_2 = -0{\cdot}137$	(WS)†
and		
	$\Omega_3 = -0{\cdot}358$	(HO)
	$\Omega_3 = -0{\cdot}103$	(WS)†

HO harmonic oscillator
WS Woods–Saxon
† Rost (1968). By using the Woods–Saxon potential of Blomqvist and Wahlborn (1960) we obtain nearly the same results.

The numerical values for the (WS) radial integrals listed above have been calculated by making use of the Woods–Saxon potentials suggested by Rost (1968), whereby the cases NO and PO have been taken. This author has determined the parameters for his Woods–Saxon potential

$$V(r) = V_C(r) - V_0 f(X_0) - V_{s.o.}\left(\frac{\hbar}{m_\pi c}\right)^2 \frac{1}{r}\frac{df}{dr}(X_{s.o.})\mathbf{l}\cdot\boldsymbol{\sigma} \qquad (14.527)$$

with

$$f(X_n) = \left[1 + \exp\left(\frac{r - R_n}{a_n}\right)\right]^{-1}$$

$$V_C(r) = \text{Coulomb potential}$$

by considering the single particle and hole energies in the lead region. Of course, further parametrizations of Woods–Saxon potentials exist in the literature and have also been applied for the transition under consideration (see Damgaard and Winther 1964; Damgaard *et al.* 1969; Behrens *et al.* 1972).

A glance at the different results for the radial integrals† $\langle 1h_{9/2}|\, \phi(r)\, |2g_{9/2}\rangle$ indicates that they react very sensitively on details of

† The validity of the $\Delta n = 0$ selection rule (n = number of radial modes) for the radial integral $\langle \pi 3s_{1/2}^{-1} \mid \nu 4s_{1/2}\rangle$ has been tested experimentally for the allowed $(1/2^+ \rightarrow 1/2^+)$ transition $^{209}\text{Pb} \rightarrow {}^{209}\text{Tl}$ (Datar *et al.* 1980). The near zero value of this radial integral implies that the formulae for the radial integrals remain valid despite the presence of the Coulomb interaction.

the radial wave functions used. The reason for that particular behaviour is that the number of radial nodes in the $2g_{9/2}$ and $1h_{9/2}$ wave functions differs. Consequently, in the radial integral there are two contributions with opposite sign and the result depends strongly on the position of the radial node point. In order to illustrate this latter point we can split the integral Ω_3 into two parts, i.e. the part inside the nucleus and the part outside. For harmonic oscillator wave functions we then get for $2g \rightarrow 1h$ (see Behrens and Bühring 1971)

$$\int_0^R g_f\left(\frac{r}{R}\right) I(1,1,1,1;r) g_i r^2 \, dr = +0{\cdot}022 \qquad (14.528a)$$

$$\int_R^\infty g_f\left(\frac{r}{R}\right) I(1,1,1,1;r) g_i r^2 \, dr = -0{\cdot}381. \qquad (14.528b)$$

Here, the main contribution comes from the region outside the nuclear radius R. This demonstrates very clearly the superiority of the electron radial wave function expansion shown in eqns (4.88)–(4.91) over methods used in earlier times (see also the discussion in Section 4.3.3). In addition, the sensitivity of the radial integral Ω_3 on details of the radial wave function poses the question of how sensitive this integral is to the form of the nuclear charge distribution.

The functions $I(1,1,1,1;r)$ differ for different charge distributions (see Fig. 4.1), but outside the nuclear radius this difference is small. Thus as demonstrated in Table 4.6 the influence of the nuclear charge distribution is small (~4%) and can be neglected for our further considerations.

For the quantities A_0, B_0, C_0 etc. we then obtain, if dominant terms only are considered (higher order corrections can approximately be neglected in this case)

$$A_0 = -\lambda[\Omega_1 - \tfrac{1}{3}(\alpha Z \Omega_3 + W_0 R \Omega_2)] \qquad (14.529a)$$

$$B_0 = \tfrac{1}{3}\lambda \Omega_2 \qquad (14.529b)$$

$$C_0 = \tfrac{1}{3}\sqrt{\tfrac{1}{11}}\,[-\Omega_1 + \tfrac{1}{3}\alpha Z \Omega_3(1+10\lambda) + \tfrac{1}{3}(W_0 R)\Omega_2(1-10\lambda)] \qquad (14.529c)$$

$$C_1 = \tfrac{20}{9}\sqrt{\tfrac{1}{11}}\,\lambda \Omega_2 \qquad (14.529d)$$

$$D_0 = \tfrac{1}{9}\sqrt{\tfrac{1}{11}}\,\Omega_2(1+10\lambda) \qquad (14.529e)$$

$$E_0 = -\frac{\sqrt{2}}{3}\sqrt{\frac{1}{11}}\,\Omega_2(1+5\lambda) \qquad (14.529f)$$

$$F_0 = -\frac{\sqrt{2}}{3}\sqrt{\frac{1}{11}}\,\Omega_2(1-5\lambda) \qquad (14.529g)$$

$$G_0 = 4\lambda\sqrt{\tfrac{1}{33}}\,\Omega_2 \qquad (14.529h)$$

$$H_0 = 4\lambda\sqrt{\tfrac{1}{33}}\,\Omega_2. \qquad (14.529i)$$

For the transition $^{209}Pb \rightarrow {}^{209}Bi$ we have

$$\begin{aligned} \alpha Z &= 0{\cdot}6057 \\ W_0 R &= 0{\cdot}0417 \\ R &= 0{\cdot}0184. \end{aligned} \tag{14.530}$$

Then, we get by choosing $\lambda = 1{\cdot}25$ the results shown in Table 14.14.

For the transition $^{209}Pb \rightarrow {}^{209}Bi$ two observables have been measured only, the shape factor and the *ft*-value (see Persson *et al.* 1971; Behrens *et al.* 1972). By inserting our results from Table 14.12 into eqns (14.476)–(14.479) we obtain the following theoretical results:

(i) shape factor

$$\text{HO: } C(W_e) = 7{\cdot}22 \times 10^{-3}[1 + 0{\cdot}118 W_e - 0{\cdot}046 \mu_1/W_e + 0{\cdot}005(1 + 0{\cdot}16\lambda_2) W_e^2] \tag{14.531}$$

$$\text{WS: } C(W_e = 1{\cdot}84 \times 10^{-2}[1 + 0{\cdot}0027 W_e + 0{\cdot}011 \mu_1/W_e + 0{\cdot}00031(1 + 0{\cdot}16\lambda_2) W_e^2]; \tag{14.532}$$

TABLE 14.14 *The quantities A_0, B_0, C_0, D_0, E_0, F_0, G_0, and H_0 calculated by using different nuclear radial wave functions for the β-transition $^{209}Pb \rightarrow {}^{209}Bi$*

$A_0 =$	0·011	(HO)
	0·135	(WS)†
$B_0 =$	−0·143	(HO)
	−0·057	(WS)†
$C_0 =$	−0·084	(HO)
	−0·013	(WS)†
$C_1 =$	−0·287	(HO)
	−0·115	(WS)†
$D_0 =$	−0·155	(HO)
	−0·062	(WS)†
$E_0 =$	0·353	(HO)
	0·141	(WS)†
$F_0 =$	−0·256	(HO)
	−0·102	(WS)†
$G_0 = H_0 =$	−0·299	(HO)
	−0·119	(WS)†

HO harmonic oscillator
WS Woods–Saxon
† Rost (1968). By using the Woods–Saxon potential of Blomqvist and Wahlborn (1960) we obtain nearly the same results.

(ii) log ft-values

$$\text{HO: } \log ft = 5{\cdot}87 \tag{14.533}$$

$$\text{WS: } \log ft = 5{\cdot}53. \tag{14.534}$$

The experimental result for the shape factor (Behrens *et al.* 1972) is shown in Fig. 14.14. The slope of this experimental shape factor has been determined as

$$a = -0{\cdot}005 \pm 0{\cdot}014$$

and

$$\log ft = 5{\cdot}622 \pm 0{\cdot}005.$$

A comparison of both the theoretical and experimental results shows that only the use of Woods–Saxon radial wave functions lead to a remarkable consistency. The use of harmonic oscillator wave functions is, on the other hand, in this case not suggested.

The six parameters of the Woods–Saxon potential are usually determined by fitting different experimental data like, for example, the energy levels for particles and holes for the vicinity of ^{208}Pb, reaction data, Coulomb energies and the charge distributions. Since the sets of parameters obtained in this way are not unique various different ones exist in the literature (see, for example, Blomqvist and Wahlborn 1960; Zaidi and Darmodjo 1967; Rost 1968; Kahana *et al.* 1969; Batty and Greenlees 1969; Batty 1970). For details of the procedure the reader is referred to the original papers mentioned above. Because of the sensitivity of the radial integrals $2g \rightarrow 1h$ to the details of the single-particle potential it is

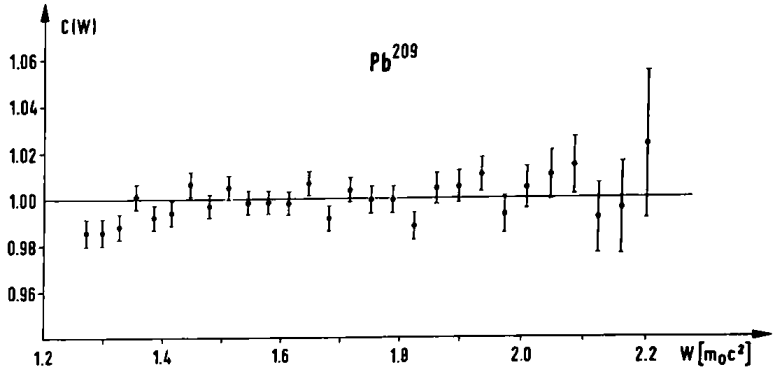


FIG. 14.14. Experimental shape factor for the decay of ^{209}Pb (taken from Behrens *et al.* 1972).

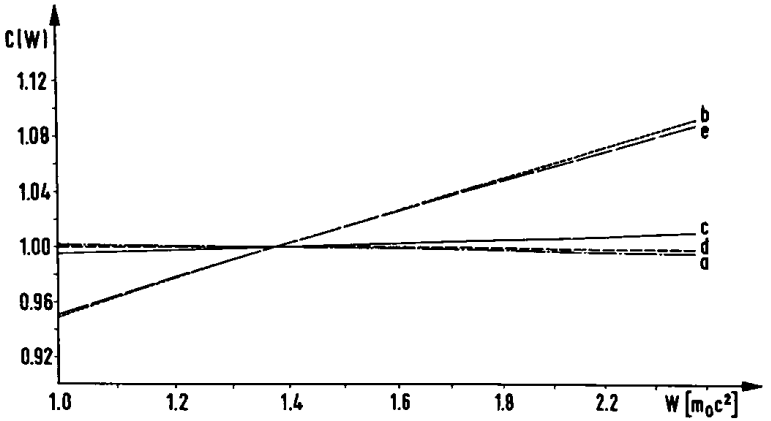


FIG. 14.15. Theoretical shape factor of the ^{209}Pb beta-spectrum. The labelled curves have been calculated by using different Woods–Saxon potentials. The labels correspond to (a) Blomqvist and Wahlborn (1960), Rost (1968) (NO, PO); (b) Zaidi and Darmodjo (1967) (neutron potential), Batty and Greenlees (1969) (proton potential); (c) Rost (1968) (NOP, POP); (d) Rost (1968) (NOT, POT), Batty and Greenlees (1969); (e) Kahana *et al.* (1969) (taken from Behrens *et al.* 1972).

not surprising that we even get different results for Woods–Saxon potentials of different authors. For the shape factor this is shown in Fig. 14.15 and for the log *ft*-value in Table 14.15. It is obvious that the shape factors and log *ft*-values calculated by using some of the above mentioned Woods–Saxon potentials disagree with the experimental data as was the case for the harmonic oscillator potential.

TABLE 14.15 *Theoretical log ft-values of the* ^{209}Pb*-decay calculated by using different Woods–Saxon potentials*† *(from Behrens et al. 1972)*

log *ft*	Woods–Saxon potential of:
5·55	Blomqvist and Wahlborn (1960)
5·87	neutrons: Zaidi and Darmodjo (1967) protons: Batty and Greenlees (1969)
5·56	Rost (1968) (NO, PO)
5·17	Rost (1968) (NOT, POT)
5·80	Rost (1968) (NOP, POP)
5·79	Kahana *et al.* (1969)
5·39	Batty and Greenlees (1969)
5·622±0·005	Experiment

† A small difference between the value for Rost (NO, PO) listed here and that mentioned before comes from the fact that higher order terms are here included.

For the case of agreement between theory and experiment our analysis shows that

$$A_0 \gg C_0 \tag{14.535}$$

$$A_0 \simeq C_1 \simeq F_0 \simeq E_0 \simeq H_0 \simeq G_0. \tag{14.536}$$

That means we then have

$$|M_0(1,1)| \gg \begin{cases} |M_1(1,1)| \\ |M_1(1,2)| \\ |M_1(2,1)| \\ |M_2(1,2)| \\ |M_2(2,1)| \end{cases}. \tag{14.537}$$

Since in these cases A_0 is essentially determined by ${}^{A}F^{(0)}_{000}$ shape factor and ft-value are governed by this form factor coefficient too. Thus, the form of the shape factor is that of the allowed transitions (for a similar analysis of the transition $^{209}\mathrm{Pb} \rightarrow {}^{209}\mathrm{Bi}$ see also Damgaard and Winther 1964; Bohr and Mottelson 1969†). It should also be kept in mind that the reason for $A_0 \gg C_0$ is that we have a near cancellation of the form factor coefficients in C_0.

As before we will additionally consider the $1/2^+ \rightarrow 1/2^-$ transitions $^{207}\mathrm{Tl} \rightarrow {}^{207}\mathrm{Pb}$ where we have single-hole configurations for the states involved ($W_0 = 3{\cdot}810 \pm 0{\cdot}011$, $t_{1/2} = (4{\cdot}77 \pm 0{\cdot}02)$ min). In the case of the ground-state–ground-state transition we are dealing with a transition of the type $3s^{-1}_{1/2} \rightarrow 3p^{-1}_{1/2}$. Now, single particle matrix elements between hole states are related to those between particle states by a phase factor $(-1)^{j_i + j_f + K}$ where K is the rank of the tensor operator, i.e. we have

$$\langle 3p^{-1}_{1/2} || T_{KLs} || 3s^{-1}_{1/2} \rangle = (-1)^{K+1} \langle 3s_{1/2} || T_{KLs} || 3p_{1/2} \rangle \tag{14.538}$$

(see, for example, Bohr and Mottelson 1969).

By making use of the expressions listed in Table 14.12 and of eqns (14.461)–(14.469) we then obtain in impulse approximation

$${}^{A}F^{(0)}_{000} = -\lambda \Omega_1 = -0{\cdot}223 \tag{14.539a}$$

$${}^{A}F^{(0)}_{011} = -\lambda \Omega_2 = -0{\cdot}681 \tag{14.539b}$$

$${}^{A}F^{(0)}_{011}(1,1,1,1) = -\lambda \Omega_3 = -0{\cdot}820 \tag{14.539c}$$

$${}^{V}F^{(0)}_{101} = -\frac{1}{\sqrt{3}} \Omega_1 = -0{\cdot}103 \tag{14.539d}$$

$${}^{A}F^{(0)}_{111} = -\lambda \sqrt{2}\, \Omega_2 = -0{\cdot}963 \tag{14.539e}$$

† It should be noted that both analyses differ, however, remarkably.

$$^{A}F_{111}^{(0)}(1,1,1,1) = -\lambda\sqrt{2}\,\Omega_3 = -1{\cdot}160 \qquad (14.539f)$$

$$^{V}F_{110}^{(0)} = \Omega_2 = 0{\cdot}545 \qquad (14.539g)$$

$$^{V}F_{110}^{(0)}(1,1,1,1) = \Omega_3 = 0{\cdot}656 \qquad (14.539h)$$

$$^{A}F_{211}^{(0)} = 0. \qquad (14.539i)$$

As before the nuclear radial wave functions calculated for the Woods–Saxon potential (Rost 1968) have been used. It should, however, be noted that in this case the radial integrals are not very sensitive to the form of the nuclear potential contrary to the case $2g_{9/2} \rightarrow 1h_{9/2}$ considered before.†

By inserting these results into eqns (14.475)–(14.479) we get the theoretical log *ft*-value and shape factor. A comparison with the experimental results is shown in the following for the *ft*-values:

$$\log ft_{\text{calc}} = 5{\cdot}19 \qquad (14.540)$$

$$\log ft_{\text{exp}} = 5{\cdot}22 \qquad (14.541)$$

and for the shape factor in Fig. 14.16. The corresponding experiment for the shape factor has been made by Trischuk and Kankeleit (1967). Compared with the transition $^{209}\text{Pb} \rightarrow {}^{209}\text{Bi}$ we have now the reversed situation that

$$A_0 = -0{\cdot}043 \qquad (14.542a)$$

$$C_0 = 0{\cdot}192 \qquad (14.542b)$$

i.e. that the rank 0 form factor coefficients cancel approximately. Thus, this transition is governed by the rank 1 form factor coefficients.

In both cases, the transitions $^{209}\text{Pb} \rightarrow {}^{209}\text{Bi}$ and $^{207}\text{Tl} \rightarrow {}^{207}\text{Pb}$, the asymmetry of electrons emitted from oriented nuclei is especially sensitive to the relative contributions of the rank 0 and rank 1 form factor coefficients (see eqns (14.480)–(14.482)). A measurement of this quantity would therefore provide a very good additional test possibility for the general behaviour‡ of these transitions in that context.

† We have

$$\Omega_1 = \begin{cases} 0{\cdot}1611 & \text{(HO)} \\ 0{\cdot}1786 & \text{(WS)} \end{cases}$$

$$\Omega_2 = \begin{cases} 0{\cdot}642 & \text{(HO)} \\ 0{\cdot}545 & \text{(WS)} \end{cases}$$

$$\Omega_3 = \begin{cases} 0{\cdot}774 & \text{(HO)} \\ 0{\cdot}656 & \text{(WS)} \end{cases}$$

HO = harmonic oscillator

WS = Woods–Saxon (Rost 1968).

‡ This observable depends essentially on the ratio A_0/C_0. Thus, in the case of ^{209}Pb it might also be sensitive to weak magnetism contributions.

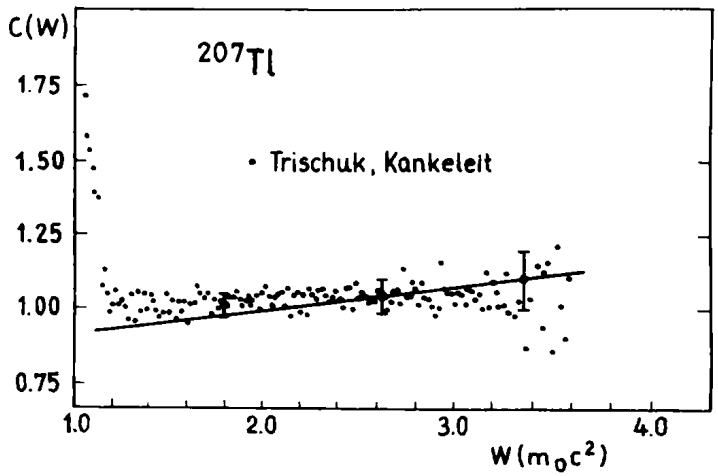


FIG. 14.16. Experimental shape factor for the decay of ^{207}Tl. The data have been taken from Trischuk and Kankeleit (1967). The curve corresponds to the theoretical calculation as described in the text.

The relativistic form factor coefficients or matrix elements, respectively, ${}^{A}F^{(0)}_{000}$ and ${}^{V}F^{(0)}_{101}$, for the transitions under consideration have been evaluated in the usual non-relativistic approximation (see eqns (8.24)–(8.26)). For the vector form factor coefficient ${}^{V}F^{(0)}_{101}$ a relation

$$ {}^{V}F^{(0)}_{101} = -\frac{1}{\sqrt{3}}\{W_0 - 2{\cdot}5)R + \tfrac{6}{5}\alpha Z\}\, {}^{V}F^{(0)}_{110} \tag{14.543} $$

exists, which is based on the CVC theory (see eqns (10.64) and (10.67)). More precisely this expression can also be written as (see Section 10.2)

$$ {}^{V}F^{(0)}_{101} = -\frac{1}{\sqrt{3}}(W_0 - 2{\cdot}5 + \Delta E_C)R\, {}^{V}F^{(0)}_{110} \tag{14.544} $$

where ΔE_C is the Coulomb displacement energy (for $^{209}\text{Pb} \rightarrow {}^{209}\text{Bi}$ we have $\Delta E_C = (18{\cdot}793 \pm 0{\cdot}011)$ MeV).

On the other hand, this relation can then also be expressed as†

$$ {}^{V}F^{(0)}_{101} = -\frac{1}{\sqrt{3}}\Delta_{T,T-1}\, R\, {}^{V}F^{(0)}_{110} \tag{14.545} $$

† On the basis of the CVC theory, the $E1$–γ-transition from the T to the $T-1$ level is also related to the form factor coefficient ${}^{V}F^{(0)}_{110}$ (see Section 10.2). We have

$$ \Gamma_\gamma(E1)_{T\rightarrow T-1} = \tfrac{2}{3}\alpha \frac{E_\gamma^3 R^2 ({}^{V}F^{(0)}_{110})^2}{T} $$

(for experiments see Shoda *et al.* 1971; Snover *et al.* 1971).

(for ^{209}Bi $\Delta_{T,T-1} = (18{\cdot}645 \pm 0{\cdot}015)$ MeV). Here, $\Delta_{T,T-1}$ is the excitation energy† of the isobaric analogue state to ^{209}Pb in ^{209}Bi (for ^{209}Pb we have $T = 45/2$). As shown in Section 10.2 the above CVC relation is exact only in the absence of the Coulomb interaction. In the presence of the Coulomb interaction its validity holds only approximately. Deviations are expected to have their origin in Coulomb effects which are proportional to $(\alpha Z)^2$ and to higher orders of αZ. In order to elucidate the differences between the results obtained from the CVC relation and from the usual non-relativistic limit we compare the ratio‡

$$\Lambda_{\mathrm{CVC}} = -\frac{\sqrt{3}\,{}^{\mathrm{V}}F_{101}^{(0)}}{\xi R\,{}^{\mathrm{V}}F_{110}^{(0)}} \tag{14.546}$$

$(\xi = \alpha Z/2R)$ which for the CVC relation (14.543) is simply given by

$$\Lambda_{\mathrm{CVC}} = 2{\cdot}4 + \frac{W_0 - 2{\cdot}5}{\xi}. \tag{14.547}$$

Based on the CVC theory (see eqns (10.63) and (10.65)) the following relation can also be derived in impulse approximation:

$$\Lambda_{\mathrm{CVC}} = \frac{2R\int_0^\infty g_f\left\{E_i - E_f - (M_n - M_p) + \frac{\alpha Z}{R}U(r)\right\}\left(\frac{r}{R}\right)g_i r^2\,\mathrm{d}r}{\alpha Z\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,\mathrm{d}r}. \tag{14.548}$$

Here we have introduced the average Coulomb potential $(\alpha Z/R)U(r)$ acting on the proton.

For the non-relativistic limit we have on the other hand

$$\Lambda_{\mathrm{nr}} = \frac{\Omega_1}{\xi R\Omega_2} = \frac{2R\int_0^\infty g_f\{E_i - E_f - (V_i - V_f)\}\left(\frac{r}{R}\right)g_i r^2\,\mathrm{d}r}{\alpha Z\int_0^\infty g_f\left(\frac{r}{R}\right)g_i r^2\,\mathrm{d}r}. \tag{14.549}$$

† It is

$$\Delta_{TT'} = \frac{a(A)}{A}\{T(T+1) - T'(T'+1)\}$$

where $a(A)$ is the corresponding coefficient in the symmetry energy term

$$E_{\mathrm{sym}} = -\frac{a(A)}{A}\{T(T+1)\}$$

of the binding energy of the whole nucleus (Bethe–Weizsäcker equation).

‡ In impulse approximation and cartesian notation (see Section 8.1.1.6), Λ reads as

$$\Lambda = \frac{\int \boldsymbol{\alpha}}{\xi R\int \mathrm{i}\frac{\mathbf{r}}{R}}.$$

Then we obtain:

(i) $^{209}\text{Pb} \rightarrow {}^{209}\text{Bi}$

CVC theory: $\Lambda_{\text{CVC}} = 2{\cdot}39$ (from eqn (14.543)),
$\Lambda_{\text{CVC}} = 2{\cdot}22$ (from eqn (14.545)),

non-relativistic limit: $\Lambda_{\text{nr}} = 0{\cdot}83$ (HO),
$\Lambda_{\text{nr}} = 3{\cdot}17$ (WS);

(ii) $^{207}\text{Tl} \rightarrow {}^{207}\text{Pb}$

CVC theory: $\Lambda_{\text{CVC}} = 2{\cdot}48$ (from eqn (14.543)),

non-relativistic limit: $\Lambda_{\text{nr}} = 1{\cdot}09$ (WS).

It is seen that the discrepancy between Λ_{CVC} and Λ_{nr} is of the order of about a factor 2, but very strongly model dependent.

By comparing both expressions, that based on the CVC theory and that based on the non-relativistic approximation, we recognize that the main difference between the two relations has its origin in (see eqn (14.527)):

(i) the somewhat different potentials V_0 for neutrons and protons;

(ii) the spin-orbit coupling potential $V_{\text{s.o.}}$.

These latter two potentials are included in the difference $V_i(r) - V_f(r)$ occurring in the non-relativistic approximation but not in the relation based on the CVC theorem. The different potentials V_0 for neutrons and protons are a consequence of the neutron excess and related to the symmetry energy. We can write

$$V_0 = V_0^* - \frac{V_1}{A} t_3 T_{3_c} \tag{14.550}$$

where t_3 is the isospin operator acting on the last nucleon outside the double magic core and T_{3_c} the corresponding one acting on the core. The above potential is a component of the more general charge exchange potential†

$$V_0 = V_0^* - \frac{V_1}{A} \mathbf{t}\mathbf{T}_c. \tag{14.551}$$

The term $V_1 \mathbf{t}\mathbf{T}_c$ represents an exchange current between the particle and the core, and can be understood as an average of two body exchange forces. The occurrence of the latter term in eqns (14.550) or (14.551), respectively, is, however, an indirect consequence of the Coulomb interaction because for the valley of beta-stable cores‡

$$T_{3_c} \simeq \frac{A}{8} \frac{\Delta E_C - (M_n - M_p)}{a(A)}.$$

† $\mathbf{t}\mathbf{T}_c = \frac{1}{2}(t_+ T_{-c} + t_- T_{+c}) + t_3 T_{3_c}$.

‡ If the Bethe–Weizsäcker equation is written as

$$B = b_{\text{Vol}} A - b_{\text{surf}} A^{2/3} - \frac{a}{A}(N-Z)^2 - \frac{3}{5}\frac{\alpha Z^2}{R}.$$

Thus, in the case where the Coulomb interaction would be switched off the term $V_1 t_3 T_{3_c}$ would disappear since the stable double magic cores would then be of the type $N = Z$, i.e. of the type $T_{3_c} = 0$. Since

$$(\mathbf{ls})_i - (\mathbf{ls})_f = -(\kappa_i - \kappa_f)/2,$$

the spin-orbit potential is also responsible for the difference between CVC and non-relativistic relation. Of course, for $\Delta j = 0$ transitions the effect of this term is remarkable only because in that case the sign of κ_i and κ_f is different. For $\Delta j = \pm 1$ transitions where $\kappa_i - \kappa_f = \pm 1$ the contribution of this term is, on the other hand, negligible. In conclusion, it is difficult to decide which of the methods should be preferred in order to evaluate the relativistic form factor coefficient ${}^{\mathrm{V}}F_{101}^{(0)}$. Both the CVC and non-relativistic relations are model dependent and only approximately valid. The exactness of the CVC relation is destroyed if the Coulomb interaction is involved. Then, this latter relation is dependent on exchange effects, too, since even in impulse approximation the Coulomb term reads exactly

$$\lambda_{\mathrm{Coul}} = \frac{\int \left[V_{\mathrm{C}}, \mathrm{i}\frac{\mathbf{r}}{R} \right]}{\xi R \int \mathrm{i}\frac{\mathbf{r}}{R}} \tag{14.552}$$

if the Coulomb potential V_{C} is not approximated by an average potential (see, for example, Fujita and Une 1976). In addition, core polarization effects, which will be treated later on, strongly influence the CVC relation (Fujita and Une 1976). In the past various authors have tried to determine Λ experimentally for a great number of β-transitions with the aim of comparing Λ_{exp} with Λ_{CVC}. No definite conclusions could, however, be drawn (for a review, see Fujita and Une 1976). In cases of $\Delta J = 0$ non-unique first forbidden transitions where both the axial vector form factor coefficient ${}^{\mathrm{A}}F_{000}^{(0)}$ and the vector form factor coefficient ${}^{\mathrm{V}}F_{101}^{(0)}$ contribute, all relativistic form factor coefficients should be calculated in the same systematic way. PCAC offers, however, no reasonable possibility to derive a useful relation between ${}^{\mathrm{A}}F_{000}^{(0)}$ and ${}^{\mathrm{A}}F_{011}^{(0)}$ (see Section 10.3). For that reason we have chosen the non-relativistic approximation for their evaluation in our examples of the transition $^{209}\mathrm{Pb} \rightarrow {}^{209}\mathrm{Bi}$ and $^{207}\mathrm{Tl} \rightarrow {}^{207}\mathrm{Pb}$. Nevertheless, it should be noticed that this latter method leads to a surprisingly good agreement between experiment and theory for these special transitions.

The single particle shell model used before for the analysis of the transitions $^{209}\mathrm{Pb} \rightarrow {}^{209}\mathrm{Bi}$ and $^{207}\mathrm{Tl} \rightarrow {}^{207}\mathrm{Pb}$ is of course a very simple one. A short look at the corresponding magnetic moments which read as

(Arima and Hyuga 1979)

^{209}Bi	$\mu_s = 2{\cdot}62$	$\mu_{exp} = 4{\cdot}117 \pm 0{\cdot}011$
^{209}Pb	$\mu_s = -1{\cdot}91$	$\mu_{exp} = -1{\cdot}33 \pm 0{\cdot}06$
^{207}Tl	$\mu_s = 2{\cdot}79$	$\mu_{exp} = 1{\cdot}83 \pm 0{\cdot}18$
^{207}Pb	$\mu_s = 0{\cdot}64$	$\mu_{exp} = 0{\cdot}593 \pm 0{\cdot}001$

(μ_s = Schmidt value) shows that this simple model has some shortcomings. Our simple model is insufficient in so far that the core cannot be considered to be really inert. Thus, we expect the nucleons in the core to also give contributions to the β-decay matrix elements. Two mechanisms are important in that context; the exchange and the core polarization effects. The exchange effects have generally been discussed in Section 8.2 and will therefore not be treated further here. All the more, special calculations of this latter effect do not exist for the transitions under consideration. The core polarization effects which are significant in many cases should now be considered in a little more detail. Classically, this effect can be understood if the core is assumed as a sphere surrounded by the active valence nucleon. Owing to the interaction between this nucleon and the core nucleons the core will be deformed, i.e. polarized. If the valence nucleon makes now a transition from one state to another one the polarization of the core changes and contributes to the transition. The transition strength is then enhanced or reduced (hindered).

Quantum mechanically, this effect happens because we have truncated the infinite set of basis states (of the nucleons in the core and of the valence nucleon) in such a way that we have only retained that state which we believe is the most important one. Just as is usually done for the residual interaction between two nucleons, where the truncation causes the effective nucleon–nucleon interaction, effective operators have to be introduced for the single particle operators which can be written as

$$\langle \phi_f' | O_{eff} | \phi_i' \rangle = \frac{\langle \phi_f | O | \phi_i \rangle}{\sqrt{\{\langle \phi_f | \phi_f \rangle \langle \phi_i | \phi_i \rangle\}}} \qquad (14.553a)$$

where the ϕ' are the truncated model space wave functions, as usually, normalized to 1, i.e.

$$\langle \phi', \phi' \rangle = 1 \qquad (14.553b)$$

and the ϕ represent the true wave functions whose norm, however, deviates from 1. We have, therefore, to extend our configuration space and admit also configurations where core and valence nucleon is excited. The most important modification of the β-decay matrix elements is now caused by the particle vibration coupling (see also Section 8.1.1.5.1). That

means the wave function for our single particle or hole with spin j must now be written as

$$|j\rangle = |(j, {}^{208}\mathrm{Pb}0^+)_j\rangle + \sum_{\lambda} \sum_{j'} a^j_{\lambda j'} |(j' {}^{208}\mathrm{Pb}\lambda^\pi)_j\rangle \tag{14.554}$$

where λ and π characterize the angular momentum and the parity of the core excited states which have, of course, also a more or less complicated microscopic particle-hole structure (for example, for $\lambda^\pi = 1^+$ the configuration $\pi 1h^{-1}_{11/2} \pi 1h_{9/2}$ and $\nu 1i^{-1}_{13/2} \nu 1i_{11/2}$, for $\lambda^\pi = 4^-, 5^-$ the configuration $\nu 3p^{-1}_{1/2} \nu 2g^{-1}_{9/2}$, for $\lambda^\pi = 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$ the configuration $\nu 1f^{-1}_{5/2}\, 2g^1_{9/2}$ etc.).

Generally a single particle can be coupled to an oscillator which is also spin and isospin dependent. The properties of the corresponding vibrations can be classified according to their operators (see, for example, Bohr and Mottelson 1975; Ejiri and Fujita 1978)

$$O_{\kappa s \lambda t} = \sum_{i=1} i^\kappa r_i^\kappa \left\{ Y_\kappa(i) \otimes \sum\nolimits^{(s)}(i) \right\}^{(\lambda)}_{\mu} I^{(t)}_{\mu_\tau} \tag{14.555}$$

where the spin and isospin dependence is given by

$$\begin{aligned} \sum\nolimits^{(s)} &= \begin{cases} 1 & s = 0 \quad \text{(non-spin-flip)} \\ \boldsymbol{\sigma} & s = 1 \quad \text{(spin-flip)} \end{cases} \\ I^{(t)}_{\mu_\tau} &= \begin{cases} 1 & t = 0 \quad \text{(isoscalar)} \\ 2t_{\mu_\tau} & t = 1 \quad \text{(isovector)} \end{cases} \end{aligned} \tag{14.556}$$

The quantities κ, λ, s, and t are the orbital angular momentum, the total angular momentum, and the spin and isospin carried by the excitation, respectively. Isoscalar and non-spin-flip quadrupole and octupole vibrations have been discussed before in Section 8.1.1.5.1 (that particle vibration coupling for the lead region is, for instance, discussed in the articles by Hamamoto (1969, 1970), Auerbach and Stein (1969), Broglia *et al.* (1970), and Bohr and Mottelson (1975)). Now, the core polarization modes $O_{\kappa s' \lambda t}$ very strongly affect β-decay matrix elements with the O_{KLs} (see Section 8.1.1.1, eqn (8.10)), i.e. those where $K = \kappa$, $L = \lambda$, $s' = s$, and $t = 1$, $\mu_\tau = \pm 1$. Isovector modes with $\mu_\tau = \pm 1$ which are especially important for the core polarization effects in nuclear β-decay are usually called charge exchange modes. These core polarization modes consist of many coherent superpositions of the particle-hole (neutron–proton) excitations of the nucleons inside the nuclear core. If we write the final state as

$$|f\rangle = \sqrt{(1-\varepsilon^2)}\, |f\rangle_{\mathrm{sp}} + \varepsilon\, |C_{KsL1}\rangle \tag{14.557}$$

where $|f\rangle_{\mathrm{sp}}$ is the single particle state, $C_{Ks\lambda 1}$ the O_{KsL1} coupled mode

state and $|\varepsilon| \ll 1$ we obtain for the transition matrix element

$$M_{KLs} = \sqrt{(1-\varepsilon^2)}\,\langle f_{sp}|\,|O_{KLs}|\,|i\rangle + \varepsilon\,\langle C_{KsL1}|\,|O_{KLs}|\,|i\rangle. \quad (14.558)$$

The latter term is now of the same order of magnitude as the former, although the mixing amplitude ε is small, since the collective state matrix element $\langle C_{KsL1}|\,|O_{KLs}|\,|i\rangle$ is large because of coherent contributions. Thus the matrix element is strongly modified such that (Ejiri and Fujita 1978)

$$\langle O_{KLs}\rangle = \frac{1}{1+\kappa}\langle O_{KLs}\rangle_{sp}. \quad (14.559)$$

This can also be expressed as a renormalization of the vector and axial vector coupling constants g_V and g_A

$$\frac{g_{eff}}{g} = \frac{1}{1+\kappa}. \quad (14.560)$$

κ is the susceptibility (polarizibility). For β-decay this topic is extensively reviewed for medium and heavy nuclei by Ejiri and Fujita (1978). For further details the reader is therefore referred to that article (for first non-unique forbidden β-transitions the renormalization g_{eff}/g if of the order 0·2–0·4).

For the transitions $^{209}\text{Pb} \rightarrow {}^{209}\text{Bi}$ and $^{207}\text{Tl} \rightarrow {}^{207}\text{Pb}$ under consideration two explicit calculations of the core polarization effects exist, that of Batkin *et al.* (1976) and the very recent and detailed one by Krmpotic *et al.* (1980). If we define

$$\tilde{a} = \frac{({}^{V}M^{(0)}_{110})_{eff}}{({}^{V}M^{(0)}_{110})_{sp}} \quad (14.561a)$$

$$\tilde{b} = \frac{({}^{A}M^{(0)}_{111})_{eff}}{({}^{A}M^{(0)}_{111})_{sp}} \quad (14.561b)$$

$$\tilde{c} = \frac{({}^{A}M^{(0)}_{011})_{eff}}{({}^{A}M^{(0)}_{011})_{sp}} \quad (14.561c)$$

$$\tilde{d} = \frac{({}^{A}M^{(0)}_{211})_{eff}}{({}^{A}M^{(0)}_{211})_{sp}} \quad (14.561d)$$

their results can be summarized as:

transition	$\tilde{a}$	$\tilde{b}$	$\tilde{c}$	$\tilde{d}$
$^{207}\text{Tl} \rightarrow {}^{207}\text{Pb}$	0·77	0·87	0·83	
	0·68	0·69	0·70	
$^{209}\text{Pb} \rightarrow {}^{209}\text{Bi}$	0·78	0·79	0·73	0·60
	1·43	0·77	0·81	0·66

The upper numbers are the results of Batkin *et al.* (1976) and the lower ones those of Krmpotic *et al.* (1980). Both results are more or less in agreement, with exception of those for $\tilde{a}$. The comparison of single-particle calculations and experimental results† shown before speaks for a renormalization in the direction of the values listed above, but the magnitude is even a little lower than that indicated above. Definite conclusions‡ cannot, however, be drawn from that comparison because firstly the single-particle calculations are also burdened by some uncertainties as discussed before and secondly exchange effects have not been taken into account, which certainly also play some role.

In the foregoing we have discussed the simple case of the beta-decay of one valence nucleon outside a double magic core. In the following we will now go over to the more complicated, but nevertheless also simple to handle situation of two valence nucleons outside a double magic core. For that purpose, we remain in the lead region and consider the ground-state–ground-state $1^- \rightarrow 0^+$ β-transition from ^{210}Bi(RaE) to ^{210}Po(RaF) where we have in the initial state a valence neutron and proton with the main configuration $\pi 1h_{9/2}\, \nu 2g_{9/2}$ and in the final state two valence protons with the main configuration $(\pi 1h_{9/2})^2$ (see Fig. 14.12). The fact that the ground state of ^{210}Bi has spin and parity 1^- and not 0^- as should be intuitively assumed, is caused by details of the effective nucleon–nucleon interaction (see, for example, Schiffer and True 1976). This effective interaction between the two valence nucleons also leads now to the consequence that the mean occupation number of the configurations $\pi 1h_{9/2}\, \nu 2g_{9/2}$ and $(\pi 1h_{9/2})^2$ is not 2 any more. Other configurations of excited single-particle states are occupied with a certain probability too, even if the configurations ($\pi 1h_{9/2}\ \nu 2g_{9/2}$) and $(\pi 1h_{9/2})^2$ remain the predominant ones. Taking into account only configurations for which the independent single-particle

† Based on the CVC theory the form factor coefficient ${}^{V}F_{110}^{(0)}$ is related to the radiative width of the $E1$ γ-transition from the ^{209}Pb isobaric analogue state to the ground state of ^{209}Bi. From that fact experimental values for a can also be derived (Shoda *et al.* 1971; Snover *et al.* 1971):

^{207}Pb $\quad \tilde{a} = 0{\cdot}56 \pm 0{\cdot}08$

^{209}Bi $\quad \tilde{a} = \ 21 \pm 1 \quad (=0{\cdot}6).$

‡ A detailed analysis on the renormalization effects for β-transitions in the lead region has been carried out by Damgaard *et al.* (1969). These authors obtained two sets of values

(i) $\dfrac{(g_V)_{\text{eff}}}{g_V} = 0{\cdot}5 \qquad \dfrac{(g_A)_{\text{eff}}}{g_A} = 0{\cdot}4$

(ii) $\dfrac{(g_V)_{\text{eff}}}{g_V} = 0{\cdot}2 \qquad \dfrac{(g_A)_{\text{eff}}}{g_A} = 0{\cdot}6.$

It should, however, be noted that this analysis is model dependent.

energies are smaller† than 3·5 MeV the wave functions for the ground states of ^{210}Bi and ^{210}Po may be written as follows:

$$\begin{aligned}^{210}\text{Bi};\,1^-\rangle = {} & a_1\,|\pi 1h_{9/2}\,\nu 2g_{9/2}\rangle + a_2\,|\pi 1h_{9/2}\,\nu 1i_{11/2}\rangle \\ & + a_3\,|\pi 2f_{7/2}\,\nu 2g_{9/2}\rangle + a_4\,|\pi 1h_{9/2}\,\nu 2g_{7/2}\rangle \\ & + a_5\,|\pi 2f_{7/2}\,\nu 2g_{7/2}\rangle + a_6\,|\pi 2f_{7/2}\,\nu 3d_{5/2}\rangle \qquad (14.562)\end{aligned}$$

$$^{210}\text{Po};\,0^+\rangle = b_1\,|(\pi 1h_{9/2})^2\rangle + b_2\,|(\pi 2f_{7/2})^2\rangle + b_3\,|(\pi 1i_{13/2})^2\rangle. \qquad (14.563)$$

Several attempts to calculate theoretically the coefficients a_i and b_i have been made in the past (Kim and Rasmussen 1963; Freed and Rhodes 1969; Nishibori 1970; Herling and Kuo 1972; Schiffer and True 1976). Let us treat the whole problem in impulse approximation. Then, for the calculation of the two-particle matrix elements we have to go back to Section 8.1.1.3. The neutron–proton wave function in the initial state can have isospin $T_v = 0$ and $T_v = 1$, the two proton wave function in the final state only $T_v = 1$ (see eqn (8.57)). Since we are dealing with a core (^{208}Pb) of 44 excess neutrons ($T_c = 22$) the total isospin $\mathbf{T} = \mathbf{T}_c + \mathbf{T}_v$ is only a good quantum number. The isospin wave function of the initial state then reads as

$$\phi_{\text{core}}(T_c, T_c)\phi_n(\tfrac{1}{2}, \tfrac{1}{2})\phi_p(\tfrac{1}{2}, -\tfrac{1}{2}) = \phi_{\text{core}}(T_c, T_c)\sum_{T_v=0}^{1} C(\tfrac{1}{2}\tfrac{1}{2}T_v; \tfrac{1}{2}-\tfrac{1}{2})\phi_2(T_v, 0) \qquad (14.564)$$

$$= \sqrt{\tfrac{1}{2}}\,\phi_{\text{core}}(T_c, T_c)\phi_2(1, 0) + \sqrt{\tfrac{1}{2}}\,\phi_{\text{core}}(T_c, T_c)\phi_2(0, 0) \qquad (14.565)$$

($\phi_2(T_v, T_{3v})$ = isospin wave function of the two valence nucleons) and for the final state

$$\phi_{\text{core}}(T_c, T_c)\phi_p(\tfrac{1}{2}, -\tfrac{1}{2})\phi_p(\tfrac{1}{2}, -\tfrac{1}{2}) = \phi_{\text{core}}(T_c, T_c)\phi_2(1, -1). \qquad (14.566)$$

By making use of eqn (8.63) we obtain further on (only matrix elements of tensor rank 1 are possible)

$$\begin{aligned}\mathfrak{M}_{1\text{LS}_{\text{two particle}}} = {} & \sum_{T_{v_i}=0}^{1} \sqrt{\frac{1}{2}}\,6\{2T_{v_i}+1\}^{1/2}\begin{pmatrix}1 & 1 & T_{v_i}\\ 1 & -1 & 0\end{pmatrix} \\ & \times \begin{Bmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ T_{v_i} & \frac{1}{2} & 1\end{Bmatrix}(-1)^{T_{v_i}+1}\begin{Bmatrix}j_p & 0 & j_p\\ 1 & j_n & 1\end{Bmatrix} \\ & \times \sqrt{\frac{2j_n+1}{3}}\,\mathfrak{M}_{1\text{Ls}}(j_n \to j_p)_{\text{single particle}} \\ = {} & -\sqrt{\frac{2}{3}}\sqrt{\frac{2j_n+1}{2j_p+1}}\,\mathfrak{M}_{1\text{Ls}}(j_n \to j_p)_{\text{single particle}}. \qquad (14.567)\end{aligned}$$

† It should be noted that the level $\nu 1j_{15/2}$ at 1·42 MeV in the nucleus ^{209}Pb may not be regarded as a pure single particle state, but rather as a mixture of the configuration $|(\nu 1j_{15/2}\,0^+)15/2^-\rangle$ and $|(\nu 2g_{9/2}\,3^-)15/2^-\rangle$ (Hamamoto 1969). For this reason the configuration $|\pi 1i_{19/2}\,n1j_{15/2}\rangle$ has not been included.

If we, for the moment, only take into account the predominant configurations $\pi 1h_{9/2}\,\nu 2g_{9/2}$ and $(\pi 1h_{9/2})^2$ in the initial and final nuclear state, respectively, we have simply to multiply the quantities C_1, D_0, E_0, and F_0 of the transition $^{209}\text{Pb} \rightarrow {}^{209}\text{Bi}$ from Table 14.14 by $-\sqrt{\frac{2}{3}}$ in order to obtain those for the transition $^{210}\text{Bi} \rightarrow {}^{210}\text{Po}$.

In the case of C_0 we have, however, some other small modifications since it is now $Z = 84$ and $W_0 = 3{\cdot}256 \pm 0{\cdot}008$ (Flothmann *et al.* 1969). Then we obtain for the transition $^{210}\text{Bi} \rightarrow {}^{210}\text{Po}$

$$C_0 = \begin{matrix} 0{\cdot}068 & \text{(HO)} \\ 0{\cdot}010 & \text{(WS)} \end{matrix}$$

$$C_1 = \begin{matrix} 0{\cdot}234 & \text{(HO)} \\ 0{\cdot}094 & \text{(WS)} \end{matrix}$$

$$D_0 = \begin{matrix} 0{\cdot}127 & \text{(HO)} \\ 0{\cdot}051 & \text{(WS)} \end{matrix}$$

$$E_0 = \begin{matrix} -0{\cdot}288 & \text{(HO)} \\ -0{\cdot}115 & \text{(WS)} \end{matrix}$$

$$F_0 = \begin{matrix} 0{\cdot}209 & \text{(HO)} \\ 0{\cdot}083 & \text{(WS)} \end{matrix}.$$

For $^{V}F_{101}^{(0)}$ the non-relativistic approximation has been used. If, on the other hand, use is made of the CVC relation we get

$$C_0 = \begin{matrix} 0{\cdot}054 & \text{(HO)} \\ 0{\cdot}012 & \text{(WS)} \end{matrix}.$$

By inserting these values for the quantities C_0, C_1, etc. into the relevant equations for the observables and carrying out a quick and rough calculation we recognize immediately that no agreement with the existing experiments is obtained. For the shape factor which exhibits a very strong deviation† from the allowed form three precise measurements exist. Their results can be summarized as shown in Table 14.16 (see also Fig. 14.17). Experimental data on the longitudinal polarization have been published by Harmsen and Holm (1962), Wegener *et al.* (1958), Alikhanov *et al.* (1959), Ullman *et al.* (1961), Schatz *et al.* (1964), and Poppensieker *et al.*

† From the beginning the peculiar behaviour of that transition has initiated many theoretical analyses and attempts to explain theoretically the unusual form of its observables. Thus, the theoretical treatment of the decay of ^{210}Bi has a long history (see, Yamada 1953; Newby and Konopinski 1959; Spector and Blin-Stoyle 1962; Daniel 1962; Spector 1963; Bühring 1963*b*; Deutsch and Lipnik 1965; Sodemann and Winther 1965; Damgaard *et al.* 1969; Fayens and Khodel 1970; Löhken *et al.* 1971; Civitarese *et al.* 1974; Behrens and Szybisz 1974; Ebert *et al.* 1975; Morita *et al.* 1976; Krmpotic *et al.* 1980). For a survey see also Morita *et al.* (1971).

TABLE 14.16 *Experimental results for the shape factor† of the* $1^- - 0^+$ *decay* $^{210}Bi \rightarrow {}^{210}Po$

E_0 (keV)	a $[m_e c^2]^{-1}$	$\gamma_1 b$ $m_e c^2$	c $(m_e c^2)^{-1}$	Author
1155 ±5	−0·431±0·011	1·42 ±0·29	0·0334±0·0068	Plassmann and Langer (1954)
1160·5±1·5	−0·456±0·050	0·301±0·107	0·0540±0·0027	Daniel (1962)
1153 ±4	−0·431±0·090	0·084±0·078	0·0488±0·0040	Flothmann *et al.* (1969) Löhken *et al.* (1971)

† The corresponding coefficients a, b, and c are not those given originally by the authors. They have been reevaluated (see Behrens and Szybisz 1976).

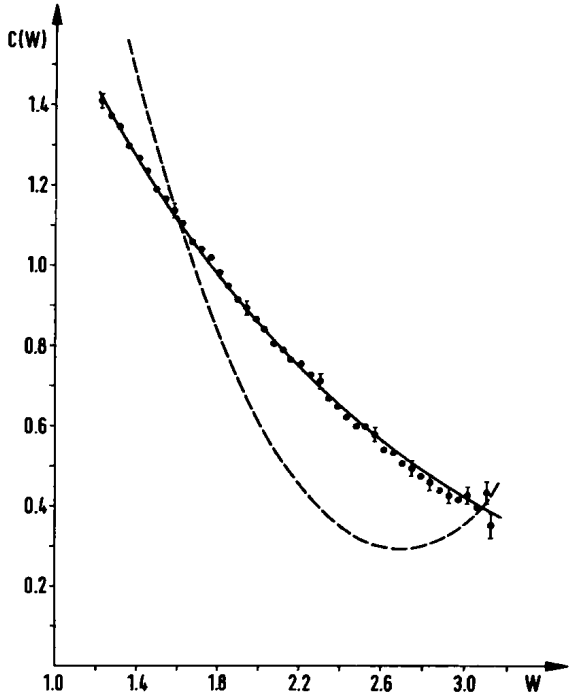


FIG. 14.17. The experimental spectrum shape for the decay of ^{210}Bi (Flothmann *et al.* 1969). The solid line has been calculated by using the fitted mixing coefficients shown in Table 14.17. Here, all higher order contributions have been included. The dashed line corresponds to that when only main terms are considered (taken from Behrens and Szybisz 1974).

† This fact shows that a violation, if any, of time reversal invariance is small: a breakdown of time reversal invariance would imply that, relative to the vector coupling constant, the axial-vector coupling constant is complex. If its imaginary part would be of the same order of magnitude as its real part, the required nearly complete cancellation between the vector and axial-vector contributions in C_0 could not occur. Therefore the imaginary part is small, if not zero, and time reversal invariance probably holds (see Alikhanov *et al.* 1959 and references quoted therein).

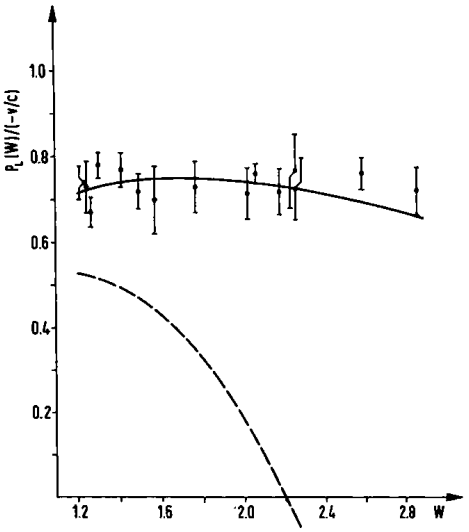


FIG. 14.18. The experimental longitudinal polarization $P_e(W_e)$ for the decay of ^{210}Bi. The data have been taken from various authors which are mentioned in the text. The solid line has been calculated by using the fitted mixing coefficients shown in Table 14.17. Here all higher order contributions have been included. The dashed line corresponds to that when only main terms are considered (taken from Behrens and Szybisz 1974).

(1977). Also, a very unusual behaviour has been found ($P_e \simeq -(0{\cdot}7 \pm 0{\cdot}1)(v/c)$) for that observable (see Fig. 14.18). The reason why the simple configurations $i\rangle = (\pi 1h_{9/2}\, \nu 2g_{9/2})$ and $f\rangle = (\pi 1h_{9/2})^2$ are not able to describe shape factor and longitudinal polarization although they are the predominant ones ($a_1 \simeq 0{\cdot}95$ and $b_1 \simeq 0{\cdot}95$) is that we have, as already mentioned before, a near cancellation of the different terms in C_0. Thus, very small changes of the form factor coefficients or matrix elements, respectively, cause drastic effects concerning the behaviour of the observables. By inspection of the relevant expressions for the shape factor $C(W_e)$ (see eqns (14.475) to (14.479)) and for the longitudinal electron polarization P_e, respectively (see eqns (14.516) to (14.519)), we see on a first look that it must be $C_0 \ll C_1$, $C_0 \ll E_0$, $C_0 \ll D_0$. The cancellation is therefore nearly complete.† Under these circumstances we cannot expect that any theoretical calculation,‡‡ which makes use of some

‡‡ It should be noted that

$$\mathfrak{M}_{KLs} = \sum_{ij} a_i b_j \mathfrak{M}_{KLs}(i \to j)$$

where the $\mathfrak{M}_{KLs}(i \to j)$ are the two particle matrix elements of the transition from configuration $i \to j$.

more refined wave functions of the type given in eqns (14.562) and (14.563) (for example, those of Kim and Rasmussen 1963, 1965), sufficiently reproduces the observables of this transition even if the order of magnitude of the corresponding matrix elements themselves is obtained satisfactorily. Thus, for a further discussion of this transition we have three possibilities:

(i) to determine the form factor coefficients by making use of a least squares fit of the spectrum shape, the electron longitudinal polarization and the *ft*-value. The form factor coefficients are treated as independent parameters in that approach (see, for example, Löhken *et al.* 1971; Civitarese *et al.* 1974);

(ii) to adjust the nuclear model parameters themselves, i.e. in our case the mixing coefficients a_i and b_i of the nuclear wave function given in eqns (14.562) and (14.563) by using the method of a least squares fit of those observables which have been mentioned before (see, for example, Behrens and Szybisz 1974);

(iii) to determine the quantities C_0, C_1, C_2, D_0, E_0, and F_0 by applying the least squares fit method for a description of the observables which are at our disposal.

Every quantitative analysis which makes use of the least squares fit method is based on a systematic search for a minimum of the total χ^2_T-function defined by

$$\chi^2_T = \chi^2[C(W_e)] + \chi^2[P_e]. \tag{14.568}$$

The $\chi^2(k)$-function for each observable k then reads as

$$\chi^2(k) = \sum_i \left[\frac{k(i)_{exp} - k(i)_{theor}}{\Delta k(i)_{exp}}\right]^2 \tag{14.569}$$

where $k(i)_{exp}$ are the experimental data with uncertainty $\Delta k(i)_{exp}$, and $k(i)_{theor}$ are theoretical values for the observable k calculated from the corresponding set of parameters (form factor coefficients or nuclear model parameters or quantities C_0, C_1 etc.).

As mentioned by Schweitzer and Simms (1972) a particular minimum of the total χ^2_T-function can only be accepted if in our case it satisfies the additional criterion

$$\frac{\chi^2[C(W_e)]}{N[C(W_e)]} \lesssim 1 \qquad \frac{\chi^2[P_e]}{N[P_e]} \lesssim 1 \tag{14.570}$$

where $N[C(W_e)]$ and $N[P_e]$ are the numbers of experimental points considered for the observables $C(W_e)$ and P_e, respectively. Otherwise, undue weight is given to the shape factor $C(W_e)$ where the number of data points is much larger compared to the longitudinal polarization P_e. Then, it could happen that a solution which yields perfect agreement with the shape factor data can have a good χ^2_T, but disagrees with electron

polarization data within some standard deviations. By the way, this latter point should always be carefully considered if more than just one observable is fitted, and the number of experimental points and their degree of uncertainty differs very much for each of these observables (see Schweizer and Simms 1972). Each value of χ^2 corresponds to a certain region in the multidimensional parameter space to which a confidence level P can be assigned by comparing this $\chi^2(P)$ value with a χ^2-distribution table. A certain degree of confidence has to be achieved otherwise either the measurements are inconsistent or the theoretical model is inappropriate.

The fitted parameters can additionally be normalized by requiring that the ft-value is correctly reproduced. This latter point means nothing other than a multiplication of the form factor coefficients or of the quantities C_0, C_1 etc. by a certain common factor (then the mixing coefficients a_i and b_i are not normalized to 1).

At the beginning we will now discuss the method suggested under (i) to fit the form factor coefficients. This method has essentially two disadvantages. Firstly, it is not able to take into account higher order contributions because then more than five form factor coefficients have to be fitted. In general, these are always too many parameters in order to get a solution, i.e. to find a suitable minimum of χ^2. In the case of the transition $^{210}\mathrm{Bi} \rightarrow {}^{210}\mathrm{Po}$ these contributions of higher order terms are, however, of special importance because of the strong cancellation effects mentioned before. Secondly, the form factor coefficients are taken as independent parameters, which is, however, not what they really are. In reality, they are strongly correlated as a consequence of an appropriate nuclear model. Very often, unphysical results for the form factor coefficients are then obtained as a result of a fit even if the χ^2 of the fit itself is reliable, especially then, when one observable contains systematic inconsistencies. Some examples for the transition $^{210}\mathrm{Bi} \rightarrow {}^{210}\mathrm{Po}$ are shown in the following where unfortunately the shape factor $C(W_e)$ only has been fitted:

(a)

$$^{\mathrm{V}}F_{101}^{(0)} = 0{\cdot}0645 \tag{14.571a}$$

$$^{\mathrm{V}}F_{110}^{(0)} = -0{\cdot}0635 \tag{14.571b}$$

$$^{\mathrm{A}}F_{111}^{(0)} = -0{\cdot}185 \tag{14.571c}$$

(Löhken *et al.* 1971)

(b)

$$^{\mathrm{V}}F_{101}^{(0)} = 0{\cdot}079^{+0{\cdot}005}_{-0{\cdot}009} \tag{14.572a}$$

$$^{\mathrm{V}}F_{110}^{(0)} = -0{\cdot}233^{+0{\cdot}066}_{-0{\cdot}046} \tag{14.572b}$$

$$^{\mathrm{A}}F_{111}^{(0)} = -0{\cdot}168^{+0{\cdot}004}_{-0{\cdot}010} \tag{14.572c}$$

(Civitarese *et al.* 1974).

Both results are normalized such that the ft-value is described correctly, but, it should be noted that in the case of Löhken *et al.* (1971) higher order effects have not been taken into account and even the older method of the expansion of the electron radial wave functions in powers of r has been applied. In the case of Civitarese *et al.* (1974) the Coulomb function treatment used has some deficiencies and the handling of higher order contributions was not the completely appropriate one.

For the next step, we will now treat the fitting procedure of the wave function mixing parameters a_i and b_i of eqns (14.562) and (14.563), i.e. the determination of the nuclear model parameters for the transition $^{210}\text{Bi} \rightarrow {}^{210}\text{Po}$. This method has, of course, the advantage that all higher order terms can easily be included and their influence can explicitly be studied. All the more, the demanded nuclear structure information† can directly be determined from the beta-decay experiment. The disadvantage of this procedure, on the other hand, lies in the fact that their results are dependent on some approximations which have mostly to be made in order to compute theoretically the matrix elements or form factor coefficients, respectively. Usually, induced, exchange and core polarization effects are not taken into account and the simple impulse approximation treatment is used. For the beta-transition ^{210}Bi–^{210}Po this latter approach has been applied (Behrens and Szybisz 1974) with the results‡ shown in Table 14.17 (the experimental data for the shape factor of Flothmann *et al.* (1969) have been used). The error of the coefficients determined in that way is smaller than 1% (see Fig. 14.19) and therefore smaller than the systematic uncertainty introduced by using different radial wave functions or by the omission of exchange and core polarization effects. The relativistic form factor coefficient ${}^{\text{V}}F_{101}^{(0)}$ was, by the way, evaluated on the basis of the CVC relation using eqn (14.548). A fit with not too different results can, however, also be obtained by using the non-relativistic approximation.

The results for the observables are shown in Figs. 14.17 and 14.18 where the solid line represents the best fit. By making use of the wave

† All experimental investigations (with very few exceptions) of first non-unique forbidden beta-transitions have been carried out with the aim to obtain some information about the underlying nuclear structure.

‡ The coefficients a_i were normalized according to

$$\sum_{i=1}^{6} a_i^2 = 1.$$

Thus, the parameters $a_2 \ldots a_5$ were only considered to be independent; a_1 follows from the normalization condition. Since it is not possible to obtain any information about the component $b_3 \,|\pi(1i_{13/2})^2)$ because it does not contribute to the transition, the normalization was chosen to $b_1^2 + b_2^2 = 1 - b_3^2 = \eta^2$ or to $(b_1/\eta)^2 + (b_2/\eta)^2 = b_1'^2 + b_2'^2 = 1$ (see Behrens and Szybisz 1974).

TABLE 14.17 *Mixing coefficients extracted from the experimental data for the beta-transition* ^{210}Bi–^{210}Po *compared with the theoretical estimations (Behrens and Szybisz 1974)*

Coefficients	HO_{fit}	HO_{theor}†	HO_{theor}‡	WS_{fit}
a_1	0·9434	0·9689		0·9783
a_2	0·1933	0·2192		0·0661
a_3	−0·0841	−0·0463		0·0799
a_4	−0·2438	−0·1034		−0·1496
a_5	0·0773	0·0162		0·0597
a_6	−0·0107	0·0051		−0·0787
b_1'	0·9526	0·9460	0·9129	0·9385
b_2'	0·3044	0·3243	0·4081	0·3453
$\frac{\chi^2[C(W_e)]}{N[C(W_e)]}$	0·41			0·47
$\frac{\chi^2[P_e]}{N[P_e]}$	0·76			0·83

HO harmonic oscillator
WS Woods–Saxon with the parameters suggested by Rost (1968)
† Kim and Rasmussen (1963, 1965)
‡ Herling and Kuo (1972)

function mixing coefficients given in Table 14.17 each and every form factor coefficient can be calculated. Corresponding values can be found in the article by Behrens and Szybisz (1974). A comparison of fitted and theoretical coefficients shown in Table 14.17 illustrates the good agreement between both as far as the general features are concerned. It also makes clear that beta-transitions of the type under consideration are a powerful tool to determine nuclear structure details.

Of course, the quantities C_0, C_1, C_2 etc. can also be evaluated. They are shown for harmonic oscillator radial wave functions and for the corresponding coefficients a_i and b_i of the second column in Table 14.15, which represent the best fit of the experiments for these wave functions, in the following. In the fit all higher order terms have been included:

	Higher order terms included	Dominant terms only	
$C_0 =$	$-7{\cdot}40\times10^{-3}$	$-4{\cdot}80\times10^{-3}$	(14.573*a*)
$C_1 =$	$7{\cdot}96\times10^{-2}$	$9{\cdot}67\times10^{-2}$	(14.573*b*)
$C_2 =$	$-4{\cdot}51\times10^{-2}$	0	(14.573*c*)
$D_0 =$	$6{\cdot}31\times10^{-2}$	$6{\cdot}96\times10^{-2}$	(14.573*d*)
$E_0 =$	$-1{\cdot}67\times10^{-1}$	$-1{\cdot}93\times10^{-1}$	(14.573*e*)
$F_0 =$	$7{\cdot}71\times10^{-3}$	$1{\cdot}21\times10^{-2}$	(14.573*f*)

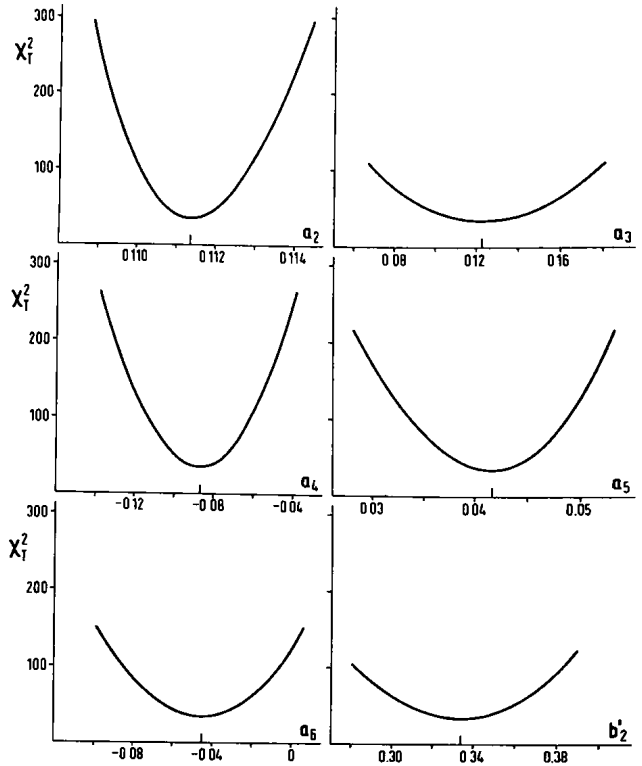


FIG. 14.19. The χ_T^2 as a function of the parameters a_i and b_i' shown in Table 14.17 around the minimum (taken from Behrens and Szybisz 1974).

It should be noted that the quantities are normalized according to $\sum_i a_i^2 = 1$ and $\sum_i b_i'^2 = 1$, but not to reproduce the correct ft-value. By inserting the above quantities into the relevant equations we would obtain

$$\log ft = 8{\cdot}265 \qquad (14.574a)$$

compared with the experimental value

$$\log ft_{\text{exp}} = 8{\cdot}097. \qquad (14.574b)$$

If the quantities C_0, C_1 etc. listed above are intended to be normalized such that the experimental ft-value is reproduced they have to be multiplied by a factor 1·21.

By inspection of the various values for the quantities C_0, C_1 etc. given above it is clearly demonstrated that it is not possible to describe the

observables by considering the dominant terms only (difference between the exact C_0 and that of the dominant terms is about a factor 2/3!). The drastic difference between the observables calculated by including all higher order terms and the identical ones by including dominant terms only is also shown in Figs. 14.17 and 14.18 by the dashed lines (however, for the Woods–Saxon type of wave functions).

By inserting the quantities C_0, C_1 etc. mentioned above into eqns (14.476) to (14.479) and (14.516) to (14.519) we obtain for the shape factor ($k_n = 6{\cdot}60 \times 10^{-5}(1 - 0{\cdot}041\mu_1)$)

$$a = -0{\cdot}453 \tag{14.575a}$$

$$\gamma_1 b = 0{\cdot}214 \tag{14.575b}$$

$$c = 0{\cdot}0544 \tag{14.575c}$$

and for the electron polarization

$$P_e = -\frac{p_e}{W_e}\Lambda_1\left\{1 - 0{\cdot}214\mu_1/[W_e C_N(W_e)] + 0{\cdot}104\left(\frac{\Lambda_2}{\Lambda_1} - 1\right)\lambda_2 \Big/ C_N(W_e)\right\} \tag{14.576}$$

where $C_N(W_e)$ is the normalized shape factor which reads as

$$C_N(W_e) = 1 + aW_e + \mu_1\gamma_1 b/W_e + cW_e^2. \tag{14.577}$$

The third term in the equation (14.576) can be neglected for $W_e > 1{\cdot}2$.

The most important form factor coefficients which belong to the fitted harmonic oscillator wave functions reported in Table 14.17 are quoted in the following (Behrens and Szybisz 1974):

$$^{V}F_{101}^{(0)} = 0{\cdot}0465 \tag{14.578a}$$

$$^{V}F_{110}^{(0)} = -0{\cdot}111 \tag{14.578b}$$

$$^{V}F_{110}^{(0)}(1,1,1,1) = -0{\cdot}144 \tag{14.578c}$$

$$^{A}F_{111}^{(0)} = -0{\cdot}178 \tag{14.578d}$$

$$^{A}F_{111}^{(0)}(1,1,1,1) = -0{\cdot}158. \tag{14.578e}$$

The other form factor coefficients which are responsible for the higher order terms can be found in the paper of Behrens and Szybisz (1974).

Often, the experimental data have been analysed to determine the form factor coefficient ratios Λ and u where Λ has been defined before (see eqn (14.546)) and u is given by

$$u = \frac{\sqrt{2}\,^{A}F_{111}^{(0)}}{^{V}F_{110}^{(0)}}. \tag{14.579}$$

For convenience of the reader we will therefore report these ratios

calculated with the best fit wave functions listed in Table 14.15:

$$u = \begin{cases} 2{\cdot}27 & \text{(HO)} \\ 3{\cdot}08 & \text{(WS)} \end{cases} \tag{14.580}$$

$$\Lambda = \begin{cases} 2{\cdot}37 & \text{(HO)} \\ 2{\cdot}26 & \text{(WS)} \end{cases}. \tag{14.581}$$

For comparison with the results of Löhken *et al.* (1971) and Civitarese *et al.* (1974) which have been quoted before we will also list the corresponding form factor coefficients normalized such that the *ft*-values of the transitions ^{210}Bi–^{210}Po are correctly reproduced.†

They then read‡

$${}^{V}F^{(0)}_{101} = \begin{cases} 0{\cdot}0565 & \text{(HO)} \\ 0{\cdot}0460 & \text{(WS)} \end{cases} \tag{14.582}$$

$${}^{V}F^{(0)}_{110} = \begin{cases} -0{\cdot}135 & \text{(HO)} \\ -0{\cdot}115 & \text{(WS)} \end{cases} \tag{14.583}$$

$${}^{A}F^{(0)}_{111} = \begin{cases} -0{\cdot}215 & \text{(HO)} \\ -0{\cdot}251 & \text{(WS)} \end{cases}. \tag{14.584}$$

The whole discussion under point (ii) demonstrates very clearly that the method of directly determining form factor coefficients (or nuclear matrix elements) by a fitting procedure gives results which are not precise enough in order to compare them with accurate theoretical calculations.

Thirdly, an alternative procedure which was suggested under point (iii) consisted in the fitting of the quantities C_0, C_1, C_2, D_0, E_0, and F_0 to the experimental data. This method has the advantage that it is completely model independent and takes care automatically of all higher order terms. As mentioned before, these are the linear combinations of form factor coefficients which govern every beta-transition. A disadvantage of this approach is the treatment of the quantities C_0, C_1, C_2 etc. as independent parameters, which is not what they really are. This latter fact lies on the

† That means the wave function normalization conditions are replaced by

$$\sum_i a_i^2 = \varepsilon_a^2$$

$$\sum_i b_i'^2 = \varepsilon_b^2$$

where we have $\varepsilon_a^2 \varepsilon_b^2 = \begin{cases} 1{\cdot}47 & \text{(HO)} \\ 4{\cdot}08 & \text{(WS)} \end{cases}$.

‡ ${}^{V}F^{(0)}_{110}(1,1,1,1) = \begin{cases} -0{\cdot}175 & \text{(HO)} \\ -0{\cdot}145 & \text{(WS)} \end{cases}$

${}^{A}F^{(0)}_{111}(1,1,1,1) = \begin{cases} -0{\cdot}158 & \text{(HO)} \\ -0{\cdot}167 & \text{(WS)} \end{cases}$.

same line as discussed before under point (i). The foregoing analysis shows, however, that this procedure of an extraction of the quantities C_0, C_1, C_2 etc. from the experimental data is superior to the method of directly fitting the form factor coefficients. Afterwards these linear combinations of form factor coefficients C_0, C_1 etc. can be compared with theoretical calculations. Unfortunately, this way has not been chosen for the transition $^{210}\text{Bi} \rightarrow {}^{210}\text{Po}$ so that corresponding results cannot be presented here.

Every theoretical interpretation of the transition $^{210}\text{Bi} \rightarrow {}^{210}\text{Po}$ has also to make allowance of exchange and core polarization effects. Unfortunately, exchange effects have not been considered for this transition. On the other hand, core polarization effects for the beta-transition $^{210}\text{Bi} \rightarrow {}^{210}\text{Po}$ have explicitly been calculated by some authors. Their results are shown in the following (for the definition of the ratios $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, and $\tilde{d}$ see eqns (14.561a–d)):

(α) Arita *et al.* (1976)†

$$\tilde{a} = \begin{cases} 0{\cdot}92 & \text{(Serber)} \\ 0{\cdot}82 & \text{(Rosenfeld)} \end{cases} \tag{14.585}$$

$$\tilde{b} = \begin{cases} 0{\cdot}80 & \text{(Serber)} \\ 0{\cdot}85 & \text{(Rosenfeld)} \end{cases}; \tag{14.586}$$

(β) Morita *et al.* (1976b)

$$\tilde{b} = 0{\cdot}86 \quad \text{(HO)} \tag{14.587}$$

$$\frac{u_{\text{eff}}}{u} = 1{\cdot}22 \quad \text{(HO)}; \tag{14.588}$$

(γ) Krmpotic *et al.* (1980)‡ (see also Ebert *et al.* 1975)

$$\tilde{a} = \begin{cases} 0{\cdot}78 & \text{(KR)} \\ 1{\cdot}12 & \text{(HK)} \end{cases} \tag{14.589}$$

$$\tilde{b} = \begin{cases} 0{\cdot}85 & \text{(KR)} \\ 0{\cdot}79 & \text{(HK)} \end{cases}. \tag{14.590}$$

We recognize that for harmonic oscillator radial wave functions a renormalization of about 20% occurs on average. Because of the sensitivity of some radial wave functions, as discussed before, it would be not surprising if the core polarization magnitude depends additionally on the type of wave functions used. It is also evident that the extraction of wave function mixing coefficients is not essentially influenced by core polarization effects

† Serber and Rosenfeld means different residual interactions (harmonic oscillator wave functions have been used).

‡ KR = Kim and Rasmussen (1963, 1965).
HK = Herling and Kuo (1972).

for the transition under consideration. Of course, there exists a connection between the nuclear wave functions gained by the analysis of the beta-transition of $^{210}Bi \rightarrow {}^{210}Po$, core polarization (exchange) effects and the magnetic moments of ^{210}Bi and ^{210}Po. As illustrated before for the single valence nucleon transitions these magnetic moments are, however, much more sensitive to such effects. A discussion of that point can be found in an article by Morita *et al.* (1976*b*). Thus, for details, the reader is referred to that paper.

After the treatment of an example with two valence nucleons outside the double magic core of ^{208}Pb we will finally also discuss one case with two valence holes. To do so we choose the ground-state–ground-state beta-decay of $^{206}Tl \rightarrow {}^{206}Pb$. This is a transition of the type $0^- \rightarrow 0^+$ with the shell model states $3p_{1/2}^{-1}\, 3s_{1/2}^{-1} \rightarrow (3p_{1/2})^{-2}$ involved. The endpoint energy of the decay is $W_0 = 3{\cdot}989 \pm 0{\cdot}008$ and its half-life $t_{1/2} = 4{\cdot}27 \pm 0{\cdot}05$ min (see Wiesner *et al.* 1972).

Tensor rank 0 form factor coefficients only contribute. In this case we have (see eqns (8.63) and (14.567))

$$\mathfrak{M}_{0\mathrm{LS}_{\text{two holes}}} = -\sqrt{2}\mathfrak{M}_{0\mathrm{LS}}(j_i^{-1} \rightarrow j_f^{-1}). \tag{14.591}$$

Thus, we have simply to multiply the form factor coefficients obtained before for the transition $^{207}Tl \rightarrow {}^{207}Pb$ (see eqns (14.539*a–i*)) by $-\sqrt{2}$ in order to get the relevant ones for this two hole transition. We then obtain

	HO	WS†	WS‡	
${}^{A}F_{000}^{(0)}$	0·284	0·316	0·324	(14.592*a*)
${}^{A}F_{011}^{(0)}$	1·135	0·963	0·884	(14.592*b*)
${}^{A}F_{011}^{(0)}(1,1,1,1)$	1·368	1·160	1·050	(14.592*c*)

where HO = harmonic oscillator radial wave functions, and WS = Woods–Saxon radial wave functions. By inserting these results into the relevant equations (see eqn (14.452)) we then arrive at

$$A_0 = \begin{cases} -0{\cdot}012 & \text{(HO)} \\ +0{\cdot}061 & \text{(WS)†} \\ +0{\cdot}090 & \text{(WS)‡} \end{cases} \tag{14.593}$$

$$\left|\frac{A_1}{A_0}\right| \ll 10^{-2}$$

$$B_0 = \begin{cases} -0{\cdot}379 & \text{(HO)} \\ -0{\cdot}321 & \text{(WS)†} \\ -0{\cdot}292 & \text{(WS)‡} \end{cases} \tag{14.594}$$

We have a strong cancellation of the different terms within A_0. Thus, this quantity is very sensitive to details of the nuclear radial wave functions

although the form factor coefficients themselves do not show such a sensitive behaviour. Despite this cancellation effect higher order contributions have, however, no remarkable influence on A_0 because they, by chance, cancel in this case too.

Two more recent experiments for the shape factor exist (Persson *et al.* 1971; Wiesner *et al.* 1972). In our comparison between theory and experiment we use that by Wiesner *et al.* (1972). These authors have also analysed their results theoretically and obtained by making use of a three parameter fit for the quantities A_0, A_1, and B_0, whereby the absolute values were fixed by the *ft*-value,

$$A_0 = 0{\cdot}185 \tag{14.595a}$$

$$A_1 = -0{\cdot}060 \tag{14.595b}$$

$$B_0 = -0{\cdot}26. \tag{14.595c}$$

From these fitted quantities it follows directly that the theoretical *ft*-values are too large by factors of about 240 for (HO), 9 for (WS),† and 4 for (WS).‡ This behaviour which is in contrast to most cases of first non-unique forbidden transitions should, however, not be overestimated because of the cancellation effects mentioned above. The corresponding experimental data of Wiesner *et al.* (1972) are shown in Fig. 14.20. The solid line in this figure represents the theoretical curve calculated by using the results computed with the Woods–Saxon wave functions of Batty and Greenlees (1969). Its form can be written as

$$C(W_e) = 0{\cdot}0081\{1 + 0{\cdot}096\mu_1/W_e\}. \tag{14.596}$$

The theoretical log *ft*-value has the value

$$\log ft = 5{\cdot}88 \tag{14.597a}$$

compared with the experimental one

$$\log ft = 5{\cdot}27. \tag{14.597b}$$

By the way, more sophisticated wave functions where configuration mixing is taken into account, as for example presented by Damgaard *et al.* (1969), do not improve the agreement between theoretical and experimental *ft*-values (Wiesner *et al.* 1972). The reason is that the cancellation effects in the quantity A_0 are not very much influenced by these more refined wave functions. $0^- - 0^+$ transitions have also often been analysed with the aim of drawing some conclusions about induced tensor and

† By Rost (1968).
‡ By Batty and Greenlees (1969).

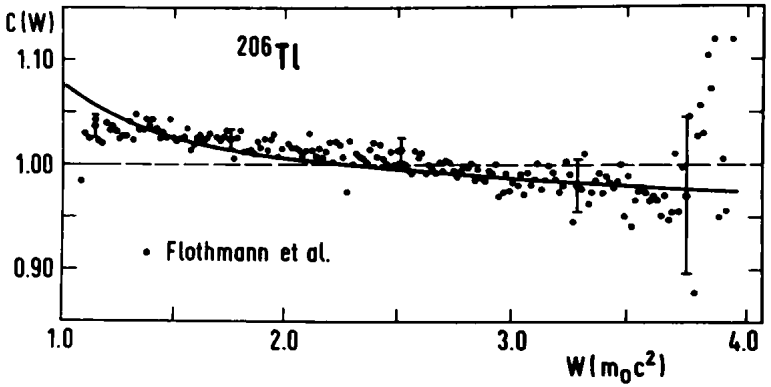


FIG. 14.20. Experimental shape factor for the decay of ^{206}Tl (Flothmann *et al.* 1970; Wiesner *et al.* 1972). The solid line represents the simple shell model calculation as described in the text.

pseudoscalar coupling constants (see, for example, Bühring 1963; Krmpotic and Tadic 1969; Eman *et al.* 1975). Reliable results were, however, not obtained. Core polarization effects have also been evaluated for the transition $^{206}\text{Tl} \rightarrow {}^{206}\text{Pb}$ (Krmpotic *et al.* 1980) and have been obtained as

$$({}^{A}\mathfrak{M}^{(0)}_{011})_{\text{eff}}/{}^{A}\mathfrak{M}^{(0)}_{011} = 0{\cdot}72. \tag{14.598}$$

However, no improvement is gained by taking into account this effect. On the other hand, exchange effects which have not up to now been considered for this transition, could indeed solve the problem.

Before leaving the lead region from which we have chosen the examples in the preceding, we once again will refer particularly to two treatments of a greater number of first non-unique forbidden beta-decays in that region, namely the one of Damgaard *et al.* (1969), which contains a more phenomenological consideration of the beta-transitions, and the other one of Krmpotic *et al.* (1980), which is more directed to the evaluation of core polarization effects. For ^{208}Tl reference should also be made to the very detailed and complete discussion of the corresponding beta-transitions made by de Raedt (1970).

Generally, in the past first non-unique forbidden transitions have experimentally and theoretically been investigated to obtain nuclear structure information about the participating nuclear states. Therefore, we will discuss briefly at least one example where it was possible to observe a greater number of different observables. For that purpose we have chosen the $2^- - 2^+$ β-decay of ^{86}Rb to ^{86}Sr (Appel *et al.* 1974). For the decay scheme see Fig. 14.21. All information about the participating

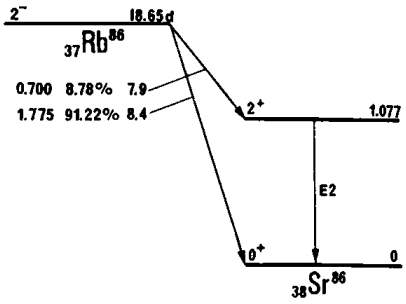


FIG. 14.21. Decay scheme of ^{86}Rb.

nuclei and the transition under consideration has been compiled and evaluated by Tepel (1978). For details not mentioned in the following we therefore refer to the summary just quoted. In the simple single-particle shell model the configuration of the initial state reads as (see Fig. 14.22)

$$|^{86}\mathrm{Rb};\,2^-\rangle = |(\nu 1g_{9/2})^{-1}(\pi 1f_{5/2})^{-1};\,2^-\rangle \tag{14.599}$$

and of the final state† (first excited state in ^{86}Sr)

$$|^{86}\mathrm{S};\,2^+\rangle = |(\nu 1g_{9/2})^{-2}_{2};\,2^+\rangle. \tag{14.600}$$

The beta-decay would then be of the type

$$\nu 1g_{9/2} \rightarrow \pi 1f_{5/2}. \tag{14.601}$$

As a consequence only matrix elements or form factor coefficients of tensor rank 2 (or higher) should contribute to the $2^- - 2^+$ non-unique forbidden transition, i.e. it should have a unique forbidden form analogous to the corresponding transition into the ground state of ^{86}Sr. Experimental results for the shape factor $C(W_e)$, $\beta-\gamma$ angular correlation and $\beta-\gamma$ (CP) correlation are shown in Fig. 14.23 which has been taken from the article of Appel *et al.* (1974). The unique forbidden form following from the simple shell model wave functions given in eqns (14.599) and (14.600) are shown as dashed lines. It is evident that they are not able to describe the observables of this beta-transition, but we expect a high sensitivity for nuclear structure details because the deviations of the simple shell model results from the experimental data are so drastic. In truth, the experimental results can be described by the following form factor coefficients which are normalized such that the correct *ft*-value

† The ground state configuration for ^{86}Sr would be

$$|^{86}\mathrm{Sr};\,0^+\rangle = |(1g_{9/2})^{-2}_{0};\,0^+\rangle$$

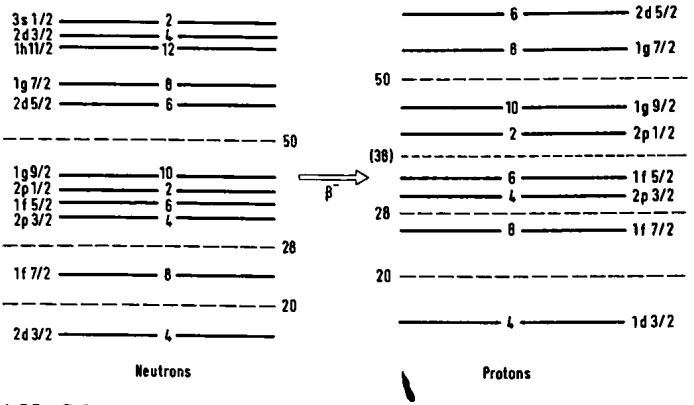


FIG. 14.22. Schematic representation of the shell model levels for ^{86}Rb and ^{86}Sr (Appel *et al.* 1974).

(log $ft = 7{\cdot}93$) is reproduced (Appel *et al.* 1974):

$$^{A}F^{(0)}_{000} = 0{\cdot}0144$$
$$^{A}F^{(0)}_{011} = 0{\cdot}0579$$
$$^{A}F^{(0)}_{011}(1,1,1,1) = 0{\cdot}0625$$
$$^{V}F^{(0)}_{101} = 0{\cdot}0030$$
$$^{V}F^{(0)}_{110} = -0{\cdot}0211$$
$$^{V}F^{(0)}_{110}(1,1,1,1) = -0{\cdot}0248$$
$$^{A}F^{(0)}_{111} = -0{\cdot}0301$$
$$^{A}F^{(0)}_{111}(1,1,1,1) = -0{\cdot}0303$$
$$^{A}F^{(0)}_{211} = -0{\cdot}365.$$

The behaviour of the observables which correspond to the form factor coefficients just listed is indicated by a solid line in Fig. 14.23. One recognizes on a first look that the form factor coefficient of tensor rank 2 $^{A}F^{(0)}_{211}$ is very much enhanced compared to the other ones of tensor rank 0 and 1. That fact was predicted by the predominant configurations as we have seen before. A detailed theoretical analysis of the beta-transitions $^{86}\text{Rb} \rightarrow {}^{86}\text{Sr}$ was carried out by Wahlborn (1964). This author explained the experimental data of the 2^{-}–2^{+} transition by assuming configuration admixtures of one-phonon quadrupole vibrational states as described in Section 8.1.1.5.1 (see eqn (8.146)). Then the following configurations for

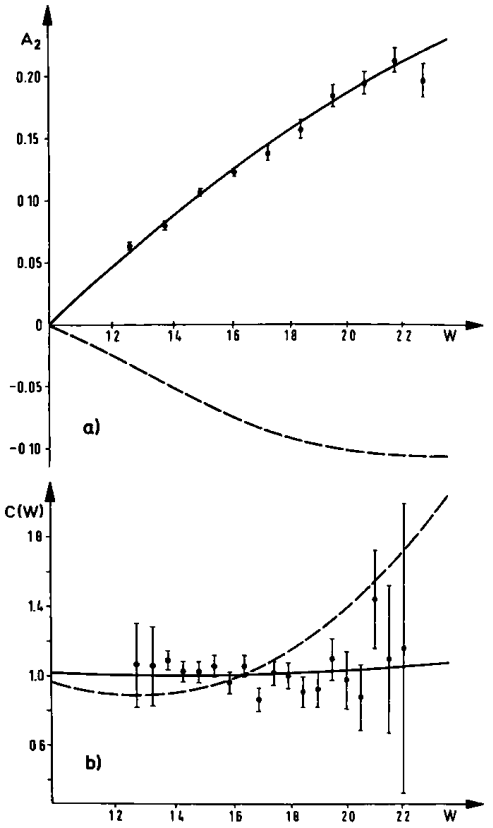


FIG. 14.23. Observables of ^{86}Rb (taken from Appel *et al.* 1974). (a) β–γ angular correlation. $A_2 = \varepsilon$ as a function of the beta energy W; (b) shape factor $C(W)$ (from Daniel *et al.* 1968) as a function of the beta energy W; (c) β–γ (CP) correlation. Polarization P_γ as a function of the angle θ between β and γ; (d) β–γ (CP) correlation. $\delta = \overline{\tilde{A}\varepsilon_D}$ (from Bosken *et al.* 1971) as a function of the beta energy W.
$\overline{\tilde{A}\varepsilon_D}$ means that the asymmetry coefficient $\tilde{A}$ has been folded with the detector analysing efficiency. The dashed line represents the simple single-particle shell model calculation. The solid line represents the results obtained by using the fitted configuration mixing coefficients.

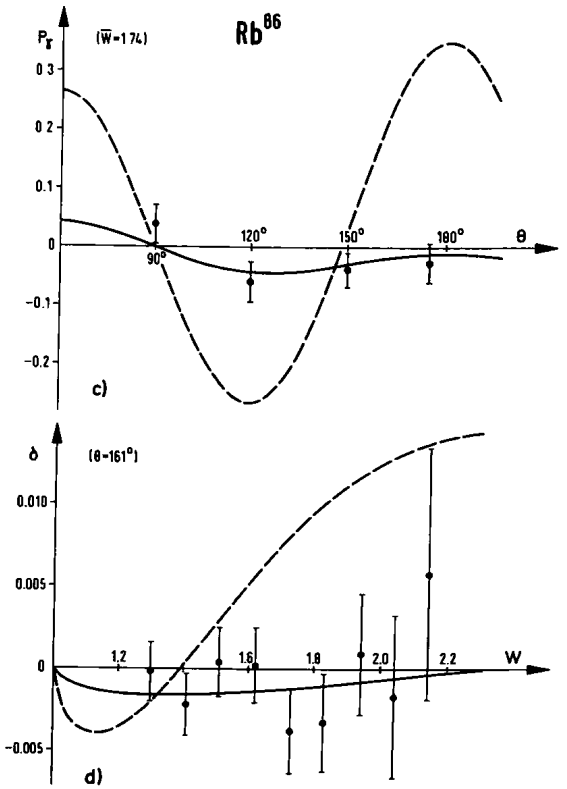


FIG. 14.23

initial and final states could be proposed† (Appel *et al.* 1974):

$$|^{86}\mathrm{Rb};2^-\rangle = a_0\,|[(\nu g_{9/2})^{-1}(\pi f_{5/2})^{-1}]2^-,00;2^-\rangle + a_1\,|[(\nu g_{9/2})_0^{-2}(\nu d_{5/2})^1(\pi f_{5/2})^{-1}]J_1',12;2^-\rangle + a_2\,|[(\nu g_{9/2})_0^{-2}(\nu g_{7/2})^1(\pi f_{5/2})^{-1}]J_2',12;2^-\rangle + a_3\,|[(\nu g_{9/2})_0^{-2}(\nu d_{3/2})^1(\pi f_{5/2})^{-1}]J_3',12;2^-\rangle \tag{14.602}$$

$$|^{86}\mathrm{Sr};2^+\rangle = b_0\,|[(\nu g_{9/2})_2^{-2}]2,00;2^+\rangle + b_1\,|[(\nu g_{9/2})_0^{-2}]0,12;2^+\rangle. \tag{14.603}$$

† In principle, the proton levels $1f_{7/2}$, $1h_{11/2}$ may also be coupled to the neutron hole $(1g_{9/2})^{-1}$ and a one-phonon quadrupole vibration. These wave function components are, however, considered to be not so important.

The mixing coefficients a_0, a_1, a_2, a_3, and b_0, b_1 can now be determined by a least squares fit to the experimental data shown in Fig. 14.23. For that approach use has to be made of eqns (8.77) and (8.147) and of the impulse approximation. As a result the following values were obtained (Appel *et al.* 1974)

$$\begin{aligned} 0{\cdot}92 \leqslant a_0 \leqslant 0{\cdot}99 \qquad & 0{\cdot}56 \leqslant b_0 \leqslant 0{\cdot}89 \\ 0{\cdot}13 \leqslant a_1 \leqslant 0{\cdot}37 \qquad & 0{\cdot}45 \leqslant b_1 \leqslant 0{\cdot}83 \\ 0{\cdot}02 \leqslant a_2 \leqslant 0{\cdot}07 & \\ -0{\cdot}09 \leqslant a_3 \leqslant -0{\cdot}02 & \end{aligned} \tag{14.604}$$

Here, harmonic oscillator radial wave functions were used and a normalization condition of

$$\sum_i a_i^2 = 1, \qquad \sum_i b_i^2 = 1 \tag{14.605}$$

in the basis space was chosen.† A simple theoretical estimation in the weak coupling model gives on the other hand

$$\begin{aligned} a_1 &\simeq 0{\cdot}35 \\ a_2 &\simeq 0{\cdot}035\,. \\ a_3 &\simeq 0 \end{aligned} \tag{14.606}$$

The agreement with the extracted coefficients is very good despite the rough approach used for this estimation. The result found for the mixing coefficients b_0 and b_1 of the first excited 2^+-level of ^{86}Sr is also fully consistent with a theoretical consideration carried out microscopically by Ogawa (1973). Further details can be taken from the corresponding publication (Appel *et al.* 1974).

Similar direct determinations of nuclear model parameters (configuration mixing coefficients, deformation parameters etc.) for the first non-unique forbidden beta-decays of ^{42}K, ^{142}Pr, and ^{170}Tm have been made (Gehrhardt 1977). By the way, the practicability of this method has been demonstrated excellently in this latter paper.

As far as the first non-unique forbidden transitions are concerned, in relevent theoretical analyses great efforts were undertaken in the past to extract the corresponding nuclear transition matrix elements‡ by fitting the different observables. In that context, the method for searching these

† The ft-value is then, however, not correctly reproduced. If this goal is to be achieved the coefficients $a_i b_j$ have to be divided by a factor of 3·46. Of course, the normalization condition of eqns (14.605) then does not hold any more.

‡ In earlier times beta-decay has always been described in terms of nuclear matrix elements because the impulse approximation has been introduced at a very early stage in the derivation of the corresponding expressions.

quantities was always based on the minimization of a reduced chi-square function χ^2 as defined before in eqns (14.568) to (14.570). In reality, form factor coefficients were, however, extracted from the observables and not, as stated in many papers, pure nuclear matrix elements since the parameters† determined include exchange, many-body effects etc. Besides, it is worth noting that often approximations have been introduced for the treatment of the Coulomb functions (sometimes inappropriately) in earlier analyses (a discussion of that point has, for instance, been given by Bühring 1968). Here, the Coulomb functions were taken at the nuclear radius R and then partly treated as constants as far as the integration within the radial integrals of the nuclear matrix elements is concerned (Morita and Morita 1958). Only the strong r-dependence, which is determined by the orbital angular momentum of the electron partial wave, was taken into account under the integration while the remaining weak r dependence (in our treatment of the functions $H_k(r)$, $h_k(r)$, $D_k(r)$, and $d_k(r)$, see eqns (4.4a–d)) was neglected. In other approaches chosen in the past the electron radial wave functions were expanded in terms of powers of the radial co-ordinate r. That is not exactly valid, as illustrated before in Section 4.3, and, by the way, was the reason why dominant and higher order terms were not separated correctly in these treatments ($I(1, 1, 1, 1; r)$ was approximated by $\frac{3}{2}-\frac{3}{10}(r/R)^2$ for $0 \leqslant r \leqslant \infty$ whereby only the first term $\frac{3}{2}$ was included into the dominant terms). In many papers dealing with the extraction of nuclear matrix elements one also finds statements like 'beta spectroscopy can give useful

† Historically, the following notations have been introduced many years ago which are often used in the literature:

$$\eta v = {}^{\wedge}F^{(0)}_{000}$$
$$\eta w = -R\,{}^{\wedge}F^{(0)}_{011}$$
$$\eta y = {}^{\vee}F^{(0)}_{101}$$
$$\eta x = -R\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(0)}_{110}$$
$$\eta u = -R\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(0)}_{111}$$
$$\eta z = 2R\sqrt{\tfrac{1}{3}}\,{}^{\wedge}F^{(0)}_{211}$$
$$V = {}^{\wedge}F^{(0)}_{000} - \xi R\,{}^{\wedge}F^{(0)}_{011}$$
$$Y = {}^{\vee}F^{(0)}_{101} + \xi R\sqrt{\tfrac{2}{3}}\,{}^{\wedge}F^{(0)}_{111} + \xi R\sqrt{\tfrac{1}{3}}\,{}^{\vee}F^{(0)}_{110}$$

with $\xi = \alpha Z/2R$.

For larger atomic numbers Z the quantities V and Y are approximately related to ours as (but not if cancellation occurs)

$$V \simeq A_0$$
$$Y \simeq -C_0.$$

The parameter η is a common normalization factor which can be determined by either selecting one form factor coefficient as a reference or by reproducing the ft-value correctly.

information on nuclear structure', but if one looks at the conclusions of these papers one often finds that the information on nuclear structure actually obtained is rather limited. The reason for such a poor result then was that the individual matrix elements determined were highly unspecified either because there were too many or because only combinations of them (like in our formalism A_0, C_0 etc.) could be fitted accurately.

Because of the importance which is usually attributed to that method we will, however, explicitly mention some more recent examples in the following, where accurate enough Coulomb functions (some of the approximations in the treatment of the Coulomb functions mentioned above have, however, always been made) were used, where the analysis was carried out appropriately and where real nuclear structure information was partly extracted. These are the decays of ^{198}Au (Schweitzer and Simms 1972; Bosch *et al.* 1973; Krmpotic and Szybisz 1974), of ^{122}Sb and ^{124}Sb (Smith *et al.* 1973; Bosch *et al.* 1976; Ko-nashi *et al.* 1976), of ^{141}Ce (Civitarese *et al.* 1973; Szybisz 1973; Szybisz *et al.* 1974), of ^{137}Xe (Szybisz 1974), of ^{148}Pm (Szybisz 1975), and of ^{140}La (Smith and Simms 1970). Further older examples are discussed and summarized in the books by Schopper (1966) and by Morita (1973).

Concluding this section we also will give reference to some other more recent attempts to calculate theoretically beta-decay observables for first non-unique forbidden transitions. Here, the theoretical considerations of the decay ^{15}C and of the mirror decays ^{17}N and ^{17}Ne should be mentioned (Towner and Hardy 1972). Another decay, namely that of ^{16}O, which is of the type 0^-–0^+, has been analysed by Holstein and Kim (1979). For the decays of the deformed nuclei ^{186}Re and ^{188}Re, which are very difficult to describe from a nuclear structure point of view, a very detailed theoretical study has been made by Bogdan *et al.* (1980), where it was attempted to explain the corresponding observables by making use of the triaxial rotor model with decoupled nucleons (earlier calculations in the framework of an axially symmetric rigid rotor have been carried out by Behrens and Bogdan (1970), and by van der Werf (1969, 1971)). The effect of induced terms like weak magnetism has also been considered in this latter article. Induced terms are here of importance because very strong cancellation between the dominant terms occurs such that in some cases ($1^- \rightarrow 2^+$ transitions) relativistic weak magnetism terms govern the transition to a large extent. It is also worth mentioning that in this latter treatment the formalism outlined in this book has been used so that the electron radial wave functions have been taken into account in a completely exact manner.

Generally, perfect agreement between experimental data and theoretical calculations for first non-unique forbidden transitions can only be

expected if firstly a reliable nuclear model for the participating nuclei exists and secondly

(i) induced,
(ii) exchange,
(iii) core polarization,

effects are taken into account.

14.3.5. Summary

From the equations and the examples of the preceding sections we can conclude that the observables of first non-unique forbidden transitions are generally functions of the electron (positron) energy even if only dominant contributions are taken into account. Some observables which are very small in the case of allowed transitions like, for example, the anisotropy a_2 of electrons emitted from oriented nuclei or the anisotropy coefficient ε of the β–γ angular correlation are one or two orders of magnitude larger. Normally, we expect that deviations from the allowed form are larger as the higher is the corresponding transition energy W_0, i.e. they can be well observed exactly if $W_0R \gtrsim \alpha Z$. Three extreme cases which are partly opposite can, however, be distinguished.

(a) The situation where

$$|A_0| \text{ or } |C_0| \gg |RB_0| \simeq |W_0RC_1| \simeq |RD_0| \simeq |W_0RE_0| \simeq |W_0RF_0| \simeq |W_0RG_0| \simeq |W_0RH_0|. \qquad (14.607)$$

Then, the first non-unique forbidden transitions behave equally as the allowed ones. This becomes immediately evident since that fact is just characteristic for allowed transitions (see Section 14.2.1.4). It is obvious that the above condition is more probably, but not necessarily, fulfilled for heavier nuclei (see eqns (14.452)–(14.460)). Then, the Coulomb energy of an electron at the nuclear radius $2\xi = \alpha Z/R$ is usually large compared to the total decay energy W_0. Thus, the terms in A_0 and C_0 connected with αZ can dominate the other ones and determine the magnitude of A_0 and C_0 together with the behaviour of the whole transistions (for $\alpha Z \to 1$ this will be immediately clear from eqns (14.452) and (14.454)). Usually, this case where $\xi \gg W_0$ is called† ξ-approximation, but, as we have seen from our example $^{209}\text{Pb} \to {}^{209}\text{Bi}$, A_0 or C_0 can also be dominated by ${}^{A}F^{(0)}_{000}$ or ${}^{V}F^{(0)}_{101}$, respectively, because of a special nuclear structure, and that despite the fact that these are relativistic types of form factor coefficients compared to the others which are non-relativistic ones.

† Then, the terms proportional to W_0R and to W_eR can be neglected and are therefore omitted.

(b) The case where the quantities A_0 and C_0 are small because of a cancellation of the different terms, although the contributing form factor coefficients themselves have normal values. Then we have

$$|A_0| \text{ or } |C_0| \simeq |RB_0| \simeq |W_0RC_1| \simeq |RD_0| \simeq |W_0RE_0| \simeq |W_0RF_0| \simeq |W_0RG_0| \simeq |W_0RH_0|. \quad (14.608)$$

Usually, this situation is denoted as cancellation effect. As we have explicitly demonstrated on our examples (^{210}Bi or ^{206}Tl) this case is not so rare as it might seem on a first look. On the other hand, we then expect strong deviations from the allowed form for all observables.

(c) The case where

$$|H_0| \simeq |G_0| \gg |A_0| \simeq |B_0| \simeq |C_0| \simeq |C_1| \simeq |D_0| \simeq |E_0| \simeq |F_0|. \quad (14.609)$$

This may happen because of a special nuclear selection rule by which the tensor rank 2 form factor coefficients are allowed, but the tensor rank 0 and 1 ones are forbidden and therefore hindered (the example ^{86}Rb discussed before was of this type). We can distinguish two types of selection rules which can be responsible for such a peculiar order of magnitude of the different quantities A_0, B_0 etc.

(i) *The j-selection rule* The essential point of this selection rule has its origin in the shell model states which predominantly built up the configurations of initial and final nuclear states. If their total angular momentum j differs by $\Delta j = |j_i - j_f| = 2$ and their orbital angular momentum l by $\Delta l = |l_i - l_f| = 1$, the matrix element $^{A}M^{(0)}_{211}$ is allowed while the other matrix elements M_{0Ls} and M_{1Ls} are forbidden (see Table 14.12; in the decay of ^{86}Rb we had $\nu 1g_{9/2} \rightarrow \pi 1f_{5/2}$). In reality, the matrix elements M_{0Ls} and M_{1Ls} are, however, not zero because of the mixing of the shell model configurations, but they are more or less strongly hindered.

(ii) *The K-selection rule* This is an additional selection rule which occurs for deformed nuclei (see Section 8.1.1.5.2). It reads as

$$K \geqslant |K_f - K_i| \quad (14.610)$$

where K is the tensor rank of the form factors, and K_i and K_f are the projections of the angular momentum J of the deformed core to the body fixed axis for initial and final nuclear state. If $|K_i - K_f|$ is now larger than $\Delta J = J_i - J_f$ where J_i and J_f are the total angular momenta of initial and final nuclear states, some form factor coefficients are suppressed. For example, if $|K_i - K_f| = 2$ and $\Delta J = 0$ those of tensor rank

$K=0$ and $K=1$ are not permitted and therefore hindered compared with ${}^{A}F_{211}^{(0)}$.

The extreme of this case corresponds, of course, to the unique forbidden behaviour since this latter type of transition is just characterized by the limit

$$|H_0| = |G_0| \neq 0$$

and (14.611)

$$|A_0| = |B_0| = |C_0| = |C_1| = |E_0| = |F_0| = 0.$$

From a nuclear physics point of view decays which deviate strongly from the allowed form are of more interest. Then, most of the observables react sensitively on details of the nuclear structure and the first non-unique forbidden transitions can be used as a tool for investigating details of the nuclear structure.

14.4. First unique forbidden transitions

The first unique forbidden transitions are defined by the selection rule (see eqn (14.32))

$$\Delta J = 2 \qquad \pi_i \pi_f = -1. \tag{14.612}$$

Thus, they can, up to a point, be considered to be crippled first non-unique forbidden transitions, as the form factor coefficients of rank 0 and 1 are forbidden by the above selection rule and only the still surviving terms of tensor rank 2 contribute. If dominant terms only are included these then are the following ones (see eqns (14.56) and (14.58)):

$$M_2(1,2) = \tfrac{1}{3} p_\nu R G_0 \tag{14.613a}$$

$$M_2(2,1) = \tfrac{1}{3} p_e R H_0 \tag{14.613b}$$

where (see eqns (14.459) and (14.460))

$$H_0 = G_0 = -{}^{A}F_{211}^{(0)}. \tag{14.614}$$

If higher order terms are additionally considered the complete eqns (14.446) and (14.449) have to be taken. Of course, one can put the question why these transitions are treated separately. The answer is that this class of forbidden transitions can be handled as simply as the allowed ones in so far as all observables with exception of the ft-value are independent of any form factor coefficients, i.e. of nuclear structure details (if higher order terms are neglected). In principle, the allowed Gamow–Teller transitions can namely just be considered as the 0th unique forbidden transition (see Section 14.1). Besides, it is also the fact

that they generally do not depend on any form factor coefficient which was the origin for their name 'unique' (with exception of the ft-value).

By inserting the above eqns (14.613) for $M_2(1, 2)$ and $M_2(2, 1)$ into the relevant general equations of Section 7.2, we then obtain the formulae (the upper sign means always β^--decay, the lower always β^+-decay):

(a) shape factor and ft-value (see eqn (7.56))

$$C(W_e) = \tfrac{1}{9} R^2 ({}^A F_{211}^{(0)})^2 (p_\nu^2 + \lambda_2 p_e^2) \tag{14.615}$$

$$ft = \frac{18\pi^3 \ln 2}{G_\beta^2 R^2 ({}^A F_{211}^{(0)})^2 (\overline{p_\nu^2} + \overline{\lambda_2 p_e^2})} \tag{14.616}$$

(the bar means averaging over the beta-spectrum);

(b) distribution of electrons emitted from oriented nuclei (see eqns (7.38) to (7.40) and Table 7.2)

$$\omega(\theta_e, W_e) = 1 + a_1 P_1(\cos\theta_e) + a_2 P_2(\cos\theta_e) + a_3 P_3(\cos\theta_e) \tag{14.617}$$

(θ_e = angle between axis of orientation and electron emission) with

$$a_1 = \pm \left\{ \frac{15 J_i}{2(J_i+1)} \right\}^{1/2} f_1(J_i) \frac{p_e}{W_e} \frac{\Lambda_1 p_\nu^2 + \frac{3}{5}\lambda_2 \Lambda_2 p_e^2}{p_\nu^2 + \lambda_2 p_e^2} \tilde{\Gamma}_{22}(1) \tag{14.618}$$

$$a_2 = -3 J_i^2 \left\{ \frac{35}{2J_i(J_i+1)(2J_i-1)(2J_i+3)} \right\}^{1/2} f_2(J_i) \frac{\lambda_2 p_e^2}{p_\nu^2 + \lambda_2 p_e^2} \tilde{\Gamma}_{22}(2) \tag{14.619}$$

$$a_3 = \mp 3 J_i^3 \left\{ \frac{70}{J_i(J_i+1)(J_i+2)(J_i-1)(2J_i-1)(2J_i+3)} \right\}^{1/2} \times f_3(J_i) \frac{p_e}{W_e} \frac{\lambda_2 \Lambda_2 p_e^2}{p_\nu^2 + \lambda_2 p_e^2} \tilde{\Gamma}_{22}(3). \tag{14.620}$$

For $f_1(J)$, $f_2(J)$, and $f_3(J)$ see eqns (14.151), (14.174), and (14.488), respectively;

(c) β–γ and β–α correlation (see eqns (7.74)–(7.76), (7.79)–(7.80), and Table 7.2)

$$\omega(W_e, \theta_{\beta\gamma}) = 1 + \tau \frac{v}{c} \tilde{A} P_1(\cos\theta_{\beta\gamma}) + \varepsilon P_2(\cos\theta_{\beta\gamma}) \tag{14.621}$$

($\theta_{\beta\gamma}$ is the angle between β-particle and γ-quantum, τ determines the sign of the γ-circular polarization and is equal to ± 1, v and c

are the velocities of the β-particles and of light) with

$$\tilde{A} = \pm \frac{(5\Lambda_1 p_\nu^2 + 3\Lambda_2\lambda_2 p_e^2)\Gamma_{22}(1)A_1(\gamma) - 6\Lambda_2\lambda_2 p_e^2(\frac{5}{2}\cos^2\theta_{\beta\gamma} - \frac{3}{2})\Gamma_{22}(3)A_3(\gamma)}{\sqrt{10}\,(p_\nu^2 + \lambda_2 p_e^2) - \sqrt{35}\,\lambda_2 p_e^2(\frac{3}{2}\cos^2\theta_{\beta\gamma} - \frac{1}{2})\Gamma_{22}(2)A_2(\gamma)} \tag{14.622}$$

$$\varepsilon = -\sqrt{\frac{7}{2}}\frac{\lambda_2 p_e^2}{p_\nu^2 + \lambda_2 p_e^2}\Gamma_{22}(2)A_2(\gamma) \tag{14.623}$$

For β–α correlations $\tilde{A}$ is zero, $\theta_{\beta\gamma}$ has to be replaced by the angle between β- and α-particle $\theta_{\beta\alpha}$ and $A_k(\gamma)$ by $A_k(\alpha)$. For $A_k(\gamma)$ see eqn (7.80) and for $A_k(\alpha)$ see eqn (7.76);

(d) electron–neutrino correlations (see eqns (7.96)–(7.98) and Table 7.4)

$$\omega(\theta_{e\nu}, W_e) = 1 + d_1 P_1(\cos\theta_{e\nu}) + d_2 P_2(\cos\theta_{e\nu}) \tag{14.624}$$

($\theta_{e\nu}$ is the angle between electron and neutrino) with

$$d_1 = -\frac{1}{5}\frac{p_e}{W_e}\frac{\Lambda_1 p_\nu^2 + \lambda_2\Lambda_2 p_e^2}{p_\nu^2 + \lambda_2 p_e^2} + 2\eta_{12}\frac{p_\nu p_e}{p_\nu^2 + \lambda_2 p_e^2} \tag{14.625}$$

$$d_2 = -\frac{2}{5}\frac{p_e}{W_e}\nu_{12}\frac{p_\nu p_e}{p_\nu^2 + \lambda_2 p_e^2}; \tag{14.626}$$

(e) electron–neutrino triple correlations (see eqns (7.13), (7.110), (7.112), (7.114), (7.116))

$$\omega_J(\theta_e, \theta_\nu, W_e) = 1 + \tilde{d}\,\frac{\mathbf{J}(\mathbf{p}_e \times \mathbf{p}_\nu)}{J p_e p_\nu} \tag{14.627}$$

with

$$\tilde{d} = 3\left\{\frac{15J_i}{2(J_i+1)}\right\}^{1/2} f_1(J_i) s e_{12}\frac{p_\nu p_e}{p_\nu^2 + \lambda_2 p_e^2}\tilde{\Gamma}_{22}(1); \tag{14.628}$$

(f) distribution of gamma radiation emitted from oriented nuclei without detecting the intermediate beta-decay (if gamma-polarization is not observed) (see eqns (7.131)–(7.132) and Table 7.2)

$$\omega_\gamma(\theta_\gamma) = \sum_{k=\text{even}} h_k P_k(\cos\theta_\gamma) \tag{14.629}$$

(θ_γ = angle between gamma quantum and axis of orientation)

$$h_k = (-1)^{J_i+J_f}\{2J_i+1\}\{2J_f+1\}^{1/2}\binom{2k}{k}J_i^k\left\{\frac{(2k+1)(2J_i-k)!}{(2J_i+k+1)!}\right\}^{1/2}$$
$$\times f_k(J_i)\begin{Bmatrix} J_i & J_i & k \\ J_f & J_f & 2 \end{Bmatrix}A_k(\gamma). \tag{14.630}$$

For $A_k(\gamma)$ see eqn (7.80);

(g) electron polarization (see eqn (7.151))

$$P_e = \mp \frac{p_e}{W_e} \Lambda_1 \left[1 + \left(\frac{\Lambda_2}{\Lambda_1} - 1 \right) \lambda_2 \frac{p_e^2}{p_\nu^2 + \lambda_2 p_e^2} \right]; \tag{14.631}$$

(h) electron capture (see eqns (7.174)–(7.176)). For the ft-value we have

$$n_x f_x t = \frac{2\pi^3 \ln 2}{G_\beta^2 C_x} \tag{14.632}$$

(the index x stands for the different atomic shells K, L_1, L_2, L_3 etc. For closed atomic shells it is $n_x = 1$) with

$$C_x = \tfrac{1}{9} R^2 ({}^A F_{211}^{(0)})^2 (p_{\nu_x})^2 \tag{14.633}$$

for capture from atomic shells with $\kappa_x = \pm 1$ (K, L_1, L_2, M_1, M_2 etc.) and

$$C_x = \tfrac{1}{9} R^2 ({}^A F_{211}^{(0)})(p_{e_x})^2 \tag{14.634}$$

for capture from atomic shells with $\kappa_x = \pm 2$ (L_3, M_3, M_4 etc.). For the L_1/K capture† ratio, for instance, then follows

$$\frac{\lambda_{L_1}}{\lambda_K} = \frac{n_{L_1} \beta_{L_1}^2 B_{L_1} p_{\nu_{L_1}}^4}{n_K \beta_K^2 B_K p_{\nu_K}^4} \tag{14.635}$$

(for the amplitudes β_x see Section 4.6.1 and for the exchange and overlap correction factors see Section 4.6.2).

Expressions for the L_2/K, M_1/K, L_2/L_1, and M_1/L_1 capture ratios are entirely analogous. The L_3/L_1 ratio is, on the other hand, given by

$$\frac{\lambda_{L_3}}{\lambda_{L_1}} = \frac{n_{L_3} (p_{e_{L_3}} \beta_{L_3})^2 B_{L_3} p_{\nu_{L_3}}^2}{n_{L_1} \beta_{L_1}^2 B_{L_1} p_{\nu_{L_1}}^4}. \tag{14.636}$$

Other $k_x = 2$ to $k_x = 1$ capture ratios are analogous.

The electron capture to positron decay ratio for K-capture reads as

$$\frac{\lambda_K}{\lambda_{\beta^+}} = \frac{\pi}{2} \frac{n_K \beta_K^2 B_K p_{\nu_K}^4}{(p_\nu^2 + \lambda_2 p_e^2) f_{\beta^+}}. \tag{14.637}$$

The corresponding ratio for L_1, L_2, M_1, M_2 capture etc. is analogous.

It should be pointed out that the Coulomb function λ_2 (see Section 4.2,

† Note that the decay constant is

$$\lambda = \frac{\ln 2}{t}.$$

eqn (4.10)) is of special importance for the first unique forbidden transitions while the influence of the other Coulomb functions Λ_1, Λ_2, and η_{12} occurring is not so remarkable. Numerical values for those functions can, by the way, be found in the comprehensive tables by Behrens and Jänecke (1969) to which we always refer in this text. The electron triple correlation plays, however, a special role in that context. Its Coulomb function se_{12} (see Section 4.2.3, eqns (4.51) and (4.58)) depends strongly on beta-energy and atomic number Z, and governs the whole behaviour of that observable.

Corresponding formulae which use the same Coulomb function definitions as we do can partly also be found in the books by Schopper (1966) and by Behrens and Jänecke (1969). For electron capture reference should be made to the review article of Bambynek *et al.* (1977) where relevant expressions are described in detail and numerical values for the atomic bound state amplitudes β_x (Section 4.6.1) together with the exchange and overlap correction factors B_x can be taken from.

Naturally, higher order terms which can be included when the complete expressions for the quantities G_0 and H_0 given in eqns (14.466) and (14.449) together with the other quantities G_1 and H_1 given in eqns (14.447) and (14.450) are taken into account, can contribute.

Let us consider as an example the shape factor $C(W_e)$ in a little more detail. Including the higher order term just mentioned we then get (Behrens *et al.* 1975)

$$C(W_e) = \tfrac{1}{9}R^2(G_0^2 p_\nu^2 + H_0^2 \lambda_2 p_e^2)[1 + a(W_e) W_e] \tag{14.638}$$

where

$$a(W_e) = 2R \frac{G_0 G_1 p_\nu^2 + H_0 H_1 \lambda_2 p_e^2}{G_0^2 p_\nu^2 + H_0^2 \lambda_2 p_e^2}. \tag{14.639}$$

Because in G_0 and H_0 the dominant term is $-{}^{A}F_{211}^{(0)}$ in a first approximation there is $G_0 \simeq H_0$. Thus, the above expression reduces to

$$C(W_e) = C_N(W_e)[1 + a(W_e) W_e] \tag{14.640}$$

where $C_N(W_e)$ is the normal unique shape factor of eqn (14.615) and $a(W_e)$ reads as

$$a(W_e) = R \frac{c_1 p_\nu^2 + c_2 \lambda_2 p_e^2}{p_\nu^2 + \lambda_2 p_e^2} \tag{14.641}$$

with c_1 and c_2 being constants. For the evaluation of experiments this expression has often been approximated by $a(W_e) = a$.

Indeed, by surveying the existing experimental material (see the compilation of Behrens and Szybisz 1976) we find that small deviations from

the normal unique form of the following order of magnitude exist

$$0 > a > -0{\cdot}06. \tag{14.642}$$

A prominent example for more recent precision experiments has, for example, been carried out for the $2^- \rightarrow 0^+$ decay of ^{90}Y (Flothmann *et al.* 1975) with the result $a = -0{\cdot}0114 \pm 0{\cdot}0003$. β–γ angular correlations have also experimentally been investigated (for the decay of ^{144}Pr, see de Raedt 1970; Wischhusen 1973; Bosch *et al.* 1973). Deviations from the unique form have, however, not been found for this latter observable. Analogous to the allowed transitions, induced terms influence the unique forbidden transitions via the higher order contributions too. Some analyses to explain the deviations of the shape factors from the unique form (other observables have also been considered) by introducing higher order terms including either induced terms or assuming induced coupling constants to be zero have been presented in the last years (Abecasis and Krmpotic 1970; Bosch *et al.* 1973; Behrens *et al.* 1975; Eman *et al.* 1975). No definite conclusions could, however, be drawn from these analyses. Certainly, the unique forbidden transitions are as suitable as the allowed ones to investigate hindrance phenomena by core polarization, configuration mixing effects etc. Thus, this type of transition was preferably used as a tool for such investigations (Towner and Warburton 1971). The corresponding effects have been summarized by Ejiri and Fujita (1978). A uniform reduction factor of $({}^{A}\mathfrak{M}^{(0)}_{211})_{\text{eff}}/({}^{A}\mathfrak{M}^{(0)}_{211})_{\text{single particle}} \simeq 0{\cdot}3$ compared to free nuclear values has been quoted by these authors which can mostly be attributed to core polarization effects. Theoretical analyses confirm this latter conclusion.

14.5. Higher forbidden transitions

General expressions for each observable and for every degree of forbiddeness have been evaluated in Chapter 7. By inserting the corresponding quantities $M_K(k_e, k_\nu)$ and $m_K(k_e, k_\nu)$, which are given

(i) in eqns (14.11)–(14.14) for non-unique forbidden transitions,

(ii) in eqn (14.14) for unique forbidden transitions.

Formulae for the observables can be derived (if dominant terms only are considered). They belong to the following selection rules (see eqns (14.31)–(14.32)):

$$\pi_i \pi_f = (-1)^K \qquad K\text{th non-unique forbidden transitions} \tag{14.643}$$

$$\pi_i \pi_f = (-1)^{K-1} \qquad (K-1)\text{th unique forbidden transitions.} \tag{14.644}$$

Beyond that, for convenience of the reader we will explicitly list some expressions:

(i) for second non-unique forbidden transitions, for example, we have

for $k_e + k_\nu = 3$

$$M_2(k_e, k_\nu) = 2\sqrt{\frac{1}{15}}\left\{\frac{1}{(2k_e-1)!\,(2k_\nu-1)!}\right\}^{1/2}(p_eR)^{k_e-1}$$
$$\times(p_\nu R)^{k_\nu-1}\left\{-\sqrt{\tfrac{5}{2}}\,{}^{\mathrm{V}}F^{(0)}_{211} \mp \frac{\alpha Z}{2k_e+1}\,{}^{\mathrm{V}}F^{(0)}_{220}(k_e, 1, 1, 1)\right.$$
$$-\left[\frac{W_eR}{2k_e+1}+\frac{p_\nu R}{2k_\nu+1}\right]{}^{\mathrm{V}}F^{(0)}_{220} \mp \frac{\alpha Z}{2k_e+1}$$
$$\left.\times\sqrt{\tfrac{3}{2}}\,{}^{\mathrm{A}}F^{(0)}_{221}(k_e, 1, 1, 1) - \left[\frac{W_eR}{2k_e+1}-\frac{p_\nu R}{2k_\nu+1}\right]\sqrt{\tfrac{3}{2}}\,{}^{\mathrm{A}}F^{(0)}_{221}\right\} \tag{14.645}$$

$$m_2(k_e, k_\nu) = -2\sqrt{\frac{1}{15}}\left\{\frac{1}{(2k_e-1)!\,(2k_\nu-1)!}\right\}^{1/2}(p_eR)^{k_e-1}$$
$$\times(p_\nu R)^{k_\nu-1}\frac{R}{2k_e+1}\{{}^{\mathrm{V}}F^{(0)}_{220} + \sqrt{\tfrac{3}{2}}\,{}^{\mathrm{A}}F^{(0)}_{221}\}, \tag{14.646}$$

for $k_e + k_\nu = 4$

$$M_2(k_e, k_\nu) = 2\sqrt{\frac{2}{5}}\left\{\frac{1}{(2k_e-1)!\,(2k_\nu-1)!}\right\}^{1/2}(p_eR)^{k_e-1}(p_\nu R)^{k_\nu-1}$$
$$\times\left\{\frac{1}{(2k_e-1)(2k_\nu-1)}\right\}^{1/2}\left\{{}^{\mathrm{V}}F^{(0)}_{220} - \frac{k_e-k_\nu}{3}\sqrt{\frac{3}{2}}\,{}^{\mathrm{A}}F^{(0)}_{221}\right\} \tag{14.647}$$

$$M_3(k_e, k_\nu) = -2\sqrt{\frac{2}{15}}\left\{\frac{1}{(2k_e-1)!\,(2k_\nu-1)!}\right\}^{1/2}$$
$$\times(p_eR)^{k_e-1}(p_\nu R)^{k_\nu-1}\,{}^{\mathrm{A}}F^{(0)}_{321}; \tag{14.648}$$

(ii) for second unique forbidden transitions, on the other hand, we have $(k_e + k_\nu = 4)$

$$M_3(k_e, k_\nu) = -2\sqrt{\frac{2}{15}}\left\{\frac{1}{(2k_e-1)!\,(2k_\nu-1)!}\right\}^{1/2}(p_eR)^{k_e-1}(p_\nu R)^{k_\nu-1}\,{}^{\mathrm{A}}F^{(0)}_{321}. \tag{14.649}$$

In that case shape factor and electron polarization can very easily be written down to (see eqn (7.56))

$$C(W_e) = \frac{1}{15^2}R^4({}^{\mathrm{A}}F^{(0)}_{321})^2(p_\nu^4 + \tfrac{10}{3}\lambda_2 p_e^2 p_\nu^2 + \lambda_3 p_e^4) \tag{14.650}$$

and (see eqn (7.151))

$$P_e = \mp \frac{p_e}{W_e} \frac{\Lambda_1 p_\nu^4 + \frac{10}{3}\Lambda_2 \lambda_2 p_e^2 p_\nu^2 + \Lambda_3 \lambda_3 p_e^4}{p_\nu^4 + \frac{10}{3}\lambda_2 p_e^2 p_\nu^2 + \lambda_3 p_e^4}. \tag{14.651}$$

If the electron energy is not too small so that $\Lambda_n \simeq \Lambda_1$ this reduces to

$$P_e = \mp \frac{p_e}{W_e} \Lambda_1;$$

(iii) for third non-unique forbidden transitions the corresponding expressions read for $k_e + k_\nu = 4$

$$M_3(k_e, k_\nu) = 2\sqrt{\frac{2}{35}} \left\{ \frac{1}{(2k_e - 1)!\,(2k_\nu - 1)!} \right\}^{1/2} (p_e R)^{k_e - 1} (p_\nu R)^{k_\nu - 1}$$
$$\times \left\{ -\sqrt{\tfrac{7}{3}}\, {}^{\mathrm{V}}F_{321}^{(0)} \mp \frac{\alpha Z}{2k_e + 1}\, {}^{\mathrm{V}}F_{330}^{(0)}(k_e, 1, 1, 1) \right.$$
$$- \left[\frac{W_e R}{2k_e + 1} + \frac{p_\nu R}{2k_\nu + 1} \right] {}^{\mathrm{V}}F_{330}^{(0)} \mp \frac{2\alpha Z}{(2k_e + 1)} \sqrt{\frac{1}{3}}$$
$$\left. \times\, {}^{\mathrm{A}}F_{331}^{(0)}(k_e, 1, 1, 1) - 2\left[\frac{W_e R}{2k_e + 1} - \frac{p_\nu R}{2k_\nu + 1} \right] \sqrt{\frac{1}{3}}\, {}^{\mathrm{A}}F_{331}^{(0)} \right\} \tag{14.652}$$

$$m_3(k_e, k_\nu) = -2\sqrt{\frac{2}{35}} \left\{ \frac{1}{(2k_e - 1)!\,(2k_\nu - 1)!} \right\}^{1/2} (p_e R)^{k_e - 1} (p_\nu R)^{k_\nu - 1}$$
$$\times \frac{R}{2k_e + 1} \left[{}^{\mathrm{V}}F_{330}^{(0)} + 2\sqrt{\tfrac{1}{3}}\, {}^{\mathrm{A}}F_{331}^{(0)} \right], \tag{14.653}$$

for $k_e + k_\nu = 5$

$$M_3(k_e, k_\nu) = 8\sqrt{\frac{1}{35}} \left\{ \frac{1}{(2k_e - 1)!\,(2k_\nu - 1)!} \right\}^{1/2} (p_e R)^{k_e - 1} (p_\nu R)^{k_\nu - 1}$$
$$\times \left\{ \frac{1}{(2k_e - 1)(2k_\nu - 1)} \right\}^{1/2} \left\{ {}^{\mathrm{V}}F_{330}^{(0)} - \frac{k_e - k_\nu}{2} \sqrt{\frac{1}{3}}\, {}^{\mathrm{A}}F_{331}^{(0)} \right\} \tag{14.654}$$

$$M_4(k_e, k_\nu) = -4\sqrt{\frac{1}{35}} \left\{ \frac{1}{(2k_e - 1)!\,(2k_\nu - 1)!} \right\}^{1/2}$$
$$\times (p_e R)^{k_e - 1} (p_\nu R)^{k_\nu - 1}\, {}^{\mathrm{A}}F_{431}^{(0)}; \tag{14.655}$$

(iv) for third unique forbidden transitions, on the other hand, we obtain ($k_e + k_\nu = 5$)

$$M_4(k_e, k_\nu) = -4\sqrt{\frac{1}{35}\left\{\frac{1}{(2k_e-1)!\,(2k_\nu-1)!}\right\}^{1/2}} \times (p_e R)^{k_e-1}(p_\nu R)^{k_\nu-1}\,{}^{A}F^{(0)}_{431}. \tag{14.656}$$

As before, the shape factor and electron polarization can easily be evaluated to (see eqn (7.56))

$$C(W_e) = \frac{1}{105^2} R^6({}^{A}F^{(0)}_{431})^2(p_\nu^6 + 7\lambda_2 p_e^2 p_\nu^4 + 7\lambda_3 p_e^4 p_\nu^2 + \lambda_4 p_e^6) \tag{14.657}$$

and to (see eqn (7.151))

$$P_e = \mp\frac{p_e}{W_e}\,\frac{\Lambda_1 p_\nu^6 + 7\Lambda_2\lambda_2 p_e^2 p_\nu^4 + 7\Lambda_3\lambda_3 p_e^4 p_\nu^2 + \Lambda_4\lambda_4 p_e^6}{p_\nu^6 + 7\lambda_2 p_e^2 p_\nu^4 + 7\lambda_3 p_e^4 p_\nu^2 + \lambda_4 p_e^6}. \tag{14.658}$$

For higher electron energies where $\Lambda_n \simeq \Lambda_1$ this reduces to $P_e = \mp(p_e/W_e)\Lambda_1$ (upper sign β^--decay, lower sign β^+-decay).

We recognize that the general structure of the higher forbidden transitions does not differ from that of the allowed and first forbidden ones. Therefore, we will not treat them in more detail and refer to the very detailed discussion of allowed and first forbidden transitions in the foregoing sections. Even so, very little experimental material exists up to now about higher forbidden transitions. Essentially, shape factors (for a survey see the compilation of Behrens and Szybisz 1976) and *ft*-values (see Raman and Gove 1973) have been determined experimentally. The longitudinal electron polarization P_e for the second forbidden decay of ^{137}Cs has been, for example, measured by Doty and Neill (1975).

For second and third non-unique forbidden transitions a few theoretical analyses for some special beta-transitions should also be mentioned, the older ones of Lipnik and Sunier (1966) and the more recent ones of Szybisz (1974) for ^{137}Cs and of Szybisz (1976) for ^{87}Rb. The second non-unique forbidden $7/2^+ \rightarrow 3/2^+$ decay of ^{137}Cs to ^{137}Ba which has a rather simple structure of the nuclear states involved ($\nu 2d_{3/2} \rightarrow \pi 1g_{7/2}$) shows also strong cancellation effects. Thus, the complete formalism has to be used. The corresponding experimental results can be explained by a quenching of the matrix elements of about a factor 5 (Szybisz 1974). In the analysis use has been made of the CVC relation between ${}^{V}F^{(0)}_{211}$ and ${}^{V}F^{(0)}_{220}$. The third forbidden $3/2^- \rightarrow 9/2^+$ decay of ^{87}Rb to ^{87}Sr can excellently be explained by assuming a single-particle transition $\pi 2p^{-1}_{3/2} \rightarrow \nu 1g^{-1}_{9/2}$ and a hindrance factor of 0·36 (see Szybisz 1976). In a certain sense, the CVC relation between ${}^{V}F^{(0)}_{321}$ and ${}^{V}F^{(0)}_{320}$ in its impulse approximation form (see eqns (10.63) and (10.65)) has been confirmed by this

latter analysis. It should also be remarked that for the decay of ^{87}Rb it is of special importance to use suitable Coulomb functions (Behrens and Jänecke 1969) because of the very low transition energy of 275 keV. In order to avoid any doubtful approximations higher order terms should also be carefully considered. The half-lives of the fourth forbidden β^--decay and electron capture of ^{50}V to ^{50}Cr and ^{50}Ti, respectively, have recently been calculated by Nishimura (1979) by carrying out a very detailed analysis.

Higher unique forbidden transitions are especially examined in order to explain hindrance phenomena and renormalization effects (see Warburton *et al.* 1970; Towner *et al.* 1971; Szybisz 1975). An interesting case in that context is the $0^+ \rightarrow 3^+$ decay of $^{10}\text{Be} \rightarrow {}^{10}\text{B}$ where good shell model wave functions exist (Cohen and Kurath 1965). Thus, special attention was given to that transition (see Warburton *et al.* 1969; Szybisz 1975). A renormalization of the axial vector coupling constant $\lambda_{Ae}/\lambda = 0{\cdot}923 \pm 0{\cdot}096$ was found from the analysis of the latter author (see also Section 8.2). For a review of core polarization effects the article by Ejiri and Fujita (1978) should be consulted where uniform reduction (quenching) factors for $(^{A}\mathfrak{M}^{(0)}_{321})_{\text{eff}}/(^{A}\mathfrak{M}^{(0)}_{321})_{\text{single particle}}$ of about $\simeq 0{\cdot}3$ and for $(^{A}\mathfrak{M}^{(0)}_{431})_{\text{eff}}/(^{A}\mathfrak{M}^{(0)}_{431})_{\text{single particle}}$ of about $\simeq 0{\cdot}4$ are quoted, respectively.

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